Series of Real Numbers

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Outline

Given any sequence of real numbers, (a_n) for $n \geq 1$, we can construct from it a new sequence, called the sequence of **Partial Sums** as follows:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \ldots + a_n \end{aligned}$$

This notation works really well when the sequence starts at $n = 1$. If the sequence was

$$(a_n)_{n \geq -3} = \{a_{-3}, a_{-2}, a_{-1}, a_0, a_1, \ldots, a_n, \ldots\}$$

we usually still start with S_1 but we let

$$\begin{aligned} S_1 &= a_{-3} \\ S_2 &= a_{-3} + a_{-2} \\ S_3 &= a_{-3} + a_{-2} + a_{-1} \end{aligned}$$

So when we talk about the first partial sum, S_1, in our minds we are thinking about a sequence whose indexing starts at $n = 1$ but, of course, we know it could apply to a sequence that starts at a different place. A similar problem occurs if the sequence is

$$(a_n)_{n \geq 3} = \{a_3, a_4, \ldots, a_n, \ldots\}$$

Then,

$$\begin{aligned} S_1 &= a_3 \\ S_2 &= a_3 + a_4 \\ S_3 &= a_3 + a_4 + a_5 \end{aligned}$$

So the only time the indexing on the partial sums nicely matches the indexing on the original sequence is when the sequence starts at $n = 1$. This is usually not an issue: we just adjust mentally for the new start point.

For our pedagogical purposes, we will just assume all the sequences in our discussions start at $n = 1$ for convenience!

For our more convenient sequences, we can use summation notation to write the partial sums. We have

$$S_n = \sum_{i=1}^{n} a_i$$

and note the choice of letter i here is immaterial. The use of i could have been changed to the use of the letter j and so forth. It is called a *dummy variable* of summation just like we have *dummy variables* of integration.

$$S_n = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k$$

and so forth. Now we are interested in whether or not the sequence of partial sums of a seqence converges. We need more notation.

Definition

Let $(a_n)_{n \geq 1}$ be any sequence and let $(S_n)_{n \geq 1}$ be its associated sequence of partial sums.
(a) If $\lim_{n \to \infty} S_n$ exists, we denote the value of this limit by S. Since this is the same as $\lim_{n \to \infty} \sum_{i=1}^{n} a_i = S$, we often use the symbol $\sum_{i=1}^{\infty} a_i$ to denote S. But remember it is a **symbol** for a limiting process. Again, note the choice of summation variable i is immaterial.

We also say $\sum_{i=1}^{\infty} a_i$ is the **infinite series** associated with the sequence $(a_n)_{n \geq 1}$.
(b) If the $\lim_{n \to \infty} S_n$ does not exist, we say the **series** $\sum_{i=1}^{\infty} a_i$ **diverges**.

Note the divergence of a series can be several things: we say the series $\sum_{i=1}^{\infty} (a_i = 1)$ **diverges** to ∞, the series $\sum_{i=1}^{\infty} (a_i = -i)$ **diverges** to $-\infty$ and the series $\sum_{i=1}^{\infty} (a_i = (-1)^i)$ **diverges by oscillation**.

Definition

$(a_n)_{n\geq 1}$ be any sequence and let $\sum_{i=1}^{\infty} |a_i|$ be the series we construct from (a_n) by taking the absolute value of each term a_i in the sequence.

(a) if $\sum_{i=1}^{\infty} |a_i|$ **converges** we say the series **converges absolutely**.

(b) (a) if $\sum_{i=1}^{\infty} |a_i|$ **diverges** but $\sum_{i=1}^{\infty} a_i$ **converges**, we say the series **converges conditionally**.

Theorem

The n^{th} term test:
Assume $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. Note this says if a_n does **not** *converge to 0 or fails to converge at all, the original series must* **diverge**.

