section 19

19.3 We will use induction to prove each formula holds for all $n \geq 1$.

(a) **Basis step:** When $n = 1$, the sum on the left goes from 1 to $3 \cdot 1 - 2 = 1$ and so equals 1. The right hand side is $\frac{1(3-1)}{2} = \frac{1}{2} = 1$, as required.

**Induction hypothesis:** Assume the formula holds for $n = k$:

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k - 1)}{2}$$

We want to show the formula holds for $n = k + 1$:

$$1 + 4 + 7 + \cdots + (3[k + 1] - 2) = \frac{[k + 1](3[k + 1] - 1)}{2}$$

or

$$1 + 4 + 7 + \cdots + (3k + 1) = \frac{(k + 1)(3k + 2)}{2}$$

Observe that

$$1 + 4 + 7 + \cdots + (3k - 2) + (3k + 1)$$

$$= [1 + 4 + 7 + \cdots + (3k - 2)] + (3k + 1)$$

$$= \frac{k(3k - 1)}{2} + (3k + 1), \text{ by the induction hypothesis}$$

$$= \frac{(3k^2 - k) + (6k + 2)}{2} = \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{(k + 1)(3k + 2)}{2}$$

Therefore by the **principle of mathematical induction**, the formula holds for all $n \geq 1$.

(b) **Basis step:** When $n = 1$, the sum on the left goes from 1 to $1^3 = 1$ and so equals 1. The right hand side is $\frac{1^2 - 2^2}{4} = \frac{4}{4} = 1$, as required.

**Induction hypothesis:** Assume the formula holds for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k + 1)^2}{4}$$

We want to show the formula holds for $n = k + 1$:

$$1^3 + 2^3 + 3^3 + \cdots + (k + 1)^3 = \frac{(k + 1)^2(k + 2)^2}{4}$$
Observe that $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k + 1)^3$

$$= [1^3 + 2^3 + 3^3 + \cdots + k^3] + (k + 1)^3$$

$$= \frac{k^2(k + 1)^2}{4} + (k + 1)^3, \text{ by the induction hypothesis}$$

$$= \frac{(k^4 + 2k^3 + k^2) + 4(k^3 + 3k^2 + 3k + 1)}{4}$$

$$= k^4 + 6k^3 + 13k^2 + 12k + 4$$

$$= \frac{(k^2 + 2k + 1)(k^2 + 4k + 4)}{4}$$

$$= \frac{(k + 1)^2(k + 2)^2}{4}$$

Therefore by the principle of mathematical induction, the formula holds for all $n \geq 1$.

(d) Basis step: When $n = 1$, the sum on the left has the one term $\frac{1}{1^2} = \frac{1}{2}$. The right hand side is $1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$, as required.

Induction hypothesis: Assume the formula holds for $n = k$:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k + 1)} = 1 - \frac{1}{k + 1}$$

We want to show the formula holds for $n = k + 1$:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k + 2}$$

Observe that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)}$$

$$= \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} \right] + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ by the induction hypothesis}$$

$$= 1 - \frac{1}{k+1} \left[ 1 - \frac{1}{(k+2)} \right] = 1 - \frac{1}{k+1} \left[ \frac{k+1}{k+2} \right] = 1 - \frac{1}{k+2}$$

Therefore by the principle of mathematical induction, the formula holds for all $n \geq 1$. 

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19.4 We will use induction to prove each inequality holds for all \( n \geq 1 \).

(a) **Basis step:** When \( n = 1 \), the left hand side is \( 2^1 = 2 \) and the right hand side is \( 2^2 - 2^0 - 1 = 4 - 1 - 1 = 2 \), so the base case holds: \( 2 \leq 2 \).

**Induction hypothesis:** Assume the inequality holds for \( n = k \):

\[
2^k \leq 2^{k+1} - 2^{k-1} - 1
\]

We want to show the inequality holds for \( n = k + 1 \):

\[
2^{k+1} \leq 2^{k+2} - 2^k - 1
\]

Observe that

\[
2^{k+1} = 2 \cdot 2^k
\]

\[
\leq 2(2^{k+1} - 2^{k-1} - 1), \text{ by the induction hypothesis}
\]

\[
= (2^{k+2} - 2^k - 2) = (2^{k+2} - 2^k - 1) - 1
\]

\[
\leq (2^{k+2} - 2^k - 1)
\]

Therefore by the **principle of mathematical induction**, the inequality holds for all \( n \geq 1 \).

(c) **Basis step:** When \( n = 1 \), the sum on the left is \( 1 + \frac{1}{2} = \frac{3}{2} \). The right hand side is \( 1 + \frac{1}{2} = \frac{3}{2} \), so the base case holds: \( \frac{3}{2} \geq \frac{3}{2} \).

**Induction hypothesis:** Assume the inequality holds for \( n = k \):

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k} \geq 1 + \frac{k}{2}
\]

We want to show the inequality holds for \( n = k + 1 \):

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \geq 1 + \frac{k + 1}{2}
\]

Observe that

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k+1}}
\]

\[
= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k}\right] + \left[\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots + \frac{1}{2^{k+1}}\right]
\]

\[
\geq \left[1 + \frac{k}{2}\right] + \left[\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots + \frac{1}{2^{k+1}}\right], \text{ by the induction hypothesis}
\]

\[
\geq \left[1 + \frac{k}{2}\right] + \left[\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}\right]
\]

\[
= \left[1 + \frac{k}{2}\right] + 2^k \left(\frac{1}{2^{k+1}}\right) = \left[1 + \frac{k}{2}\right] + \frac{1}{2} = 1 + \frac{k + 1}{2}
\]
Therefore by the principle of mathematical induction, the inequality holds for all $n \geq 1$. 