MthSc 119, Section 11

11.1 (a) *reflexive*, not *irreflexive*, *symmetric*, *antisymmetric*, *transitive*.
(b) not *reflexive*, *irreflexive*, not *symmetric*, *antisymmetric*, not *transitive*.
(c) not *reflexive*, not *irreflexive*, not *symmetric*, *antisymmetric*, *transitive*.
(d) not *reflexive*, not *irreflexive*, *symmetric*, not *antisymmetric*, not *transitive*.
(e) *reflexive*, not *irreflexive*, *symmetric*, not *antisymmetric*, *transitive*.

11.2 (a) $R = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z}, \text{and } |x - y| \leq 2\}$.
(b) Here is a proof that R is reflexive:
Let $x \in \mathbb{Z}$. Since $|x - x| = 0$, it follows that x is near itself. Therefore xRx, showing that R is reflexive.
(c) R is not irreflexive. A counterexample is $3R3$.
(d) Here is a proof that R is symmetric.
Let xRy. By the definition of R, x is near y, which means that $|x - y| \leq 2$. Since $|x - y| \leq 2$, obviously $|y - x| \leq 2$. Therefore y is near x. Hence yRx, showing that R is symmetric.
(e) R is not antisymmetric. For a counterexample, note that $3R1$ and that $1R3$ but $1 \neq 3$.
(f) R is not transitive. For a counterexample, note that 1 is near 3 and 3 is near 5, but 1 is not near 5.

11.3 (a) $R^{-1} = \{(2, 1), (3, 2), (4, 3)\}$.
(b) $R^{-1} = R$.
(c) $R^{-1} = \{(x, y) : x, y \in \mathbb{Z}, x - y = -1\}$.

11.5 *reflexive* (true): ARA holds since $|A| = |A|$ for any subset A.
irreflexive (false): a counterexample is $A = \emptyset$, as $|\emptyset| = 0 = |\emptyset|$ shows ARA.
symmetric (true): Suppose ARB holds. Then $|A| = |B|$ and so $|B| = |A|$; thus BRA holds.
antisymmetric (false): a counterexample is $A = \{1, 4\}$, $B = \{2, 3\}$. Then ARB and BRA both hold, but $A \neq B$.
transitive (true): Suppose ARB holds and BRC holds. This means $|A| = |B|$ and $|B| = |C|$; it follows that $|A| = |C|$ and thus ARC holds.

11.6 *reflexive* (true): ARA holds since $A \subseteq A$ for any subset A.
irreflexive (false): a counterexample is $A = \{1\}$, since $A \subseteq A$ holds.
symmetric (false): a counterexample is $A = \{1\}$, $B = \{1, 2\}$. Then $A \subseteq B$

so ARB. However, BRA does not hold since B is not a subset of A.
antisymmetric (true): Suppose that ARB and BRA both hold. This means $A \subseteq B$ and $B \subseteq A$, so that $A = B$.
transitive (true): Suppose ARB holds and BRC holds. This means $A \subseteq B$ and $B \subseteq C$. We can then show that $A \subseteq C$: let $x \in A$; since $A \subseteq B$, we have $x \in B$; since $B \subseteq C$ and $x \in B$, we have $x \in C$. Thus ARC holds.

11.11 Let $A = \{1, 2\}$ and define the relation R to be the empty set (contains no ordered pairs): $R = \emptyset$. Then R is symmetric since the statement "If xRy, then yRx" holds vacuously, as the hypothesis is false. Likewise the transitive property "If xRy and yRz, then xRz" holds vacuously. Finally R is not reflexive since that property requires xRx to hold for all $x \in A$: but (for example) $1R1$ is false since the ordered pair $(1, 1)$ does not appear in R.

The proposed proof is wrong because it assumes there is an ordered pair (x, y) in the relation R.