Bolzano Weierstrass Theorems I

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Outline

Theorem

Bolzano Weierstrass Theorem
Every bounded sequence with an infinite range has at least one convergent subsequence.

Proof

As discussed, we have already shown a sequence with a bounded finite range always has convergent subsequences. Now we prove the case where the range of the sequence of values $\{a_1, a_2 \ldots, \}$ *has infinitely many distinct values. We assume the sequences start at* $n = k$ *and by assumption, there is a positive number* B *so that* $-B \leq a_n \leq B$ *for all* $n \geq k$*. Define the interval* $J_0 = [\alpha_0, \beta_0]$ *where* $\alpha_0 = -B$ *and* $\beta_0 = B$*. Thus at this starting step,* $J_0 = [-B, B]$*. Note the length of* J_0*, denoted by* ℓ_0 *is* $2B$*.*
Let $\mathcal{S}$ *be the range of the sequence which has infinitely many points and for convenience, we will let the phrase* infinitely many points *be abbreviated to* **IMPs***.*

Proof

Step 1:
Bisect $[\alpha_0, \beta_0]$ *into two pieces* u_0 *and* u_1. *That is the interval* J_0 *is the union of the two sets* u_0 *and* u_1 *and* $J_0 = u_0 \cup u_1$. *Now at least one of the intervals* u_0 *and* u_1 *contains IMPs of* $\mathcal{S}$ *as otherwise each piece has only finitely many points and that contradicts our assumption that* $\mathcal{S}$ *has IMPS. Now both may contain IMPS so select one such interval containing IMPS and call it* J_1. *Label the endpoints of* J_1 *as* α_1 *and* β_1; *hence,* $J_1 = [\alpha_1, \beta_1]$. *Note* $\ell_1 = \beta_1 - \alpha_1 = \frac{1}{2}\ell_0 = B$ *We see* $J_1 \subseteq J_0$ *and*

$$-B = \alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0 = B$$

Since J_1 *contains IMPS, we can select a sequence value* a_{n_1} *from* J_1.
Step 2:
Now bisect J_1 *into subintervals* u_0 *and* u_1 *just as before so that* $J_1 = u_0 \cup u_1$. *At least one of* u_0 *and* u_1 *contain IMPS of* $\mathcal{S}$.

Proof

Choose one such interval and call it J_2. *Label the endpoints of* J_2 *as* α_2 *and* β_2]; *hence,* $J_2 = [\alpha_2, \beta_2]$. *Note* $\ell_2 = \beta_2 - \alpha_2 = \frac{1}{2}\ell_1$ *or* $\ell_2 = (1/4)\ell_0 = (1/2^2)\ell_0 = (1/2)B$. *We see* $J_2 \subseteq J_1 \subseteq J_0$ *and*

$$-B = \alpha_0 \le \alpha_1 \le \alpha_2 \le \beta_2 \le \beta_1 \le \beta_0 = B$$

Since J_2 *contains IMPS, we can select a sequence value* a_{n_2} *from* J_2. *It is easy to see this value is different from* a_{n_1}, *our previous choice.*
You should be able to see that we can continue this argument using induction.

Proposition:
$\forall p \ge 1$, $\exists$ an interval $J_p = [\alpha_p, \beta_p]$ with the length of J_p, $\ell_p = B/(2^{p-1})$ satisfying $J_p \subseteq J_{p-1}$, J_p contains IMPS of $\mathcal{S}$ and

$$\alpha_0 \le \ldots \le \alpha_{p-1} \le \alpha_p \le \beta_p \le \beta_{p-1} \le \ldots \le \beta_0$$

. Finally, there is a sequence value a_{n_p} in J_p, different from $a_{n_1}, \ldots, a_{n_{p-1}}$.

Proof

We have already established the proposition is true for the basis step J_1 and indeed also for the next step J_2.

Inductive: *We assume the interval J_q exists with all the desired properties. Since by assumption, J_q contains IMPs, bisect J_q into u_0 and u_1 like usual. At least one of these intervals contains IMPs of $\mathcal{S}$. Call the interval J_{q+1} and label $J_{q+1} = [\alpha_{q+1}, \beta_{q+1}]$. We see immediately that*

$$\ell_{q+1} = (1/2)\ell_q = (1/2)(1/2^{q-1})B = (1/2^q)B$$

with $\ell_{q+1} = \beta_{q+1} - \alpha_{q+1}$ with

$$\alpha_q \leq \alpha_{q+1} \leq \beta_{q+1} \leq \beta_q.$$

This shows the nested inequality we want is satisfied.
Finally, since J_{q+1} contains IMPs, we can choose $a_{n_{q+1}}$ distinct from the other a_{n_i}'s. So the inductive step is satisfied and by the POMI, the proposition is true for all n. $\square$

Proof

- *From our proposition, we have proven the existence of three sequences,* $(\alpha_p)_{p\geq 0}$, $(\beta_p)_{p\geq 0}$ *and* $(\ell_p)_{p\geq 0}$ *which have various properties.*

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- *The sequence* ℓ_p *satisfies* $\ell_p = (1/2)\ell_{p-1}$ *for all* $p \geq 1$. *Since* $\ell_0 = 2B$, *this means* $\ell_1 = B$, $\ell_2 = (1/2)B$, $\ell_3 = (1/2^2)B$ *leading to* $\ell_p = (1/2^{p-1})B$ *for* $p \geq 1$.

