

# Dirchlet's Function and Limit and Continuity Arguments

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## Outline

Dirichlet's Function

Limit Examples

Dirichlet's Function is defined like this:  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{q}, & x = \frac{p}{q}, (p, q) = 1, p, q \in \mathbb{N} \\ 0, & x \in \mathbb{I}\mathbb{R} \end{cases}$$

So  $f(9/12) = 1/4$  as  $9/12 = 3/4$  when common terms are removed. This is a very strange function and we can prove

$f$  is continuous at each  $x \in \mathbb{I}\mathbb{R} \cap [0, 1]$  and is not continuous at each  $x \in \mathbb{Q} \cap [0, 1]$ .

### Proof

(Case  $x_0 \in \mathbb{I}\mathbb{R}$ ):

We will show  $f$  is continuous at these  $x$  values. Choose  $\epsilon > 0$  arbitrarily. Consider

$$|f(x) - f(x_0)| = \begin{cases} |0 - 0|, & x \in \mathbb{I}\mathbb{R} \cap [0, 1] \\ |1/q - 0|, & x = p/q, (p, q) = 1 \end{cases}$$

### Proof

Let

$$S_n = \{x : 0 < x < 1, x = p/q, (p, q) = 1 \text{ with } 1/q \geq 1/n\} \\ = \{0 < p/q < 1, (p, q) = 1, \text{ with } n \geq q\}$$

How many elements are in  $S_n$ ? Consider that table below:

1/1	{1/2	1/3	...	...	...	1/n}
2/1	2/2	{2/3	2/4	...	...	2/n}
3/1	3/2	3/3	{3/4	...	...	3/n}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
j/1	j/2	j/j	{j/(j+1)...	...	...	j/n}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\frac{(n-1)}{n/1}$	$\frac{(n-1)}{n/2}$			$\frac{(n-1)}{(n-1)/n}$		$\frac{\{(n-1)\}}{n/n}$

The fractions in the braces in red are the ones in  $(0, 1)$  although they are not necessarily in lowest terms. The first row has  $n - 1$  elements, the second  $n - 2$  and so on until we get to the  $n - 1$  row which just has one red element. So the size of  $S_n = |S_n| \leq \sum_{j=1}^{n-1} j = (n - 1)n/2$ .

## Proof

The important thing is that  $S_N$  contains only a finite number of fractions in lowest terms. Now choose  $N > 1/\epsilon$ . Then  $|S_N| \leq N(N-1)/2$ . Label the elements of  $S_N$  as follows:

$$S_N = \{r_1, \dots, r_p\}$$

where  $p \leq N(N-1)/2$ . Each fraction  $r_i = p_i/q_i$  in lowest terms with  $q_i \leq N$ . One of these fractions is closest to  $x_0$ . Call this one  $r_{min} = p_{min}/q_{min}$  and choose  $\delta = (1/2)|r_{min} - x_0|$ .

Note if  $r = p/q$ ,  $(p, q) = 1$  is any rational number in  $(x_0 - \delta, x_0 + \delta)$ ,  $r < r_{min}$  and so can't be in  $S_N$ !! (See the picture in the handwritten notes). Since  $r \notin S_N$ , the denominator of  $r = p/q$  must satisfy  $N < q$ ; i.e.  $1/q < 1/N < \epsilon$ . Combining, we see we have found a  $\delta > 0$  so that

$$0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| = \begin{cases} 0 < \epsilon, & x \in \mathbb{I} \\ |1/q - 0| < 1/N < \epsilon, & x \in \mathbb{Q} \end{cases}$$

## Proof

This shows  $f$  is continuous at any irrational number  $x_0$  in  $[0, 1]$ .

(Case  $x_0 \in \mathbb{Q} \cap [0, 1]$ ):

Then  $x_0 = p_0/q_0$ ,  $(p_0, q_0) = 1$ . Consider

$$\begin{aligned} |f(x) - f(x_0)| &= \begin{cases} |0 - 1/q_0|, & x \in \mathbb{I} \cap [0, 1] \\ |1/q - 1/q_0|, & x = p/q, (p, q) = 1 \end{cases} \\ &= \begin{cases} |0 - 1/q_0|, & x \in \mathbb{I} \cap [0, 1] \\ |\frac{q_0 - q}{q_0 q}|, & x = p/q, (p, q) = 1 \end{cases} \end{aligned}$$

