1. Define the sequence $e_0, e_1, e_2, \ldots$ as follows:

$$e_0 = 1, \quad e_1 = 4 \quad \text{and} \quad e_n = 4(e_{n-1} - e_{n-2}) \quad \text{for} \quad n \geq 2.$$ 

a) What are the first five terms of the sequence (i.e., what are $e_0$, $e_1$, $e_2$, $e_3$ and $e_4$)?

$$e_0 = 1, \quad e_1 = 4, \quad e_2 = 12, \quad e_3 = 32 \quad \text{and} \quad e_4 = 80.$$ 

b) Prove using mathematical induction (strong version):

$$e_n = (n + 1)2^n \quad \text{for all integers} \quad n \geq 0.$$ 

We will use strong induction to show that $e_n = (n + 1)2^n$ for all $n \in \mathbb{N}$.

**Basis cases:** When $n = 0$, $e_0 = 1$ and $(0 + 1)2^0 = 1 \times 1 = 1$, as required. When $n = 1$, $e_1 = 4$ and $(1 + 1)2^1 = 2 \times 2 = 4$, as required.

**Strong Induction hypothesis:** Suppose that the formula $e_n = (n + 1)2^n$ holds for all values of $n$ from 0 to $k$. We need to prove that $e_{k+1} = [(k + 1) + 1]2^{k+1} = (k + 2)2^{k+1}$.

Observe that

$$e_{k+1} = 4(e_k - e_{k-1}), \quad \text{by definition}$$

$$= 4[(k + 1)2^k - k2^{k-1}], \quad \text{by the induction hypothesis}$$

$$= 2(k + 1)2^{k+1} - k2^{k+1}$$

$$= [2(k + 1) - k]2^{k+1}$$

$$= (k + 2)2^{k+1}$$

as required. Therefore by the principle of mathematical induction, strong version, the inequality holds for all $n \in \mathbb{N}$. QED