# Upper and Lower Bounds

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August 29, 2018

# Outline

Let $S$ be a set of real numbers. We need to make precise the idea of a set of real numbers being **bounded**.

**Definition**

We say a set $S$ is bounded above if there is a number $M$ so that $x \leq M$ for all $x$ in $S$. We call $M$ an **upper bound** of $S$ or just an **u.b.**

**Example**

If $S = \{y : y = x^2 \text{ and } -1 \leq x \leq 2\}$, there are many u.b.'s of $S$. Some choices are $M = 5$, $M = 4.1$. Note $M = 1.9$ is **not** an u.b.

**Example**

If $S = \{y : y = \tanh(x) \text{ and } x \in \Re\}$, there are many u.b.'s of $S$. Some choices are $M = 2$, $M = 2.1$. Note $M = 0$ is **not** an u.b. Draw a picture of this graph too.

Let $S$ be a set of real numbers.

**Definition**

We say a set $S$ is bounded below if there is a number $m$ so that $x \geq m$ for all $x$ in $S$. We call $m$ a **lower bound** of $S$ or just a **l.b.**

**Example**

If $S = \{y : y = x^2 \text{ and } -1 \leq x \leq 2\}$, there are many l.b.'s of $S$. Some choices are $m = -2$, $m = -0.1$. Note $m = 0.3$ is **not** a l.b.

**Example**

If $S = \{y : y = \tanh(x) \text{ and } x \in \Re\}$, there are many l.b.'s of $S$. Some choices are $m = -1.1$, $m = -1.05$. Note $m = -0.87$ is **not** a l.b. Draw a picture of this graph again.

Let $S$ be a set of real numbers.

## Definition

We say a set $S$ is bounded if $S$ is bounded above and bounded below. That is, there are finite numbers $m$ and $M$ so that $m \leq x \leq M$ for all $x \in S$. We usually overestimate the bound even more and say $S$ is bounded if we can find a number $B$ so that $|x| \leq B$ for all $x \in S$. A good choice of such a $B$ is to let $B = \max(|m|, |M|)$ for any choice of l.b. $m$ and u.b. $M$.

## Example

If $S = \{y : y = x^2 \text{ and } -1 \leq x < 2\}$. here $S = [0, 4)$ and so for $m = -2$ and $M = 5$, a choice of $B$ is $B = 5$. Of course, there are many other choices of $B$.

## Example

If $S = \{y : y = \tanh(x) \text{ and } x \in \Re\}$, we have $S = (-1, 1)$ and for $m = -1.1$ and $M = 1.2$, a choice of $B$ is $B = 1.2$.

The next material is more abstract! We need to introduce the notion of **least upper bound** and **greatest lower bound**.
We also call the **least upper bound** the **l.u.b.**. It is also called the **supremum** of the set $S$. We use the notation $\sup(S)$ as well.
We also call the **greatest lower bound** the **g.l.b.**. It is also called the **infimum** of the set $S$. We use the notation $\inf(S)$ as well.

### Definition

The **least upper bound**, **l.u.b.** or sup of the set $S$ is a number $U$ satisfying

1. $U$ is an upper bound of $S$
2. If $M$ is any other upper bound of $S$, then $U \leq M$.

The **greatest lower bound**, **g.l.b.** or inf of the set $S$ is a number $u$ satisfying

1. $u$ is a lower bound of $S$
2. If $m$ is any other lower bound of $S$, then $u \geq m$.

**Example**

If $S = \{y : y = x^2$ and $-1 \leq x < 2\}$. here $S = [0, 4)$ and so $\inf(S) = 0$ and $\sup(S) = 4$.

**Example**

If $S = \{y : y = \tanh(x)$ and $x \in \Re\}$, we have *inf*$(S) = -1$ and $\sup(S) = 1$. Not the *inf* and sup of a set $S$ need **NOT** be in $S$!

