Dirchlet's Function and Limit and Continuity Arguments

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Outline

Dirichlet's Function is defined like this: $f : [0,1] \to \Re$ by

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{q}, & x = \frac{p}{q}, (p,q) = 1, p, q \in \mathbb{N} \\ 0, & x \in \mathbb{IR} \end{cases}$$

So $f(9/12) = 1/4$ as $9/12 = 3/4$ when common terms are removed. This is a very strange function and we can prove

f is continuous at each $x \in \mathbb{IR} \cap [0,1]$ and is not continuous at each $x \in \mathbb{Q} \cap [01,]$.

Proof

(Case $x_0 \in \mathbb{IR}$):
We will show f is continuous at these x values. Choose $\epsilon > 0$ arbitrarily. Consider

$$|f(x) - f(x_0)| = \begin{cases} |0 - 0|, & x \in \mathbb{IR} \cap [0,1] \\ |1/q - 0|, & x = p/q, \ (p,q) = 1 \end{cases}$$

Proof

Let

$$\begin{aligned} S_n &= \{x : 0 < x < 1,\ x = p/q,\ (p,q) = 1 \textit{ with } 1/q \geq 1/n\} \\ &= \{0 < p/q < 1,\ (p,q) = 1,\ \ \textit{with } n \geq q\} \end{aligned}$$

How many elements are in S_n? Consider that table below:

$1/1$	$\{1/2$	$1/3$	$\dots$	$\dots$	$\dots$	$1/n\}$
$2/1$	$2/2$	$\{2/3$	$2/4$	$\dots$	$\dots$	$2/n\}$
$3/1$	$3/2$	$3/3$	$\{3/4$	$\dots$	$\dots$	$3/n\}$
$\vdots$		$\ddots$				
$j/1$	$j/2$		j/j	$\{j/(j+1)\dots$	$\dots$	$j/n\}$
$\vdots$						
$(n-1)/n$	$(n-1)/2$				$(n-1)/(n-1)$	$\{(n-1)/n\}$
$n/1$	$n/2$				$(n-1)/n$	n/n

The fractions in the braces in red are the ones in $(0,1)$ although they are not necessarily in lowest terms. The first row has $n-1$ elements, the second $n-2$ and so one until we get to the $n-1$ row which just has one red element. So the size of $S_n = |S_n| \leq \sum_{j=1}^{n-1} j = (n-1)n/2$.

Proof

The important thing is that S_n contains only a finite number of fractions in lowest terms. Now choose $N > 1/\epsilon$. Then $|S_N| \leq N(N-1)/2$. Label the elements of S_N as follows:

$$S_N = \{r_1, \ldots, r_p\}$$

where $p \leq N(N-1)/2$. Each fraction $r_i = p_i/q_i$ in lowest terms with $q_i \leq N$. One of these fractions is closest to x_0. Call this one $r_{min} = p_{min}/q_{min}$ and choose $\delta = (1/2)|r_{min} - x_0|$.

Note if $r = p/q, (p,q) = 1$ is any rational number in $(x_0 - \delta, x_0 + \delta)$, $r < r_{min}$ and so can't be in S_N!! (See the picture in the handwritten notes). Since $r \notin S_N$, the denominator of $r = p/q$ must satisfy $N < q$; i.e. $1/q < 1/N < \epsilon$. Combining, we see we have found a $\delta > 0$ so that

$$0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| = \begin{cases} 0 < \epsilon, & x \in \mathbb{IR} \\ |1/q - 0| < 1/N < \epsilon, & x \in \mathbb{Q} \end{cases}$$

Proof

This shows f is continuous at any irrational number x_0 in $[0,1]$.
(Case $x_0 \in \mathbb{Q} \cap [0,1]$):
Then $x_0 = p_0/q_0$, $(p_0, q_0) = 1$. Consider

$$|f(x) - f(x_0)| = \begin{cases} |0 - 1/q_0|, & x \in \mathbb{IR} \cap [0,1] \\ |1/q - 1/q_0|, & x = p/q,\ (p,q) = 1 \end{cases}$$

$$= \begin{cases} |0 - 1/q_0|, & x \in \mathbb{IR} \cap [0,1] \\ |\frac{q_0 - q}{q\, q_0}|, & x = p/q,\ (p,q) = 1 \end{cases}$$

