MthSc 119, sections 16 and 17

16.1 Let the groups be $B_1, \ldots, B_4$. Then $|B_i| = 1000$, $|B_i \cap B_j| = 100$, $|B_i \cap B_j \cap B_k| = 10$, and $|B_1 \cap B_2 \cap B_3 \cap B_4| = 1$. Applying inclusion-exclusion gives $|B_1 \cup B_2 \cup B_3 \cup B_4| = 4 \cdot 1000 - \binom{4}{2}100 + \binom{4}{3}10 - 1 = 4000 - 600 + 40 - 1 = 3439$.

16.5 We first count the number of six-digit numbers that are "bad": they have (at least) three consecutive digits the same. Let B_1 be the set of six-digit numbers with digits $1, 2, 3$ identical; B_2 be the set of six-digit numbers with digits $2, 3, 4$ identical; B_3 be the set of six-digit numbers with digits $3, 4, 5$ identical; and B_4 be the set of six-digit numbers with digits $4, 5, 6$ identical. We apply inclusion-exclusion to count the "bad" numbers. Note that $|B_i| = 10 \cdot 10^3 = 10^4$. Also $|B_1 \cap B_2| = |B_2 \cap B_3| = |B_3 \cap B_4| = 10^3$, $|B_1 \cap B_3| = |B_2 \cap B_4| = 10^2$, and $|B_1 \cap B_4| = 10^2$. Likewise $|B_1 \cap B_2 \cap B_3| = |B_2 \cap B_3 \cap B_4| = 10^2$, and all other triple intersections have size 10. Finally $|B_1 \cap B_2 \cap B_3 \cap B_4| = 10$. This gives $|B_1 \cup B_2 \cup B_3 \cup B_4| = 4 \cdot 10^4 - (3 \cdot 10^3 + 3 \cdot 10^2) + (2 \cdot 10^2 + 2 \cdot 10) - 10 = 40000 - 3300 + 220 - 10 = 36910$. Since there are 10^6 six-digit numbers of which 36910 are "bad", there are $10^6 - 36910 = 963090$ "good" numbers (with no three consecutive digits the same).

17.1 (a) If x^2 is not odd, then x is not odd; or If x^2 is even, then x is even.
(b) If $2^p - 2$ is not divisible by p, then p is not prime.
(c) If x^2 is not positive, then x is zero.
(e) If the car does not start, then the battery is not fully charged.
(f) If not C, then not A and not B.

17.2 Suppose the if-then statement is $A \rightarrow B$. The contrapositive is therefore $\neg B \rightarrow \neg A$. The contrapositive of this latter statement is $\neg(\neg A) \rightarrow \neg(\neg B)$, which is logically equivalent to $A \rightarrow B$, the original statement.

17.3 To prove "A if and only if B" we need to show: (1) $A \Rightarrow B$ and (2) $B \Rightarrow A$. However instead of (2) we can prove the equivalent contrapositive: $\neg A \Rightarrow \neg B$.

17.4 (a) Let A, B, and C be sets with $A \subseteq B$ and $B \subseteq C$. Suppose, for sake of contradiction, that A is not a subset of C.
(b) Let x and y be two negative integers. Suppose, for sake of contradiction,

that $x + y \geq 0$.
(d) Let p and q be primes with $p + q$ also a prime. Suppose, for sake of contradiction, that $p \neq 2$ and $q \neq 2$.
(f) Let C_1 and C_2 be distinct circles. Suppose, for sake of contradiction, that C_1 and C_2 intersect in three or more points.

17.5 Suppose, for sake of contradiction, that the integers x and $x+1$ are both even. Then there is an integer a with $x = 2a$ and an integer b with $x+1 = 2b$. Adding 1 to both sides of the equation $x = 2a$, we get $x + 1 = 2a + 1$. Since $x + 1 = 2b$, it follows that $2b = 2a + 1$. Subtracting $2a$ from both sides of the equation $2b = 2a+1$ and then dividing both sides by 2, we get $b-a = \frac{1}{2}$. But $\frac{1}{2}$ is not an integer, whereas $b - a$ is an integer. $\Rightarrow\Leftarrow$ Therefore consecutive integers cannot both be even. QED

17.7 Let p and q be primes for which $p + q$ is also prime. Suppose, for sake of contradiction, that neither p nor q is 2. Since p is prime, p has no other positive divisors other than itself and 1. It follows that $2 \not| p$ and hence that p is not even; thus p must be odd. Likewise, q is odd. Since the sum of two odd numbers is even, it follows that $p+q$ is even. Hence $2|(p+q)$. But since $p > 2$ and $q > 2$, we know that $p + q > 4$. So $p + q$ has a positive divisor, namely 2, that is equal to neither 1 nor $p + q$. $\Rightarrow\Leftarrow$ Therefore if the sum of two prime numbers is also prime, one of primes must be 2. QED

17.8 Let A and B be sets. Suppose, for the sake of contradiction, that $(A-B)\cap(B-A) \neq \emptyset$. This means there is an $x \in (A-B)\cap(B-A)$. Thus $x \in A - B$ and $x \in B - A$. Since $x \in A - B$ we know that x is in A but not in B. However, $x \in B - A$ implies that $x \in B$. Thus $x \notin B$ and $x \in B$. $\Rightarrow\Leftarrow$ Therefore $(A - B) \cap (B - A) = \emptyset$. QED