**Example**

If $S = \{y : \cos(2n\pi/3), \quad \forall n \in \mathbb{N}\}$, The only possible values in $S$ are $\cos(2\pi/3) = -1/2$, $\cos(4\pi/3) = -1/2$ and $\cos(6\pi/3) = 1$. There are no other values and these 2 values are endlessly repeated in a cycle. Here $\inf(S) = -1/2$ and $\sup(S) = 1$.

## Comment

*If a set $S$ has no finite lower bound, we set $\inf(S) = -\infty$. If a set $S$ has no finite upper bound, we set $\sup(S) = \infty$.*

## Comment

*If the set $S = \emptyset$, we set $\inf(S) = \infty$ and $\sup(S) = -\infty$.*

## Definition

We say $Q \in S$ is a maximum of $S$ if $\sup(S) = Q$. This is the same, of course, as saying $x \leq Q$ for all $x$ in $S$ which is the usual definition of an upper bound. But this is different as $Q$ is in $S$. We call $Q$ a **maximizer** or **maximum element** of $S$.
We say $q \in S$ is a minimum of $S$ if $\inf(S) = q$. Again, this is the same as saying $x \geq q$ for all $x$ in $S$ which is the usual definition of a lower bound. But this is different as $q$ is in $S$. We call $q$ a **minimizer** **minimal element** of $S$.

There is a fundamental **axiom** about the behavior of the real numbers which is very important.

### Axiom

**The Completeness Axiom**

*Let $S$ be a set of real numbers which is nonempty and bounded above. Then the supreumum of $S$ exists and is finite.*

*Let $S$ be a set of real numbers which is nonempty and bounded below. Then the infimum of $S$ exists and is finite.*

### Comment

*So nonempty bounded sets of real numbers always have a finite infimum and supremum. This does not say the set has a finite minimum and finite maximum. Another way of saying this is that we don't know if $S$ has a minimizer and maximizer.*

**Theorem**

*Let $S$ be a nonempty set of real numbers which is bounded above. Then $\sup(S)$ exists and is finite. Then $S$ has a maximal element if and only if (IFF) $\sup(S) \in S$.*

**Proof**

*($\Leftarrow$): Assume $\sup(S)$ is in $S$. By definition, $\sup(S)$ is an upper bound of $S$ and so must satisfy $x \leq \sup(S)$ for all $x$ in $S$. This says $\sup(S)$ is a maximizer of $S$.*
*($\Rightarrow$): Let $Q$ denote a maximizer of $S$. Then by definition $x \leq Q$ for all $x$ in $S$ and is an upper bound. So by the definition of a supremum, $\sup(S) \leq Q$. Since $Q$ is a maximizer, $Q$ is in $S$ and from the definition of upper bound, we have $Q \leq \sup(S)$ as well. This says $\sup(S) \leq Q \leq \sup(S)$ or $\sup(S) = Q$.* $\square$

### Theorem

*Let $S$ be a nonempty set of real numbers which is bounded below. Then $\inf(S)$ exists and is finite. Then $S$ has a minimal element $\Leftrightarrow \inf(S) \in S$.*

### Proof

*($\Leftarrow$): Assume $\inf(S)$ is in $S$. By definition, $\inf(S)$ is a lower bound of $S$ and so must satisfy $x \geq \inf(S)$ for all $x$ in $S$. This says $\inf(S)$ is a minimizer of $S$.*
*($\Rightarrow$): Let $q$ denote a minimizer of $S$. Then by definition $x \geq q$ for all $x$ in $S$ and is a lower bound. So by the definition of an infimum, $q \leq \inf(S)$. Since $q$ is a minimizer, $q$ is in $S$ and from the definition of lower bound, we have $\inf(S) \leq q$ as well. This says $\inf(S) \leq q \leq \inf(S)$ or $\inf(S) = q$.* $\square$