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- $$\begin{aligned} -B = \alpha_0 &\leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p \\ &\leq \ldots \leq \\ \beta_p &\leq \ldots \leq \beta_2 \leq \ldots \leq \beta_0 = B \end{aligned}$$

Proof

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- *The sequence* ℓ_p *satisfies* $\ell_p = (1/2)\ell_{p-1}$ *for all* $p \geq 1$. *Since* $\ell_0 = 2B$, *this means* $\ell_1 = B$, $\ell_2 = (1/2)B$, $\ell_3 = (1/2^2)B$ *leading to* $\ell_p = (1/2^{p-1})B$ *for* $p \geq 1$.
- $$\begin{aligned} -B = \alpha_0 &\leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p \\ &\leq \ldots \leq \\ \beta_p &\leq \ldots \leq \beta_2 \leq \ldots \leq \beta_0 = B \end{aligned}$$
- *Note* $(\alpha_p)_{p \geq 0}$ *is bounded above by* B *and* $(\beta_p)_{p \geq 0}$ *is bounded below by* $-B$. *Hence, by the completeness axiom,* $\inf (\beta_p)_{p \geq 0}$ *exists and equals the finite number* β; *also* $\sup (\alpha_p)_{p \geq 0}$ *exists and is the finite number* α.

Proof

- *So if we fix p, it should be clear the number* β_p *is an upper bound for all the* α_p *values (look at our inequality chain again and think about this). Thus* β_p *is an upper bound for* $(\alpha_p)_{p \geq 0}$ *and so by definition of a supremum,* $\alpha \leq \beta_p$ *for all p. Of course, we also know since* α *is a supremum, that* $\alpha_p \leq \alpha$*. Thus,* $\alpha_p \leq \alpha \leq \beta_p$ *for all p.*

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- *A similar argument shows if we fix p, the number α_p is an lower bound for all the β_p values and so by definition of an infimum, $\alpha_p \leq \beta \leq \beta_p$ for all the α_p values*

Proof

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- *A similar argument shows if we fix p, the number* α_p *is an lower bound for all the* β_p *values and so by definition of an infimum,* $\alpha_p \leq \beta \leq \beta_p$ *for all the* α_p *values*
- *This tells us* α *and* β *are in* $[\alpha_p, \beta_p] = J_p$ *for all p. Next we show* $\alpha = \beta$*.*

Proof

- *Let* $\epsilon > 0$ *be arbitrary. Since* α *and* β *are in* J_p *whose length is* $\ell_p = (1/2^{p-1})B$*, we have* $|\alpha - \beta| \leq (1/2^{p-1})B$*. Pick* P *so that* $1/(2^{P-1}) < \epsilon$*. Then* $|\alpha - \beta| < \epsilon$*. But* $\epsilon > 0$ *is arbitrary. Hence, by a previous propostion,* $\alpha - \beta = 0$ *implying* $\alpha = \beta$*.*

□

Proof

- *Let* $\epsilon > 0$ *be arbitrary. Since* α *and* β *are in* J_p *whose length is* $\ell_p = (1/2^{p-1})B$*, we have* $|\alpha - \beta| \leq (1/2^{p-1})B$*. Pick* P *so that* $1/(2^{P-1}) < \epsilon$*. Then* $|\alpha - \beta| < \epsilon$*. But* $\epsilon > 0$ *is arbitrary. Hence, by a previous propostion,* $\alpha - \beta = 0$ *implying* $\alpha = \beta$*.*
- *We now must show* $a_{n_k} \to \alpha = \beta$*. This shows we have found a subsequence which converges to* $\alpha = \beta$*. We know* $\alpha_p \leq a_{n_p} \leq \beta_p$ *and* $\alpha_p \leq \alpha \leq \beta_p$ *for all* p*. Pick* $\epsilon > 0$ *arbitrarily. Given any* p*, we have*

$$\begin{aligned}
|a_{n_p} - \alpha| &= |a_{n_p} - \alpha_p + \alpha_p - \alpha|, && \textit{add and subtract trick} \\
&\leq |a_{n_p} - \alpha_p| + |\alpha_p - \alpha| && \textit{triangle inequality} \\
&\leq |\beta_p - \alpha_p| + |\alpha_p - \beta_p| && \textit{definition of length} \\
&= 2|\beta_p - \alpha_p| = 2\,(1/2^{p-1})B.
\end{aligned}$$

Choose P *so that* $(1/2^{P-1})B < \epsilon/2$*. Then,* $p > P$ *implies* $|a_{n_p} - \alpha| < 2\,\epsilon/2 = \epsilon$*. Thus,* $a_{n_k} \to \alpha$*.*

□

Theorem

Bolzano Weierstrass Theorem in $\Re^2$
Every bounded sequence of vectors with an infinite range has at least one convergent subsequence.