Let  $\epsilon_0 = (1/2)(1/q_0)$ . We only have to look at the top part. The top says  $|f(x) - f(x_0)| = 1/q_0 > (1/2)(1/q_0) = \epsilon_0$ . Thus  $f(x) = 1/q_0 > \epsilon_0$  for all  $x \in \mathbb{I}$ . Hence, no matter how small  $\delta > 0$  we choose, we can always find irrational numbers  $x_\delta$  in  $\hat{B}_\delta(x_0)$  with  $|f(x_\delta) - f(x_0)| = 1/q_0 > \epsilon_2$ . This shows  $f$  can not be continuous at any  $x_0$  in  $\mathbb{Q}$ .  $\square$

This is a very interesting function! Again, these ideas of continuity and limit are **pointwise** concepts!!

### Example

Let

$$f(x) = \begin{cases} x+1, & x > 1 \\ 3, & x = 1 \\ -3x, & x < 1 \end{cases}$$

We can study this function's behavior a number of ways.

(Case:  $\epsilon - \delta$  approach):

Let's look at the point  $x = 1$  first.

**Maybe the limiting value here is 2.**

We check

$$|f(x) - 2| = \begin{cases} |x+1-2|, & x > 1 \\ |3-2|, & x = 1 \\ |-3x-2|, & x < 1 \end{cases} = \begin{cases} |x-1|, & x > 1 \\ |1|, & x = 1 \\ |-3(x-1)-5|, & x < 1 \end{cases}$$

where we are writing all the terms we have in terms of  $x - 1$  factors.

### Example

(Continued)

Now by the backwards triangle inequality,  $|3(x-1)-5| \geq 5-3|x-1|$ .

So

$$\begin{cases} |f(x) - 2| = |x - 1|, & x > 1 \\ |f(x) - 2| = |1|, & x = 1 \\ |f(x) - 2| \geq 5 - 3|x - 1|, & x < 1 \end{cases}$$

If we make the top term small by restricting our attention to  $x$  in  $(1 - \delta, 1) \cup (1, 1 + \delta)$ , then we have

$$\begin{cases} |f(x) - 2| < \delta, & x \in (1, 1 + \delta) \\ |f(x) - 2| \geq 5 - 3\delta, & x \in (1 - \delta, 1) \end{cases}$$

where we have dropped what happens at  $x = 1$  itself as it is not important for the existence of the limit.

### Example

(Continued)

So no matter what  $\epsilon$  we pick, for any  $0 < \delta < \epsilon$ , we have

$$\begin{cases} |f(x) - 2| < \epsilon, & x \in (1, 1 + \delta) \\ |f(x) - 2| \geq 5 - 3\epsilon, & x \in (1 - \delta, 1) \end{cases}$$

and although the top piece is *small* the bottom piece is not! So the  $\lim_{x \rightarrow 1} f(x)$  can not be 2.

**Maybe the limiting value here is  $-3$ .**

We check

$$|f(x) - (-3)| = \begin{cases} |x + 1 + 3|, & x > 1 \\ |3 + 3|, & x = 1 \\ |-3x + 3|, & x < 1 \end{cases} = \begin{cases} |(x - 1) + 5|, & x > 1 \\ |6|, & x = 1 \\ |-3(x - 1)|, & x < 1 \end{cases}$$

where again we are writing all the terms we have in terms of  $x - 1$  factors.

### Example

(Continued)

Now by the backwards triangle inequality,  $|(x - 1) - 5| \geq 5 - |x - 1|$ . So

$$\begin{cases} |f(x) - (-3)| \geq 5 - |x - 1|, & x > 1 \\ |f(x) - (-3)| = |6|, & x = 1 \\ |f(x) - (-3)| = 3|x - 1|, & x < 1 \end{cases}$$

If we make the bottom term small by restricting our attention to  $x$  in  $(1 - \delta, 1) \cup (1, 1 + \delta)$ , then we have

$$\begin{cases} |f(x) - (-3)| \geq 5 - \delta, & x \in (1, 1 + \delta) \\ |f(x) - (-3)| \leq 3\delta, & x \in (1 - \delta, 1) \end{cases}$$

where we have dropped what happens at  $x = 1$  itself as it is not important for the existence of the limit.