## Lemma

**Infimum Tolerance Lemma**: *Let $S$ be a nonempty set of real numbers that is bounded below. Let $\epsilon > 0$ be arbitrarily chosen. Then*

$$\exists\, y \in S \;\ni\; \inf(S) \leq y \;<\; \inf(S) \,+\, \epsilon$$

## Proof

*We do this by contradiction. Assume this is not true for some $\epsilon > 0$. Then for all $y$ in $S$, we must have $y \geq \inf(S) + \epsilon$. But this says $\inf(S) + \epsilon$ must be a lower bound of $S$. So by the definition of infimum, we must have $\inf(S) \geq \inf(S) + \epsilon$ for a positive epsilon which is impossible. Thus our assumption is wrong and we must be able to find at least one $y$ in $S$ that satisfies $\inf(S) \leq y \;<\; \inf(S) \,+\, \epsilon$.* □

## Lemma

**Supremum Tolerance Lemma**: *Let $S$ be a nonempty set of real numbers that is bounded above. Let $\epsilon > 0$ be arbitrarily chosen. Then*

$$\exists\, y \in S \;\ni\; \sup(S) - \epsilon < y \leq \sup(S)$$

## Proof

*We do this by contradiction. Assume this is not true for some $\epsilon > 0$. Then for all $y$ in $S$, we must have $y \leq \sup(S) - \epsilon$. But this says $\sup(S) - \epsilon$ must be an upper bound of $S$. So by the definition of supremum, we must have $\sup(S) \leq \sup(S) - \epsilon$ for a positive epsilon which is impossible. Thus our assumption is wrong and we must be able to find at least one $y$ in $S$ that satisfies $\sup(S) - \epsilon < y \leq \sup(S)$.* □

## Example

Let $f(x,y) = x + 2y$ and let $S = [0,1] \times [1,3]$ which is also $S_x \times S_y$ where $S_x = \{x : 0 \le x \le 1\}$ and $S_y = \{y : 1 \le y \le 3\}$. Note $\inf_{(x,y)\in[0,1]\times[1,3]} f(x,y) = 0 + 2 = 2$ and $\sup_{(x,y)\in[0,1]\times[1,3]} f(x,y) = 1 + 6 = 7$.

$$\begin{aligned}
\inf_{1\le y\le 3} f(x,y) &= \inf_{1\le y\le 3} (x+2y) = x+2 \\
\sup_{0\le x\le 1} f(x,y) &= \sup_{0\le x\le 1} (x+2y) = 1+2y \\
\sup_{0\le x\le 1} \inf_{1\le y\le 3} (x+2y) &= \sup_{0\le x\le 1} (x+2) = 3 \\
\inf_{1\le y\le 3} \sup_{0\le x\le 1} (x+2y) &= \inf_{1\le y\le 3} (1+2y) = 3
\end{aligned}$$

so in this example,

$$\inf_{y\in S_y} \sup_{x\in S_x} f(x,y) = \sup_{x\in S_x} \inf_{y\in S_y} f(x,y)$$

.

## Example

Let

$$f(x, y) = \begin{cases} 0, & (x, y) \in (1/2, 1] \times (1/2, 1] \\ 2, & (x, y) \in (1/2, 1] \times [0, 1/2] \text{ and } [0, 1/2] \times (1/2, 1] \\ 1, & (x, y) \in [0, 1/2] \times [0, 1/2] \end{cases}$$

and let $S = [0, 1] \times [0, 1]$ which is also $S_x \times S_y$ where $S_x = \{x : 0 \leq x \leq 1\}$ and $S_y = \{y : 0 \leq y \leq 1\}$. Note $\inf_{(x,y) \in [0,1] \times [0,1]} f(x, y) = 0$ and $\sup_{(x,y) \in [0,1] \times [0,1]} f(x, y) = 2$. Then, we also can find

$$\inf_{0 \leq y \leq 1} f(x, y) = \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 0, & 1/2 < x \leq 1 \end{cases}$$

and

$$\sup_{0 \leq x \leq 1} f(x, y) = \begin{cases} 2, & 0 \leq y \leq 1/2 \\ 2, & 1/2 < y \leq 1 \end{cases}$$

