32.1 (a) \( \gcd(20, 25) = 5. \)
(b) \( \gcd(0, 10) = 10. \)
(c) \( \gcd(123, -123) = 123. \)
(d) Here we use the Euclidean Algorithm, noting first that \( \gcd(-89, -98) = \gcd(89, 98). \):

\[
\begin{align*}
98 & = 1 \times 89 + 9 \\
89 & = 9 \times 9 + 8 \\
9 & = 1 \times 8 + 1 \\
8 & = 8 \times 1 + 0
\end{align*}
\]

We conclude that \( \gcd(89, 98) = 1. \)
(e) Here we use the Euclidean Algorithm:

\[
\begin{align*}
54321 & = 1086 \times 50 + 21 \\
50 & = 2 \times 21 + 8 \\
21 & = 2 \times 8 + 5 \\
8 & = 1 \times 5 + 3 \\
5 & = 1 \times 3 + 2 \\
3 & = 1 \times 2 + 1 \\
2 & = 2 \times 1 + 0
\end{align*}
\]

We conclude that \( \gcd(54321, 50) = 1. \) Unless you’re required to use the Euclidean Algorithm here, though, it is not hard to see that the only prime divisors of 50 are 2 and 5, neither of which are divisors of 54321, hence the greatest common divisor of 50 and 54321 must be 1.
(f)

\[
\begin{align*}
29341 & = 16 \times 1739 + 1517 \\
1739 & = 1 \times 1517 + 222 \\
1517 & = 6 \times 222 + 185 \\
222 & = 1 \times 185 + 37 \\
185 & = 5 \times 37 + 0
\end{align*}
\]

1
We conclude that gcd(29341, 1739) = 37.

32.2
(d) Using back substitution together with our work above for question 32.1 (d), we get

\[ 1 = 9 - 1 \times 8 \]
\[ = 9 - (89 - 9 \times 9) \]
\[ = 10 \times 9 - 89 \]
\[ = 10(98 - 89) - 89 \]
\[ = 10 \times 98 - 11 \times 89 \]

So \( 1 = 10 \times 98 - 11 \times 89 \).

(e) Using back substitution together with our work above for question 32.1 (e), we get

\[ 1 = 3 - 2 \]
\[ = 3 - (5 - 3) \]
\[ = 2 \times 3 - 5 \]
\[ = 2(8 - 5) - 5 \]
\[ = 2 \times 8 - 3 \times 5 \]
\[ = 2 \times 8 - 3(21 - 2 \times 8) \]
\[ = 8 \times 8 - 3 \times 21 \]
\[ = 8(50 - 2 \times 21) - 3 \times 21 \]
\[ = 8 \times 50 - 19 \times 21 \]
\[ = 8 \times 50 - 19(54321 - 1086 \times 50) \]
\[ = -19(54321) + 20642(50) \]

Therefore \( 1 = -19(54321) + 20642(50) \).

33.1
(a) 6
(b) 2
(c) 0
(d) 7
(e) undefined because $12 \not\in \mathbb{Z}_{10}$.

Other practice problems.
1. Find $8^{-1}$ in $\mathbb{Z}_{25}$. Show your work.

$$25 = 3 \times 8 + 1$$
$$8 = 8 \times 1 + 0$$

Back-substitution, which in this case is very simple, gives us $1 = 25 - 3 \times 8$. It follows that $8^{-1} = 3$.

2. Which integers in $\mathbb{Z}_{26}$ have inverses?
1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25.

4. Alice uses the Caesar Cypher and sends Bob the message below. What is she saying?

$PHUUBFKULVWPDV$

MERRY CHRISTMAS