Let $\epsilon_0 = (1/2)(1/q_0)$. We only have to look at the top part. The top says $|f(x) - f(x_0)| = 1/q_0 > (1/2)(1/q_0) = \epsilon_0$. Thus $f(x) = 1/q_0 > \epsilon_0$ for all $x \in \mathbb{IR}$. Hence, no matter how small $\delta > 0$ we choose, we can always find irrational numbers x_δ in $\hat{B}_\delta(x_0)$ with $|f(x_\delta) - f(x_0)| = 1/q_0 > \epsilon_2$. This shows f can not be continuous at any x_0 in $\mathbb{Q}$. □

This is a very interesting function! Again, these ideas of continuity and limit are **pointwise** concepts!!

Example

Let

$$f(x) = \begin{cases} x+1, & x > 1 \\ 3, & x = 1 \\ -3x, & x < 1 \end{cases}$$

We can study this function's behavior a number of ways.
(Case: $\epsilon - \delta$ approach):
Let's look at the point $x = 1$ first.

Maybe the limiting value here is 2.
We check

$$|f(x) - 2| = \begin{cases} |x+1-2|, & x > 1 \\ |3-2|, & x = 1 \\ |-3x-2|, & x < 1 \end{cases} = \begin{cases} |x-1|, & x > 1 \\ |1|, & x = 1 \\ |-3(x-1)) - 5|, & x < 1 \end{cases}$$

where we are writing all the terms we have in terms of $x - 1$ factors.

Example

(Continued)
Now by the backwards triangle inequality, $|3(x-1)-5| \geq 5-3|x-1|$. So

$$\begin{cases} |f(x)-2| = |x-1|, & x > 1 \\ |f(x)-2| = |1|, & x = 1 \\ |f(x)-2| \geq 5-3|x-1|, & x < 1 \end{cases}$$

If we make the top term small by restricting our attention to x in $(1-\delta, 1) \cup (1, 1+\delta)$, then we have

$$\begin{cases} |f(x)-2| < \delta, & x \in (1, 1+\delta) \\ |f(x)-2| \geq 5-3\delta, & x \in (1-\delta, 1) \end{cases}$$

where we have dropped what happens at $x = 1$ itself as it is not important for the existence of the limit.

Example

(Continued)
So no matter what ϵ we pick, for any $0 < \delta < \epsilon$, we have

$$\begin{cases} |f(x) - 2| < \epsilon, & x \in (1, 1+\delta) \\ |f(x) - 2| \geq 5 - 3\epsilon, & x \in (1-\delta, 1) \end{cases}$$

and although the top piece is *small* the bottom piece is not! So the $\lim_{x \to 1} f(x)$ can not be 2.

Maybe the limiting value here is -3.
We check

$$|f(x) - (-3)| = \begin{cases} |x+1+3|, & x > 1 \\ |3+3|, & x = 1 \\ |-3x+3|, & x < 1 \end{cases} = \begin{cases} |(x-1)+5|, & x > 1 \\ |6|, & x = 1 \\ |-3(x-1)|, & x < 1 \end{cases}$$

where again we are writing all the terms we have in terms of $x - 1$ factors.

Example

(Continued)
Now by the backwards triangle inequality, $|(x-1)-5| \geq 5-|x-1|$. So

$$\begin{cases} |f(x)-(-3)| \geq 5-|x-1|, & x>1 \\ |f(x)-(-3)| = |6|, & x=1 \\ |f(x)-(-3)| = 3|x-1|, & x<1 \end{cases}$$

If we make the bottom term small by restricting our attention to x in $(1-\delta,1)\cup(1,1+\delta)$, then we have

$$\begin{cases} |f(x)-(-3)| \geq 5-\delta, & x \in (1,1+\delta) \\ |f(x)-(-3)| \leq 3\delta, & x \in (1-\delta,1) \end{cases}$$

where we have dropped what happens at $x=1$ itself as it is not important for the existence of the limit.