## Example

(Continued)

$$\sup_{0\le x\le 1}\ \inf_{0\le y\le 1}\ f(x,y) \quad = \quad \sup_{0\le x\le 1} \begin{cases} 1, & 0\le x\le 1/2 \\ 0, & 1/2 < x \le 1 \end{cases} = 1.$$

and

$$\inf_{0\le y\le 1}\ \sup_{0\le x\le 1}\ f(x,y) \quad = \quad \inf_{0\le y\le 1} \begin{cases} 2, & 0\le y\le 1/2 \\ 2, & 1/2 < y \le 1 \end{cases} = 2$$

so in this example

$$\inf_{y\in S_y}\ \sup_{x\in S_x}\ f(x,y) \neq \sup_{x\in S_x}\ \inf_{y\in S_y}\ f(x,y)$$

and in fact

$$\sup_{x\in S_x}\ \inf_{y\in S_y}\ f(x,y) < \inf_{y\in S_y}\ \sup_{x\in S_x}\ f(x,y)$$

- The moral here is that **order** matters. For example, in an applied optimization problem, it is not always true that

$$\min_x \max_y f(x, y) = \max_y \min_x f(x, y)$$

where $x$ and $y$ come from some domain set $S$.

- The moral here is that **order** matters. For example, in an applied optimization problem, it is not always true that

$$\min_x \max_y f(x,y) = \max_y \min_x f(x,y)$$

where $x$ and $y$ come from some domain set $S$.

- So it is probably important to find out when the order does not matter because it might be easier to compute in one ordering than another.

## Theorem

*Let $S$ be a nonempty bounded set of real numbers. Then* $\inf(S)$ *and* $\sup(S)$ *are unique.*

## Proof

*By the completeness axiom, since $S$ is bounded and nonempty, we know* $\inf(S)$ *and* $\sup(S)$ *are finite numbers. Let $u_2$ satisfy the definition of supremum also. Then, we know $u_2 \leq M$ for all upper bounds $M$ of $S$ and in particular since* $\sup(S)$ *is an upper bound too, we must have* $u_2 \leq \sup(S)$. *But since* $\sup(S)$ *is a supremum, by definition, we also know* $\sup(S) \leq u_2$ *as $u_2$ is an upper bound. Combining, we have* $u_2 \leq \sup(S) \leq u_2$ *which tells us* $u_2 = \sup(S)$. *A similar argument shows the* $\inf(S)$ *is also unique.* □

# Homework 4

4.1 Let

$$f(x,y) = \begin{cases} 3, & (x,y) \in (1/2,1] \times (1/2,1] \\ -2, & (x,y) \in (1/2,1] \times [0,1/2] \text{ and } [0,1/2] \times (1/2,1] \\ 4, & (x,y) \in [0,1/2] \times [0,1/2] \end{cases}$$

and let $S = [0,1] \times [0,1]$ which is also $S_x \times S_y$ where $S_x = \{x : 0 \le x \le 1\}$ and $S_y = \{y : 0 \le y \le 1\}$. Find

1. $\inf_{(x,y)\in S} f(x,y)$, and $\sup_{(x,y)\in S} f(x,y)$,
2. $\inf_{y\in S_y} \sup_{x\in S_x} f(x,y)$ and $\sup_{x\in S_x} \inf_{y\in S_y} f(x,y)$.
3. $\inf_{x\in S_x} \sup_{y\in S_y} f(x,y)$ and $\sup_{y\in S_y} \inf_{x\in S_x} f(x,y)$.

4.2 Let $S = \{z : z = e^{-x^2-y^2} \text{ for } (x,y) \in \Re^2\}$. Find $\inf(S)$ and $\sup(S)$. Does the minimum and maximum of $S$ exist and if so what are their values?