

Matrix Solutions to Linear Systems of ODEs

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Outline

- 1 Symmetric Systems of ODEs
- 2 Writing The Solution Another Way

Example

For the symmetric system below

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -20 & 12 \\ 12 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- Find the characteristic equation
- Find the general solution
- Solve the IVP

Solution

- *The characteristic equation is*

$$\det \left(r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -20 & 12 \\ 12 & 5 \end{bmatrix} \right) = 0$$

- *Thus*

$$\begin{aligned} 0 &= \det \left(\begin{bmatrix} r+20 & -12 \\ -12 & r-5 \end{bmatrix} \right) \\ &= (r+20)(r-5) - 144 \\ &= r^2 + 15r - 244 \end{aligned}$$

- *Hence, **eigenvalues** or **roots** of the characteristic equation are $r_1 = 9.83$ and $r_2 = -24.83$.*

Solution

- For eigenvalue $r_1 = 9.83$, substitute the value into

$$\begin{bmatrix} r + 20 & -12 \\ -12 & r - 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 29.83 & -12 \\ -12 & 4.83 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

- Picking the top row, we get $29.83V_1 - 12V_2 = 0$ implying $V_2 = 6.18V_1$.
- Letting $V_1 = a$, we find $V_1 = a$ and $V_2 = 6.18a$: so

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 6.18 \end{bmatrix}$$

- Let $a = 1/\| [1 \ 6.18]^T \| = 1/6.26 = .16$ and choose the unit eigenvector

$$\mathbf{E}_1 = 0.16 \begin{bmatrix} 1 \\ 6.18 \end{bmatrix} = \begin{bmatrix} .16 \\ .99 \end{bmatrix}$$

Solution

- So one of the solutions is

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \mathbf{E}_1 e^{9.83t} = \begin{bmatrix} .16 \\ .99 \end{bmatrix} e^{9.83t}.$$

- For eigenvalue $r_2 = -24.83$, substitute the value into

$$\begin{bmatrix} r + 20 & -12 \\ -12 & r - 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -4.83 & -12 \\ -12 & -29.83 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

- Picking the top row, we get $-4.83V_1 - 29.83V_2 = 0$ implying $V_2 = -.16V_1$.
- Letting $V_1 = b$, we find $V_1 = b$ and $V_2 = -.16b$: so

$$\mathbf{v} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = b \begin{bmatrix} 1 \\ -.16 \end{bmatrix}$$

Solution

- Let $b = 1/\| [1 \quad -.16]^T \| = 1/1.01 = .99$ and choose the unit eigenvector

$$\mathbf{E}_2 = .99 \begin{bmatrix} 1 \\ -.16 \end{bmatrix} = \begin{bmatrix} .99 \\ -.16 \end{bmatrix}$$

- Note $\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = (.16)(.99) + (.99)(-.16) = 0$.
- So the other solution is

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \mathbf{E}_2 e^{-29.83t} = \begin{bmatrix} .99 \\ -.16 \end{bmatrix} e^{-29.83t}.$$

- The general solution

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= A \mathbf{E}_1 e^{9.83t} + B \mathbf{E}_2 e^{-29.83t} \\ &= A \begin{bmatrix} .16 \\ .99 \end{bmatrix} e^{9.83t} + B \begin{bmatrix} .99 \\ -.16 \end{bmatrix} e^{-29.83t} \end{aligned}$$

Solution

- Find A and B : use the ICs.

$$\begin{aligned} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} = A \begin{bmatrix} .16 \\ .99 \end{bmatrix} e^0 + B \begin{bmatrix} .99 \\ -.16 \end{bmatrix} e^0 \\ &= A \begin{bmatrix} .16 \\ .99 \end{bmatrix} + B \begin{bmatrix} .99 \\ -.16 \end{bmatrix} \end{aligned}$$