Example

(Continued)
So no matter what ϵ we pick, for any $0 < \delta < \epsilon/3$, we have

$$\begin{cases} |f(x) + 3| \geq 5 - \epsilon/3, & x \in (1, 1+\delta) \\ |f(x) + 3| < \epsilon, & x \in (1-\delta, 1) \end{cases}$$

and although the bottom piece is *small* the top piece is not! So the $\lim_{x \to 1} f(x)$ can not be -3.

Maybe the limiting value here is $a \neq 2, -3$.
We check

$$|f(x) - a| = \begin{cases} |x + 1 - a|, & x > 1 \\ |3 - a|, & x = 1 \\ |-3x - a|, & x < 1 \end{cases}$$

where we are writing all the terms we have in terms of $x - 1$ factors.

Example

(Continued)
In terms of $x - 1$ factors, this becomes

$$|f(x) - a| = \begin{cases} |x - 1 + 2 - a|, & x > 1 \\ |3 - a|, & x = 1 \\ |-3(x-1)) - (3 + a)|, & x < 1 \end{cases}$$

Now by the backwards triangle inequality, $|(x-1) + (2-a)| = |(x-1) - (a-2)| \geq |2-a| - |x-1|$ and $|-3(x-1) - (3+a)| \geq |3+a| - 3|x-1|$. Now the distance from a to 2 is $|a-2|$ which we call d_1. The distance from a to -3 is $|a - (-3)| = |a+3|$ which we call d_2. Using these estimates, we find

$$\begin{cases} |f(x) - a| \geq d_1 - |x-1|, & x > 1 \\ |f(x) - a| = |3 - a|, & x = 1 \\ |f(x) - a| \geq d_2 - 3|x-1|, & x < 1 \end{cases}$$

Example

(Continued)
If we make the top term small by restricting our attention to x in $(1-\delta, 1) \cup (1, 1+\delta)$, then we have

$$\begin{cases} |f(x) - a| \geq d_1 - \delta, & x \in (1, 1+\delta) \\ |f(x) - a| \geq d_2 - 3\delta, & x \in (1-\delta, 1) \end{cases}$$

where yet again we have dropped what happens at $x = 1$ itself as it is not important for the existence of the limit. If we pick any positive $\epsilon < (1/2)\min\{d_1, d_2/3\}$ and for any $\delta > 0$, we have

$$\begin{cases} |f(x) - a| \geq d_1 - d_1/2 = d_1/2 > \epsilon, & x \in (1, 1+\delta) \\ |f(x) - a| \geq d_2 - 3d_2/6 = d_2/2 > \epsilon, & x \in (1-\delta, 1) \end{cases}$$

and both the top piece and the bottom piece are never small! So the $\lim_{x\to 1} f(x)$ can not be $a \neq 2, -3$ either.

Thus, $\lim_{x\to 1} f(x)$ **does not exist and** f **can not be continuous at** 1.

Example

(Continued)
The method just done is tedious! Let's try using the $\underline{\lim}_{x \to 1} f(x)$ and $\overline{\lim}_{x \to 1} f(x)$ approach instead.

Any sequence (x_n) with $x_n \neq 1 \to 1$ from the left of 1 uses the bottom part of the definition of f. It is easy to see $-3x_n \to -3$ here so -3 is a cluster point of f at 1.

Any sequence (x_n) with $x_n \neq 1 \to 1$ from the right of 1 uses the top part of the definition of f. It is easy to see $x_n + 1 \to 2$ here so 2 is a cluster point of f at 1.

