MthSc 119, Sections 6 and 7 — Model Solutions

6.1 (1 point) 2^k

6.2 (1 point) 26^3

6.4 (a) (1 point) $(100)_5 = 100 \times 99 \times 98 \times 97 \times 96$
(b) (1 point) $5 \times 100 = 500$.

6.7 (a) (1 point) $26 \times 26 \times 26 \times 10 \times 10 \times 10$.
(b) (1 point) $26 \times 25 \times 24 \times 10 \times 9 \times 8$.

6.10 (1 point) There are $36 = 26 + 10$ possible characters. The number of one-character file names is 36, the number of two-character file names is $36 \times 36 = 36^2$, and so forth. In total there are $36 + 36^2 + \cdots + 36^8$ possible file names.

6.11 (1 point) The first digit can be any of 9 possibilities (1–9). Once the first digit is chosen, there are 9 possibilities for the second digit (any digit 0–9 except for the digit in the first position). Similarly, with the first and second digits chosen, there are 9 possibilities for the third digit, and so on. This gives by the multiplication principle $9 \times 9 \times 9 \times 9 \times 9 = 9^5$ possible five-digit numbers without consecutive digits.

6.14 (1 point) Suppose the line begins on the left with a boy. There are 10 choices for this boy; call him B. Then for the next position there are 10 choices (any girl). Once the first two positions B, G are given, the third position can be any boy other than B: 9 choices. The fourth position can be any girl other than G: 9 choices. Continuing we get $10 \times 10 \times 9 \times 9 \times 8 \times 8 \times 7 \times 7 \times \cdots \times 1 \times 1 = 10! \times 10!$ possible lines starting on the left with a boy. By similar reasoning there are $10! \times 10!$ possible lines starting on the left with a girl. Combining these two numbers gives $2 \times 10! \times 10!$ possible lines.

6.15 (1 point) The four cards drawn in order define a list of length four. The first card can be any of the 52. Then second card cannot have the same suit as the first (there are 3 possible suits then) and cannot have the same value as the first (there are 12 possible values then), giving $3 \times 12 = 36$

choices for the second card. The third card cannot have the same suit as either of the first two cards (2 possible suits remain) or the same value as either of the first two cards (11 possible values remain), giving $2 \times 11 = 22$ choices. The fourth card can be chosen from only 1 remaining suit and has 10 possible values, so there are 10 choices for the last card. Altogether there are $52 \times 36 \times 22 \times 10$ ways to select the cards so that no two have the same suit or value.

7.1 (a) (1 point) The $6+8+5=19$ books can be arranged in 19! different orders.

(b) (1 point) Now suppose all books in the same language must be grouped together. First look at the sequence of languages (going from left to right) that occur on the shelves. There are 3 choices for the first language, 2 choices for the second, and 1 for the third giving 3! orders of languages. Once the grouping of languages has been set, there are 6! ways to arrange the French books within the French group, 8! ways to arrange the Russian books, and 5! ways to arrange the Spanish books. Altogether there are $3! \times 6! \times 8! \times 5!$ different orderings of books on the shelf.

7.7 (1.5 points) (a) $\prod_{k=1}^{4}(2k+1) = (2+1)(4+1)(6+1)(8+1) = 3 \cdot 5 \cdot 7 \cdot 9 = 945$.
(b) $\prod_{k=-3}^{4} k = -3 \cdot -2 \cdot -1 \cdot 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 0$.
(c) $\prod_{k=1}^{n} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n}{n-1} \cdot \frac{n+1}{n} = n+1$.
(d) $\prod_{k=1}^{n} \frac{1}{k} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \cdots \cdot \frac{1}{n} = \frac{1}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} = \frac{1}{n!}$.

7.9 Let k be an integer with $2 \leq k \leq 1000$. Since $1000! = k(1 \cdot 2 \cdot \cdots \cdot (k-1) \cdot (k+1) \cdot \cdots 999 \cdot 1000)$, we know that k is a divisor of 1000!. Since k is also a divisor of itself, it follows that k is a divisor of $1000! + k$. Hence, since $1 < k < (1000! + k)$, $1000! + k$ is composite.

Now 1001 is divisible by 11 and, since $11|1000!$, 11 is a divisor of $1000! + 1001$. Furthermore both 1000! and 1002 are divisible by 2. It follows that both $1000! + 1001$ and $1000! + 1002$ are composite.

Therefore all integers from $1000! + 2$ to $1000! + 1002$ are composite. QED