- Thus,

$$\begin{aligned} [\mathbf{E}_1 \quad \mathbf{E}_2] \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} &\Rightarrow [\mathbf{E}_1^T \quad \mathbf{E}_2^T] [\mathbf{E}_1 \quad \mathbf{E}_2] \begin{bmatrix} A \\ B \end{bmatrix} = [\mathbf{E}_1^T \quad \mathbf{E}_2^T] \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} A \\ B \end{bmatrix} &= [\mathbf{E}_1^T \quad \mathbf{E}_2^T] \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{aligned}$$

- Letting $\mathbf{D} = [-1 \quad 2]^T$, $A = \langle \mathbf{E}_1, \mathbf{D} \rangle$ and $B = \langle \mathbf{E}_2, \mathbf{D} \rangle$.

The coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} -20 & 12 \\ 12 & 5 \end{bmatrix}$$

and the eigenvalues of \mathbf{A} are $\lambda_1 = 9.83$ and $\lambda_2 = -29.83$. Then we know if

$$\mathbf{P} = [\mathbf{E}_1 \quad \mathbf{E}_2]$$

then

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \implies \mathbf{A} = \mathbf{P} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{P}^T$$

Let

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \implies \mathbf{A} = \mathbf{P} \mathbf{\Lambda} \mathbf{P}^T$$



$$\begin{aligned} \mathbf{A}^2 &= (\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T) (\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T) = \mathbf{P}\mathbf{\Lambda}(\mathbf{P}^T\mathbf{P})\mathbf{\Lambda}\mathbf{P}^T \\ &= \mathbf{P}\mathbf{\Lambda}^2\mathbf{P}^T \end{aligned}$$

$$\begin{aligned} \mathbf{A}^3 &= (\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T) (\mathbf{A}^2) = (\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T) (\mathbf{P}\mathbf{\Lambda}^2\mathbf{P}^T) = \mathbf{P}\mathbf{\Lambda}(\mathbf{P}^T\mathbf{P})\mathbf{\Lambda}^2\mathbf{P}^T \\ &= \mathbf{P}\mathbf{\Lambda}^3\mathbf{P}^T \end{aligned}$$

- It is easy to see by POMI that $\mathbf{A}^n = \mathbf{P}\mathbf{\Lambda}^n\mathbf{P}^T$.
- Recall for the scalar t , $\mathbf{A}t = t\mathbf{A}$ simply multiplies each entry of \mathbf{A} by the number t . Hence, from the above we have $\mathbf{A}^n t = \mathbf{P}\mathbf{\Lambda}^n\mathbf{P}^T t$.
- Consider

$$\mathbf{A}t = \mathbf{P} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{P}^T t = \mathbf{P} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} E_{11} t & E_{12} t \\ E_{21} t & E_{22} t \end{bmatrix} = \mathbf{P} \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} \mathbf{P}^T$$

- Thus,

$$\mathbf{A}t = \mathbf{P} \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

- A similar calculation shows

$$\mathbf{A}^2 t^2 = \mathbf{P} \begin{bmatrix} \lambda_1^2 t^2 & 0 \\ 0 & \lambda_2^2 t^2 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \mathbf{P} \begin{bmatrix} \lambda_1^2 t^2 & 0 \\ 0 & \lambda_2^2 t^2 \end{bmatrix} \mathbf{P}^T$$

- And the function

$$\begin{aligned} W_3(t) &= \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 t^2 / 2! + \mathbf{A}^3 t^3 / 3! \\ &= \mathbf{P} \begin{bmatrix} 1 + \lambda_1 t + \lambda_1^2 t^2 / 2 + \lambda_1^3 t^3 / 6 & 0 \\ 0 & 1 + \lambda_2 t + \lambda_2^2 t^2 / 2 + \lambda_2^3 t^3 / 6 \end{bmatrix} \mathbf{P}^T \\ &= \mathbf{P} \begin{bmatrix} \sum_{k=0}^3 (\lambda_1^k t^k) / k! & 0 \\ 0 & \sum_{k=0}^3 (\lambda_2^k t^k) / k! \end{bmatrix} \mathbf{P}^T \end{aligned}$$