### Example

(Continued)

So no matter what  $\epsilon$  we pick, for any  $0 < \delta < \epsilon/3$ , we have

$$\begin{cases} |f(x) + 3| \geq 5 - \epsilon/3, & x \in (1, 1 + \delta) \\ |f(x) + 3| < \epsilon, & x \in (1 - \delta, 1) \end{cases}$$

and although the bottom piece is *small* the top piece is not! So the  $\lim_{x \rightarrow 1} f(x)$  can not be  $-3$ .

**Maybe the limiting value here is  $a \neq 2, -3$ .**

We check

$$|f(x) - a| = \begin{cases} |x + 1 - a|, & x > 1 \\ |3 - a|, & x = 1 \\ |-3x - a|, & x < 1 \end{cases}$$

where we are writing all the terms we have in terms of  $x - 1$  factors.

### Example

(Continued)

In terms of  $x - 1$  factors, this becomes

$$|f(x) - a| = \begin{cases} |x - 1 + 2 - a|, & x > 1 \\ |3 - a|, & x = 1 \\ |-3(x - 1) - (3 + a)|, & x < 1 \end{cases}$$

Now by the backwards triangle inequality,

$|(x - 1) + (2 - a)| = |(x - 1) - (a - 2)| \geq |2 - a| - |x - 1|$  and  
 $|-3(x - 1) - (3 + a)| \geq |3 + a| - 3|x - 1|$ . Now the distance from  $a$  to  $2$  is  $|a - 2|$  which we call  $d_1$ . The distance from  $a$  to  $-3$  is  $|a - (-3)| = |a + 3|$  which we call  $d_2$ . Using these estimates, we find

$$\begin{cases} |f(x) - a| \geq d_1 - |x - 1|, & x > 1 \\ |f(x) - a| = |3 - a|, & x = 1 \\ |f(x) - a| \geq d_2 - 3|x - 1|, & x < 1 \end{cases}$$

## Example

(Continued)

If we make the top term small by restricting our attention to  $x$  in  $(1 - \delta, 1) \cup (1, 1 + \delta)$ , then we have

$$\begin{cases} |f(x) - a| \geq d_1 - \delta, & x \in (1, 1 + \delta) \\ |f(x) - a| \geq d_2 - 3\delta, & x \in (1 - \delta, 1) \end{cases}$$

where yet again we have dropped what happens at  $x = 1$  itself as it is not important for the existence of the limit. If we pick any positive  $\epsilon < (1/2) \min\{d_1, d_2/3\}$  and for any  $\delta > 0$ , we have

$$\begin{cases} |f(x) - a| \geq d_1 - d_1/2 = d_1/2 > \epsilon, & x \in (1, 1 + \delta) \\ |f(x) - a| \geq d_2 - 3d_2/6 = d_2/2 > \epsilon, & x \in (1 - \delta, 1) \end{cases}$$

and both the top piece and the bottom piece are never small! So the  $\lim_{x \rightarrow 1} f(x)$  can not be  $a \neq 2, -3$  either.

**Thus,  $\lim_{x \rightarrow 1} f(x)$  does not exist and  $f$  can not be continuous at 1.**

## Example

(Continued)

The method just done is tedious! Let's try using the  $\lim_{x \rightarrow 1} f(x)$  and  $\overline{\lim}_{x \rightarrow 1} f(x)$  approach instead.

Any sequence  $(x_n)$  with  $x_n \neq 1 \rightarrow 1$  from the left of 1 uses the bottom part of the definition of  $f$ . It is easy to see  $-3x_n \rightarrow -3$  here so  $-3$  is a cluster point of  $f$  at 1.

Any sequence  $(x_n)$  with  $x_n \neq 1 \rightarrow 1$  from the right of 1 uses the top part of the definition of  $f$ . It is easy to see  $x_n + 1 \rightarrow 2$  here so 2 is a cluster point of  $f$  at 1.

Any sequence  $(x_n)$  with  $x_n \neq 1 \rightarrow 1$  containing an infinite number of points both to the left of 1 and to the right of 1, has a subsequence  $(x_n^1)$  converging to  $-3$  and a subsequence  $(x_n^2)$  converging to 2 as

## Example

(Continued)

$$f(x_k) = \begin{cases} x_k^2 + 1, & x_k^2 \in (x_n), x_k^2 > 1 \\ -3x_k^1, & x_k^1 \in (x_n), x_k^1 < 1 \end{cases}$$

The top converges to 2 and the bottom converges to  $-3$  and hence this type of subsequence  $(x_n)$  can not converge. We conclude  $S(1) = \{-3, 2\}$  and so  $\lim_{x_n \rightarrow 1} f(x) = -3$  and  $\overline{\lim}_{x_n \rightarrow 1} f(x) = 2$ . Since these are not equal, we know  $\lim_{x \rightarrow 1} f(x)$  does not exist.