Any sequence (x_n) with $x_n \neq 1 \to 1$ containing an infinite number of points both to the left of 1 and to the right of 1, has a subsequence (x_n^1) converging to -3 and a subsequence (x_n^2) converging to 2 as

Example

(Continued)

$$f(x_k) = \begin{cases} x_k^2 + 1, & x_k^2 \in (x_n), x_k^2 > 1 \\ -3x_k^1, & x_k^1 \in (x_n), x_k^1 < 1 \end{cases}$$

The top converges to 2 and the bottom converges to -3 and hence this type of subsequence (x_n) can not converge. We conclude $S(1) = \{-3, 2\}$ and so $\underline{\lim}_{x_n \to 1} f(x) = -3$ and $\overline{\lim}_{x_n \to 1} f(x) = 2$. Since these are not equal, we know $\lim_{x \to 1} f(x)$ does not exist.

Thus, f is not continuous at 1.

It should be easy for you to see that the existence of the limit to the left of 1 and to the right of 1 is straightforward to establish. For example, at the point $x = 4$, $f(x) = x + 1$ and $f(4) = 5$. We have $|f(x) - 5| = |x + 1 - 5| = |x - 4|$. Given $\epsilon > 0$, if we choose $\delta = \epsilon$, we have $|x - 4| < \delta \Rightarrow |f(x) - 5| = |x - 4| < \epsilon$.
This shows f is continuous at $x = 4$. Similar arguments work for all points $x \neq 1$. So this f is continuous at all x except 1.

Another approach is to use right and left sided limits and continuity.

Definition

Let f be locally defined near p. We say the **right hand limit** of f as x approaches p exists and equals b if

$$\forall \epsilon > 0 \, \exists \, \delta > 0 \, \ni p < x < p + \delta \Rightarrow |f(x) - b| < \epsilon$$

We denote the value b by the symbol $\lim_{x \to p^+} f(x) = b$.

We say the **left hand limit** of f as x approaches p exists and equals a if

$$\forall \epsilon > 0 \, \exists \, \delta > 0 \, \ni p - \delta < x < p \Rightarrow |f(x) - a| < \epsilon$$

We denote the value a by the symbol $\lim_{x \to p^-} f(x) = a$.

Theorem

Let A be a real number. Then we have

$$\lim_{x \to p} f(x) = A \iff \forall (x_n), x_n \neq p, x_n \to p, \lim_{n \to \infty} f(x_n) = A.$$

Proof

($\Rightarrow$): We assume $\lim_{x \to p} f(x) = A$. Pick any $(x_n), x_n \neq p, x_n \to p$. Then for an arbitrary $\epsilon > 0$, we know there is a positive δ so that $x \in \hat{B}_\delta(p) \Rightarrow |f(x) - A| < \epsilon$. Since $x_n \to p$, there is a N so that $n > N \to |x_n - p| < \delta$. Combining, we see $n > N \Rightarrow |x_n - p| < \delta \Rightarrow |f(x_n) - A| < \epsilon$. Thus, $f(x_n) \to A$ too.

($\Leftarrow$):
We assume $\forall (x_n), x_n \neq p, x_n \to p$, $\lim_{n \to \infty} f(x_n) = A$. Let's do this by contradiction. We know f is locally defined in some $\hat{B}_r(p)$. Assume $\lim_{x \to p} f(x)$ does not equal A. Pick a sequence $\{1/N, 1/(N+1), \ldots\}$ so that $(p - 1/n, p + 1/n) \subset B_r(p)$ for all $n > N$.