- In general

$$W_n(t) = \mathbf{P} \begin{bmatrix} \sum_{k=0}^n (\lambda_1^k t^k) / k! & 0 \\ 0 & \sum_{k=0}^n (\lambda_2^k t^k) / k! \end{bmatrix} \mathbf{P}^T$$

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- We know $e^{\lambda_1 t} = \lim_{n \rightarrow \infty} \sum_{k=0}^n (\lambda_1^k t^k)/k!$ and $e^{\lambda_2 t} = \lim_{n \rightarrow \infty} \sum_{k=0}^n (\lambda_2^k t^k)/k!$

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- Define the matrix

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

- The system of ODEs $\mathbf{X}' = \mathbf{A}\mathbf{X}$ can be written as $\mathbf{X}' = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T\mathbf{X}$.
Let $\mathbf{Y} = \mathbf{P}^t\mathbf{X}$.

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- Then the system becomes $\mathbf{Y}' = \mathbf{\Lambda}\mathbf{Y}$. with general solution

$$\mathbf{Y}(t) = \begin{bmatrix} \alpha e^{\lambda_1 t} & 0 \\ 0 & \beta e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

for arbitrary α and β . Define

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- This gives $\mathbf{X}(t) = \mathbf{P}e^{\mathbf{\Lambda}t}\mathbf{\Theta}$
- Now define the matrix $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{\Lambda}t}\mathbf{P}^T$. Then $e^{\mathbf{A}t}\mathbf{P} = \mathbf{P}e^{\mathbf{\Lambda}t}$ and so $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{P}\mathbf{\Theta}$.

- Note

$$P\Theta = [\mathbf{E}_1 \quad \mathbf{E}_2] \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = [\alpha\mathbf{E}_1 \quad \beta\mathbf{E}_2]$$

- Note

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- So for initial data, $\mathbf{X}_0 = [x_0^1 \quad x_0^2]^T$, we have

$$\begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = [\alpha\mathbf{E}_1 \quad \beta\mathbf{E}_2]$$

implying $\alpha = \langle \mathbf{X}_0, \mathbf{E}_1 \rangle = x_0^1$ and $\beta = \langle \mathbf{X}_0, \mathbf{E}_2 \rangle = x_0^2$

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- Thus, the solution to the initial value problem is $\mathbf{X}(t) = e^{At} \mathbf{X}_0$.

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implying $\alpha = \langle \mathbf{X}_0, \mathbf{E}_1 \rangle = x_0^1$ and $\beta = \langle \mathbf{X}_0, \mathbf{E}_2 \rangle = x_0^2$

- Thus, the solution to the initial value problem is $\mathbf{X}(t) = e^{At} \mathbf{X}_0$.
- And the general solution to the dynamics has the form $\mathbf{X}(t) = e^{At} \mathbf{C}$ where \mathbf{C} is an arbitrary vector.

- The general solution to the scalar ODE $x' = \lambda x$ is $x(t) = e^{\lambda t}c$ and we now know we can write the general solution to the vector system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ as $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ also as long as we interpret the exponential matrix $e^{\mathbf{A}t}$ right.

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- Our argument was for 2×2 symmetric matrices, but essentially the same argument is used in the general $n \times n$ case but the canonical form of \mathbf{A} we need is called the **Jordan Canonical Form** which is discussed in more advanced classes.

Homework 40

40.1 Prove that $\mathbf{A}^n = \mathbf{P}\mathbf{\Lambda}^n\mathbf{P}^T$ by POMI.

40.2 Show via POMI

$$\mathbf{A}^n t^n = \mathbf{P} \begin{bmatrix} \lambda_1^n t^n & 0 \\ 0 & \lambda_2^n t^n \end{bmatrix} \mathbf{P}^T$$

40.3 Calculate $e^{\mathbf{A}t}$ for $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$