**Thus,  $f$  is not continuous at 1.**

It should be easy for you to see that the existence of the limit to the left of 1 and to the right of 1 is straightforward to establish. For example, at the point  $x = 4$ ,  $f(x) = x + 1$  and  $f(4) = 5$ . We have  $|f(x) - 5| = |x + 1 - 5| = |x - 4|$ . Given  $\epsilon > 0$ , if we choose  $\delta = \epsilon$ , we have  $|x - 4| < \delta \Rightarrow |f(x) - 5| = |x - 4| < \epsilon$ .

**This shows  $f$  is continuous at  $x = 4$ .** Similar arguments work for all points  $x \neq 1$ . So this  $f$  is continuous at all  $x$  except 1.

Another approach is to use right and left sided limits and continuity.

### Definition

Let  $f$  be locally defined near  $p$ . We say the **right hand limit** of  $f$  as  $x$  approaches  $p$  exists and equals  $b$  if

$$\forall \epsilon > 0 \exists \delta > 0 \ni p < x < p + \delta \Rightarrow |f(x) - b| < \epsilon$$

We denote the value  $b$  by the symbol  $\lim_{x \rightarrow p^+} f(x) = b$ .

We say the **left hand limit** of  $f$  as  $x$  approaches  $p$  exists and equals  $a$  if

$$\forall \epsilon > 0 \exists \delta > 0 \ni p - \delta < x < p \Rightarrow |f(x) - a| < \epsilon$$

We denote the value  $a$  by the symbol  $\lim_{x \rightarrow p^-} f(x) = a$ .

## Theorem

Let  $A$  be a real number. Then we have

$$\lim_{x \rightarrow p} f(x) = A \iff \forall (x_n), x_n \neq p, x_n \rightarrow p, \lim_{n \rightarrow \infty} f(x_n) = A.$$

## Proof

( $\Rightarrow$ ): We assume  $\lim_{x \rightarrow p} f(x) = A$ . Pick any  $(x_n), x_n \neq p, x_n \rightarrow p$ . Then for an arbitrary  $\epsilon > 0$ , we know there is a positive  $\delta$  so that  $x \in \hat{B}_\delta(p) \Rightarrow |f(x) - A| < \epsilon$ . Since  $x_n \rightarrow p$ , there is a  $N$  so that  $n > N \Rightarrow |x_n - p| < \delta$ . Combining, we see  $n > N \Rightarrow |x_n - p| < \delta \Rightarrow |f(x_n) - A| < \epsilon$ . Thus,  $f(x_n) \rightarrow A$  too.

( $\Leftarrow$ ):

We assume  $\forall (x_n), x_n \neq p, x_n \rightarrow p, \lim_{n \rightarrow \infty} f(x_n) = A$ . Let's do this by contradiction. We know  $f$  is locally defined in some  $\hat{B}_r(p)$ . Assume  $\lim_{x \rightarrow p} f(x)$  does not equal  $A$ . Pick a sequence  $\{1/N, 1/(N+1), \dots\}$  so that  $(p - 1/n, p + 1/n) \subset B_r(p)$  for all  $n > N$ .

## Proof

Then we know there is a positive  $\epsilon_0$  so that for each  $n > N$ , there is an  $x_n \neq p$  in  $(p - 1/n, p + 1/n)$  with  $|f(x_n) - A| > \epsilon_0$ . This defines a sequence  $(x_n)$  which converges to  $p$  and each  $x_n \neq p$ . Thus, by assumption, we know  $f(x_n) \rightarrow A$  which contradicts the construction we just did. So our assumption is wrong and  $\lim_{x \rightarrow p} f(x) = A$ .  $\square$

## Theorem

Let  $A$  be a real number. Then

$$\lim_{x \rightarrow p} f(x) = A \iff \left\{ \lim_{x \rightarrow p^+} f(x) = \lim_{x \rightarrow p^-} f(x) = A \right\}$$