Proof

Then we know there is a positive ϵ_0 *so that for each* $n > N$*, there is an* $x_n \neq p$ *in* $(p - 1/n, p + 1/n)$ *with* $|f(x_n) - A| > \epsilon_0$*. This defines a sequence* (x_n) *which converges to* p *and each* $x_n \neq p$*. Thus, by assumption, we know* $f(x_n) \rightarrow A$ *which contradicts the construction we just did. So our assumption is wrong and* $\lim_{x \rightarrow p} f(x) = A$*.* □

Theorem

Let A *be a real number. Then*

$$\lim_{x \rightarrow p} f(x) = A \iff \left\{ \lim_{x \rightarrow p^+} f(x) = \lim_{x \rightarrow p^-} f(x) = A \right\}$$

Proof

(⇒)
We assume $\lim_{x \to p} f(x) = A$*. Let* (x_n) *be any sequence with* $x_n \neq p$ *that converges to* p *from below* p*. We then know by the previous theorem*

$$(\lim_{x \to p} f(x) = A \Longleftrightarrow \forall (x_n), x_n \neq p, x_n \to p, \lim_{n \to \infty} f(x_n) = A)$$

that $\lim_{n \to \infty} f(x_n) = A$ *also. Since this is true for all such sequences, we apply the previous theorem again*

$$(\lim_{x \to p-} f(x) = A \Longleftrightarrow \forall (x_n), x_n \neq p, x_n \to p^-, \lim_{n \to \infty} f(x_n) = A)$$

to the left hand limit of f *at* p *to see* $\lim_{x \to p^-} f(x) = A$*. A similar argument shows* $\lim_{x \to p^+} f(x) = A$*.*

(⇐)
We assume $\lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x) = A$*. We know* f *is locally defined in some* $\hat{B}_r(p)$*. Assume* $\lim_{x \to p} f(x)$ *does not equal* A*. Again pick a sequence* $\{1/N, 1/(N+1), \ldots\}$ *so that* $(p - 1/n, p + 1/n) \subset B_r(p)$ *for all* $n > N$*. Then we know there is a positive* ϵ_0 *so that for each* $n > N$*, there is an* $x_n \neq p$ *in* $(p - 1/n, p + 1/n)$ *with* $|f(x_n) - A| > \epsilon_0$*.*

Proof

This defines a sequence (x_n) *which converges to* p *and each* $x_n \neq p$. *This sequence contains infinitely many* x_{n_k} *to the left of* p *and/ or to the right of* p. *Hence, we can find a subsequence* (x_{n_k}) *with converges to* p *from either below or above. For this subsequence, we then know* $f(x_{n_k}) \rightarrow A$ *as both the right hand and left hand limits exist and equal* A. *But* $|f(x_{n_k}) - A| > \epsilon_0$ *for all* k. *This is not possible, so our assumption was wrong and* $\lim_{x \rightarrow p} f(x) = A$. □

Definition

Let f be locally defined at p.

We say f is **continuous from the right** at p if $\lim_{x \to p^+} f(x) = f(p)$.

We say f is **continuous from the left** at p if $\lim_{x \to p^-} f(x) = f(p)$.

Example

Let's look at our a new version of our old friend:

$$f(x) = \begin{cases} x+1, & x > 1 \\ 2, & x = 1 \\ -3x & x < 1 \end{cases}$$

It is easy to see $\lim_{x \to 1^-} f(x) = -3 \neq f(1) = 2$ and $\lim_{x \to 1^+} f(x) = 2 = f(1)$. So f is right continuous at 1 but f is not left continuous at 1.

Homework 15

15.1 This is the tedious way.

$$f(x) = \begin{cases} 3x+1, & x > 2 \\ 4, & x = 2 \\ 2x-5 & x < 2 \end{cases}$$

Do a proper $\epsilon - \delta$ argument to show f is not continous at 2 but it is continuous at any other x. Your estimates here are done in terms of $x-2$ factors.

15.2 This is the easier way.

$$f(x) = \begin{cases} 3x+1, & x > 2 \\ 4, & x = 2 \\ 2x-5 & x < 2 \end{cases}$$

Use the limit inferior and limit superior ideas to show f is not continous at 2 but it is continuous at any other x.

Homework 15

15.3 This is even easier.

$$f(x) = \begin{cases} 3x+1, & x > 2 \\ 4, & x = 2 \\ 2x-5 & x < 2 \end{cases}$$

Use the right and left limit ideas to show f is not continous at 2 but it is continuous at any other x. Determine if f is right or left continuous at 2.