MthSc 119, Some extra practice problems

19.8 (a) The next three terms are $a_4 = 31$, $a_5 = 63$ and $a_6 = 127$.
We will use induction to prove that $a_n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.
Basis step: When $n = 0$, $a_0 = 1$ and $2^{0+1} - 1 = 2 - 1 = 1$, as required.
Induction hypothesis: Suppose $a_k = 2^{k+1} - 1$. We need to prove that $a_{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$. Observe that

$$\begin{aligned} a_{k+1} &= 2a_k + 1, \text{ by definition} \\ &= 2(2^{k+1} - 1) + 1, \text{ by the induction hypothesis} \\ &= 2^{k+2} - 2 + 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

as required. Therefore by the **principle of mathematical induction**, the equality holds for all $n \in \mathbb{N}$. QED

(c) $c_0 = 3$, $c_1 = 4$, $c_2 = 6$, $c_3 = 9$, $c_4 = 13$ and $c_5 = 18$.
We will use induction to prove that $c_n = \frac{n^2+n+6}{2}$ for all $n \in \mathbb{N}$.
Basis step: When $n = 0$, $c_0 = 3$ and $\frac{0^2+0+6}{2} = \frac{6}{2} = 3$, as required.
Induction hypothesis: Suppose that $c_k = \frac{k^2+k+6}{2}$. We need to show that $c_{k+1} = \frac{(k+1)^2+(k+1)+6}{2} = \frac{k^2+3k+8}{2}$. Observe that

$$\begin{aligned} c_{k+1} &= c_k + (k+1), \text{ by definition} \\ &= \frac{k^2+k+6}{2} + (k+1), \text{by induction hypothesis} \\ &= \frac{k^2+k+6}{2} + \frac{2(k+1)}{2} \\ &= \frac{k^2+k+6+2k+2}{2} \\ &= \frac{k^2+3k+8}{2} \end{aligned}$$

as required. Therefore by the **principle of mathematical induction**, the statement holds for all $n \in \mathbb{N}$. QED

(d) In order for the recursive definition $d_n = 5d_{n-1} - 6d_{n-2}$ to work when $n = 2$, we need to define both d_0 and d_1.

$d_0 = 2$, $d_1 = 5$, $d_2 = 13$, $d_3 = 35$, $d_4 = 97$ and $d_5 = 275$.

We will use strong induction to show that $d_n = 2^n + 3^n$ for all $n \in \mathbb{N}$.
Basis cases: When $n = 0$, $d_0 = 2$ and $2^0 + 3^0 = 1 + 1 = 2$ as required. When $n = 1$, $d_1 = 5$ and $2^1 + 3^1 = 2 + 3 = 5$ as required.
Strong induction hypothesis: Suppose the formula $d_n = 2^n + 3^n$ holds for all values of n from 0 to k. We need to prove that $d_{k+1} = 2^{k+1} + 3^{k+1}$. Observe that

$$\begin{aligned} d_{k+1} &= 5d_k - 6d_{k-1}, \text{ by definition} \\ &= 5[2^k + 3^k] - 6[2^{k-1} + 3^{k-1}], \text{ by induction hypothesis} \\ &= (5-3)2^k + (5-2)3^k \\ &= 2^{k+1} + 3^{k+1} \end{aligned}$$

as required. Therefore by the **principle of mathematical induction, strong version**, the statement holds for all $n \in \mathbb{N}$. QED

(e) $e_0 = 1$, $e_1 = 4$, $e_2 = 12$, $e_3 = 32$, $e_4 = 80$ and $e_5 = 192$.
We will use strong induction to show that $e_n = (n+1)2^n$ for all $n \in \mathbb{N}$.
Basis cases: When $n = 0$, $e_0 = 1$ and $(0+1)2^0 = 1 \times 1 = 1$, as required. When $n = 1$, $e_1 = 4$ and $(1+1)2^1 = 2 \times 2 = 4$, as required.
Strong Induction hypothesis: Suppose that the formula $e_n = (n+1)2^n$ holds for all values of n from 0 to k. We need to prove that $e_{k+1} = [(k+1)+1]2^{k+1} = (k+2)2^{k+1}$. Observe that

$$\begin{aligned} e_{k+1} &= 4(e_k - e_{k-1}), \text{ by definition} \\ &= 4[\,(k+1)2^k - k2^{k-1}\,], \text{ by the induction hypothesis} \\ &= 2(k+1)2^{k+1} - k2^{k+1} \\ &= [2(k+1) - k]2^{k+1} \\ &= (k+2)2^{k+1} \end{aligned}$$

as required. Therefore by the **principle of mathematical induction, strong version**, the inequality holds for all $n \in \mathbb{N}$. QED
20.14 For each element a of A we must have $f(a) \in \{0, 1\}$. Since exactly k of the n elements of A must map to 1, this can be done in $\binom{n}{k}$ ways.