## Proof

( $\Rightarrow$ )

We assume  $\lim_{x \rightarrow p} f(x) = A$ . Let  $(x_n)$  be any sequence with  $x_n \neq p$  that converges to  $p$  from below  $p$ . We then know by the previous theorem

$$(\lim_{x \rightarrow p} f(x) = A \iff \forall (x_n), x_n \neq p, x_n \rightarrow p, \lim_{n \rightarrow \infty} f(x_n) = A)$$

that  $\lim_{n \rightarrow \infty} f(x_n) = A$  also. Since this is true for all such sequences, we apply the previous theorem again

$$(\lim_{x \rightarrow p^-} f(x) = A \iff \forall (x_n), x_n \neq p, x_n \rightarrow p^-, \lim_{n \rightarrow \infty} f(x_n) = A)$$

to the left hand limit of  $f$  at  $p$  to see  $\lim_{x \rightarrow p^-} f(x) = A$ . A similar argument shows  $\lim_{x \rightarrow p^+} f(x) = A$ .

( $\Leftarrow$ )

We assume  $\lim_{x \rightarrow p^+} f(x) = \lim_{x \rightarrow p^-} f(x) = A$ . We know  $f$  is locally defined in some  $\tilde{B}_r(p)$ . Assume  $\lim_{x \rightarrow p} f(x)$  does not equal  $A$ . Again pick a sequence  $\{1/N, 1/(N+1), \dots\}$  so that  $(p - 1/n, p + 1/n) \subset B_r(p)$  for all  $n > N$ . Then we know there is a positive  $\epsilon_0$  so that for each  $n > N$ , there is an  $x_n \neq p$  in  $(p - 1/n, p + 1/n)$  with  $|f(x_n) - A| > \epsilon_0$ .

## Proof

This defines a sequence  $(x_n)$  which converges to  $p$  and each  $x_n \neq p$ . This sequence contains infinitely many  $x_{n_k}$  to the left of  $p$  and/or to the right of  $p$ . Hence, we can find a subsequence  $(x_{n_k})$  which converges to  $p$  from either below or above. For this subsequence, we then know  $f(x_{n_k}) \rightarrow A$  as both the right hand and left hand limits exist and equal  $A$ . But  $|f(x_{n_k}) - A| > \epsilon_0$  for all  $k$ . This is not possible, so our assumption was wrong and  $\lim_{x \rightarrow p} f(x) = A$ .  $\square$

### Definition

Let  $f$  be locally defined at  $p$ .

We say  $f$  is **continuous from the right** at  $p$  if  $\lim_{x \rightarrow p^+} f(x) = f(p)$ .

We say  $f$  is **continuous from the left** at  $p$  if  $\lim_{x \rightarrow p^-} f(x) = f(p)$ .

### Example

Let's look at our a new version of our old friend:

$$f(x) = \begin{cases} x + 1, & x > 1 \\ 2, & x = 1 \\ -3x & x < 1 \end{cases}$$

It is easy to see  $\lim_{x \rightarrow 1^-} f(x) = -3 \neq f(1) = 2$  and  $\lim_{x \rightarrow 1^+} f(x) = 2 = f(1)$ . So  $f$  is right continuous at 1 but  $f$  is not left continuous at 1.

## Homework 15

15.1 This is the tedious way.

$$f(x) = \begin{cases} 3x + 1, & x > 2 \\ 4, & x = 2 \\ 2x - 5 & x < 2 \end{cases}$$

Do a proper  $\epsilon - \delta$  argument to show  $f$  is not continuous at 2 but it is continuous at any other  $x$ . Your estimates here are done in terms of  $x - 2$  factors.

15.2 This is the easier way.

$$f(x) = \begin{cases} 3x + 1, & x > 2 \\ 4, & x = 2 \\ 2x - 5 & x < 2 \end{cases}$$

Use the limit inferior and limit superior ideas to show  $f$  is not continuous at 2 but it is continuous at any other  $x$ .

## Homework 15

15.3 This is even easier.

$$f(x) = \begin{cases} 3x + 1, & x > 2 \\ 4, & x = 2 \\ 2x - 5 & x < 2 \end{cases}$$

Use the right and left limit ideas to show  $f$  is not continuous at 2 but it is continuous at any other  $x$ . Determine if  $f$  is right or left continuous at 2.