Proof

We will just sketch the argument. The sequence of vectors looks like

$$x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix}$$

where each element in the sequence is a two dimensional vector. Since this sequence is bounded, there are positive numbers B_1 and B_2 so that

$$-B_1 \leq x_{1n} \leq B_1 \quad \text{and} \quad -B_2 \leq x_{2n} \leq B_2$$

Proof

The same argument we just used for the Bolzano - Weierstrass Theorem in $\Re$ *works. We find a vector* $[\alpha_1, \alpha_2]'$ *and subsequences* x^1_{1n} *and* x^1_{2n} *with* $x^1_{1n} \rightarrow \alpha_1$ *and* $x^1_{2n} \rightarrow \alpha_2$*. And we can easily see* $[\alpha_1, \alpha_2]'$ *is a vector living in the rectangle* $[-B_1, B_1] \times [-B_2.B_2]$.

Note the argument here is to bisect each side of the rectangle $[-B_1, B_1] \times [-B_2.B_2]$*. This gives 4 new subrectangles and at least one of these pieces must contain IMPs of the original vector sequence. You pick one of these pieces that has IMPs and then bisect that piece on each axis into 4 new pieces, pick a piece that has IMPs and so on.*

Convergence arguments are indeed a bit different as we have to measure distance between vectors using the usual Euclidean norm $||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ *for two vectors* x *and* y*.* $\square$

A little thought shows

Theorem

Bolzano Weierstrass Theorem in $\Re^3$
Every bounded sequence of vectors with an infinite range has at least one convergent subsequence.

Proof

We now bisect each edge of a cube and there are now 8 pieces at each step, at least one of which has IMPs. The vectors are now 3 dimensional but the argument is quite similar. □

A little thought also shows

Theorem

Bolzano Weierstrass Theorem in $\Re^4$
Every bounded sequence of vectors with an infinite range has at least one convergent subsequence.

Proof

We now bisect each edge of what is called a 4 dimensional hypercube and there are now 16 pieces at each step, at least one of which has IMPs. The vectors are now 4 dimensional but the argument is quite similar. □

POMI allows us to extend the result to

Theorem

Bolzano Weierstrass Theorem in $\Re^n$
Every bounded sequence of vectors with an infinite range has at least one convergent subsequence.

Proof

We have done the basis step and in the induction step we assume it is true for $n-1$ and show it is true for n. We now bisect each of the n edges of what is called a n dimensional hypercube and there are now 2^n pieces at each step, at least one of which has IMPs. The vectors are now n dimensional but the argument is again quite similar. $\square$

A more general type of result can also be shown which deals with sets which are bounded and contain infinitely many elements.

Definition

Let S be a nonempty set. We say the real number a is an **accumulation** points of S if given any $r > 0$, the set

$$B_r(a) = \{x : |x - a| < r\}$$

contains at least one point of S different from a. The set $B_r(a)$ is called the **ball** or **circle** centered at a with radius r.

Example

$S = (0, 1)$. Then 0 is an accumulation point of S as the circle $B_r(0)$ always contains points greater than 0 which are in S, Note $B_r(0)$ also contains points less than 0. Note 1 is an accumulation point of S also. Note 0 and 1 are not in S so accumulation points don't have to be in the set. Also note all points in S are accumulation points too. Note the set of all accumulation points of S is the interval $[0, 1]$.

Example

$S = ((1/n)_{n\geq 1}$. Note 0 is an accumulation point of S because every circle $B_r(0)$ contains points of S different from 0. Also, if you pick a particular $1/n$ in S, the distance from $1/n$ to its neighbors is either $1/n - 1/(n+1)$ or $1/n - 1/(n-1)$. If you let r be half the minimum of these two distances, the circle $B_r(1/n)$ does not contain any other points of S. So no point of S is an accumulation point. So the set of accumulation points of S is just one point, $\{0\}$.

Homework 8

8.1 Let $S = (2, 5)$. Show 2 and 5 are accumulation points of S.

8.2 Let $S = (\cos(n\pi/4))_{n\geq 1}$. Show S has no accumulation points.

8.3 This one is a problem you have never seen. So it requires you look at it right! Let (a_n) be a bounded sequence and let (b_n) be a sequence that converges to 0. Then $a_n b_n \to 0$. This is an $\epsilon - N$ proof. Note this is **not** true if (b_n) converges to a nonzero number.

8.4 If you know $(a_n b_n)$ converges does that imply both (a_n) and (b_n) converge?