section 19

19.3 We will use induction to prove each formula holds for all $n \geq 1$.

(a) **Basis step:** When $n = 1$, the sum on the left goes from 1 to $3 \cdot 1 - 2 = 1$ and so equals 1. The right hand side is $\frac{1\cdot(3-1)}{2} = \frac{1\cdot 2}{2} = 1$, as required.
Induction hypothesis: Assume the formula holds for $n = k$:

$$1 + 4 + 7 + \cdots + (3k-2) = \frac{k(3k-1)}{2}$$

We want to show the formula holds for $n = k+1$:

$$1 + 4 + 7 + \cdots + (3[k+1]-2) = \frac{[k+1](3[k+1]-1)}{2}$$

or

$$1 + 4 + 7 + \cdots + (3k+1) = \frac{(k+1)(3k+2)}{2}$$

Observe that $1 + 4 + 7 + \cdots + (3k-2) + (3k+1)$

$$\begin{aligned} &= [1 + 4 + 7 + \cdots + (3k-2)] + (3k+1) \\ &= \frac{k(3k-1)}{2} + (3k+1), \text{ by the induction hypothesis} \\ &= \frac{(3k^2-k)+(6k+2)}{2} = \frac{3k^2+5k+2}{2} \\ &= \frac{(k+1)(3k+2)}{2} \end{aligned}$$

Therefore by the **principle of mathematical induction**, the formula holds for all $n \geq 1$.

(b) **Basis step:** When $n = 1$, the sum on the left goes from 1 to $1^3 = 1$ and so equals 1. The right hand side is $\frac{1^2 \cdot 2^2}{4} = \frac{4}{4} = 1$, as required.
Induction hypothesis: Assume the formula holds for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

We want to show the formula holds for $n = k+1$:

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Observe that $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$

$$\begin{aligned}
&= [1^3 + 2^3 + 3^3 + \cdots + k^3] + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3, \text{ by the induction hypothesis} \\
&= \frac{(k^4 + 2k^3 + k^2) + 4(k^3 + 3k^2 + 3k + 1)}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \\
&= \frac{(k^2 + 2k + 1)(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

Therefore by the **principle of mathematical induction**, the formula holds for all $n \geq 1$.

(d) **Basis step:** When $n = 1$, the sum on the left has the one term $\frac{1}{1\cdot 2} = \frac{1}{2}$. The right hand side is $1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$, as required.
Induction hypothesis: Assume the formula holds for $n = k$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}$$

We want to show the formula holds for $n = k + 1$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+2}$$

Observe that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)}$

$$\begin{aligned}
&= \left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ by the induction hypothesis} \\
&= 1 - \frac{1}{k+1}\left[1 - \frac{1}{(k+2)}\right] = 1 - \frac{1}{k+1}\left[\frac{k+1}{(k+2)}\right] = 1 - \frac{1}{k+2}
\end{aligned}$$

Therefore by the **principle of mathematical induction**, the formula holds for all $n \geq 1$.

19.4 We will use induction to prove each inequality holds for all $n \geq 1$.

(a) **Basis step:** When $n = 1$, the left hand side is $2^1 = 2$ and the right hand side is $2^2 - 2^0 - 1 = 4 - 1 - 1 = 2$, so the base case holds: $2 \leq 2$.
Induction hypothesis: Assume the inequality holds for $n = k$:

$$2^k \leq 2^{k+1} - 2^{k-1} - 1$$

We want to show the inequality holds for $n = k + 1$:

$$2^{k+1} \leq 2^{k+2} - 2^k - 1$$

Observe that

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\leq 2(2^{k+1} - 2^{k-1} - 1), \text{ by the induction hypothesis} \\ &= (2^{k+2} - 2^k - 2) = (2^{k+2} - 2^k - 1) - 1 \\ &\leq (2^{k+2} - 2^k - 1) \end{aligned}$$

Therefore by the **principle of mathematical induction**, the inequality holds for all $n \geq 1$.

(c) **Basis step:** When $n = 1$, the sum on the left is $1 + \frac{1}{2} = \frac{3}{2}$. The right hand side is $1 + \frac{1}{2} = \frac{3}{2}$, so the base case holds: $\frac{3}{2} \geq \frac{3}{2}$.
Induction hypothesis: Assume the inequality holds for $n = k$:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$$

We want to show the inequality holds for $n = k + 1$:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k+1}} \geq 1 + \frac{k+1}{2}$$

Observe that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k+1}}$

$$\begin{aligned} &= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k}\right] + \left[\frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \cdots + \frac{1}{2^{k+1}}\right] \\ &\geq \left[1 + \frac{k}{2}\right] + \left[\frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \cdots + \frac{1}{2^{k+1}}\right], \text{ by the induction hypothesis} \\ &\geq \left[1 + \frac{k}{2}\right] + \left[\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}\right] \\ &= \left[1 + \frac{k}{2}\right] + 2^k\left(\frac{1}{2^{k+1}}\right) = \left[1 + \frac{k}{2}\right] + \frac{1}{2} = 1 + \frac{k+1}{2} \end{aligned}$$

Therefore by the **principle of mathematical induction**, the inequality holds for all $n \geq 1$.