MthSc 119, Sections 13 and 14

13.2 (a) 6! (b) $\frac{8!}{2!}$ (c) $\frac{11!}{2!^3}$ (d) $\frac{7!}{2!\times 3!}$ (e) $\frac{11!}{4!^2\times 2!}$

13.3 With the first and last letters fixed, there are five letters U, C, C, E, S to be rearranged, which can be done in $\frac{5!}{2!}$ ways.

13.4 Think of rearranging the five consonants F, C, T, S, L together with the six vowels (denoted by the generic X) into an eleven-letter word, such as XCXXLXTFXSX. Then substitute for the X's in order the vowels A, E, I, O, U, Y to obtain the anagram ACEILOTFUSY, which has all six vowels in alphabetical order. The problem is then equivalent to finding all anagrams of F, C, T, S, L, X, X, X, X, X, X; there are $\frac{11!}{6!}$ such anagrams.

14.3 (a) You can select the k positions for the 1s in exactly $\binom{n}{k}$ ways.
(b) Here the k positions for the 1s can be selected in $\binom{n}{k}$ ways; once this is done, the other $n-k$ positions can contain either a 0 or a 2, giving 2^{n-k} possibilities. By the rule of multiplication, there are $2^{n-k}\binom{n}{k}$ such sequences.

14.4 (a) 50!
(b) $\binom{50}{10}$
(c) $(50)_3 = 50\times 49\times 48$.

14.9 We want to show that $\binom{n}{k} = \binom{n}{n-k}$. We use Theorem 14.12:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-[n-k])!} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

14.11 Use the binomial theorem to expand $(1-1)^n$, where $x=1$ and $y=-1$:

$$(1-1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-1)^1 + \binom{n}{2}1^{n-2}(-1)^2 + \cdots + \binom{n}{n-1}1^1(-1)^{n-1} + \binom{n}{n}(-1)^n$$

Since $(1-1)^n = 0^n = 0$, we can simplify to obtain

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

This shows that the alternating sum of binomial coefficients equals 0.

14.14 Each rectangle is defined by its top and bottom boundaries, and also by its left and right boundaries. In an $m \times n$ chessboard, there are m squares along the side and n squares along the top. Since there are $m+1$ lines along the side, there are $\binom{m+1}{2}$ ways to select the top and bottom boundaries from these $m+1$ lines. Likewise there are $n+1$ lines along the top, and thus $\binom{n+1}{2}$ ways to select the left and right boundaries. By the multiplication principle, there are $\binom{m+1}{2}\binom{n+1}{2}$ different rectangles that can be formed.

Notice that when $m = n = 2$, this formula gives $\binom{3}{2}\binom{3}{2} = 3 \cdot 3 = 9$ rectangles in a 2×2 chessboard, agreeing with the statement in the book.

14.27 Since the order of drawing the cards does not matter, this is the number of ways of selecting a subset of size 5 (the hand) from a larger set of size 52 (the deck), or $\binom{52}{5} = 2{,}598{,}960$.

14.28 (a) The numerical value (denomination) for the four cards has 13 possibilities; the remaining card can be any of the 48 left (after the four cards with the same denomination are chosen). This gives $13 \cdot 48 = 624$ hands.

(b) The denomination (for the three cards) can be chosen in 13 ways; there are $\binom{4}{3}$ ways of selecting the suits of these cards. The other two cards must have two other denominations, and these denominations can be chosen in $\binom{12}{2}$ ways; the suits of these two remaining cards can be chosen in $4 \cdot 4$ ways. Altogether there are $13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4 \cdot 4$ hands.

(c) The suit can be chosen in 4 ways. Then there are $\binom{13}{5}$ ways of selecting the cards from that chosen suit. Altogether there are $4 \cdot \binom{13}{5}$ hands.

(d) The denomination for the three cards can be chosen in 13 ways. Once this is done, there are 12 remaining denominations for the pair of cards. The suits for the triple can be selected in $\binom{4}{3}$ ways, and the suits for the pair can be selected in $\binom{4}{2}$ ways. This gives $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}$ hands.

(e) The five consecutive cards can begin with a two, three, . . . , or ten, giving 9 starting places for the straight. The suit of each of the five cards can be any of 4 values, so there are $9 \cdot 4^5$ hands in total.

(f) Again there are 9 starting places for the straight to begin. Since all five cards must be the same suit, this common suit can be chosen in 4 ways. Altogether there are $9 \cdot 4$ hands.