Proof

Since $\sum_{n=1}^{\infty} a_n$ converges, we know $S_n \to S$ for some S. Thus, given $\epsilon > 0$, there is an N so that

$$n > N \Longrightarrow |S_n - S| < \epsilon/2$$

Now pick any $\hat{n} > N + 1$. Then $\hat{n} - 1$ and $\hat{n}$ are both greater than N. Thus, since $a_{\hat{n}}$ is the difference between two succesive terms in (S_n), we have

Proof

$$\begin{aligned} |a_{\hat{n}}| &= |S_{\hat{n}} - S + S - S_{\hat{n}-1}| \\ &\leq |S_{\hat{n}} - S| + |S - S_{\hat{n}-1}| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Since the choice of $\hat{n} > N$ was arbitrary, we see we have shown $|a_n| < \epsilon$ for $n > N$ with the choice of $\epsilon > 0$ arbitrary. Hence, $\lim a_n = 0$. $\square$

Now if the sequence (S_n) converges to S, we also know the sequence is a Cauchy Sequence. We also know since $\Re$ is **complete** that if (S_n) is a Cauchy Sequence, it must converge. So we can say

$$(S_n) \text{ converges} \iff (S_n) \text{ is a Cauchy Sequence}$$

Now (S_n) is a Cauchy Sequence means given $\epsilon > 0$ there is an N so that

$$n,\ m > N \implies |S_n - S_m| < \epsilon$$

For the moment, assume $n > m$. Then

$$S_n - S_m = \left(\sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \right) - \sum_{i=1}^{m} a_i = \sum_{i=m+1}^{n} a_i$$

and for $m < n$, we would get

$$S_m - S_n = \left(\sum_{i=1}^{n} a_i + \sum_{i=n+1}^{m} a_i \right) - \sum_{i=1}^{n} a_i = \sum_{i=n+1}^{m} a_i$$

We can state this as a Theorem!

Theorem

The Cauchy Criterion For Series:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \left(\forall \epsilon > 0\ \exists\ N \ni \left| \sum_{i=m+1}^{n} a_i \right| < \epsilon \text{ if } n > m > N \right)$$

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges absolutely then it also converges.

Proof

We know $\sum_{n=1}^{\infty} |a_n|$ converges. Let $\epsilon > 0$ be given. By the Cauchy Criterion for series, there is an N so that

$$\big|\, |a_{m+1}| + \ldots + |a_n| \,\big| \quad < \quad \epsilon \text{ if } n > m > N$$

The absolute values here are not necessary so we have

$$|a_{m+1}| + \ldots + |a_n| \quad < \quad \epsilon \text{ if } n > m > N$$

But the triangle inequality then tells us that

$$|a_{m+1} + \ldots + a_m| \leq |a_{m+1}| + \ldots + |a_n| \quad < \quad \epsilon \text{ if } n > m > N$$

Proof

Thus $|\sum_{i=m+1}^{n} a_i| < \epsilon$ when $n > m > N$. The series $\sum_{n=1}^{\infty} a_n$ thus satisfies the Cauchy Criterion for series and we see $\sum_{n=1}^{\infty} a_n$ converges.
□

Another note on notation. The difference between the limit of a series and a given partial sum is $S - S_n$. Using the series notation for S, this gives

$$S - S_n \quad = \quad \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i$$

This is often just written as $\sum_{i=n+1}^{\infty} a_i$ and is called the $(n+1)^{st}$ remainder term for the series. We have to be careful about manipulating this as it is really just a **symbol**.

We will show that a series of the form $\sum_{n=0}^{\infty} a_n t^n$ makes sense. Of course, this series **depends** on **t** which we haven't really discussed. Consider the ODE

$$\begin{aligned} (1+t)x''(t) + t^2 x'(t) + 2x(t) &= 0 \\ x(0) = 1 & \\ x'(0) = 3 & \end{aligned}$$

We can show the solution of this problem can be written as $\sum_{n=0}^{\infty} a_n t^n$ for a unique choice of numbers a_n and we can also show this series will **converge** on an interval $(-R, R)$ for some $R > 0$.

The number R is called the **radius of convergence** of the series $\sum_{n=0}^{\infty} a_n t^n$.

Since we want $x(0) = 1$, we see

$$\left(a_0 + a_1 t + a_2 t^2 + \ldots \right)_{t=0} = 1 \Longrightarrow a_0 = 1$$

We can also show

$$\begin{aligned} \Big(\sum_{n=0}^{\infty} a_n t^n)' &= \sum_{n=1}^{\infty} n\, a_n t^{n-1} \\ \Big(\sum_{n=0}^{\infty} a_n t^n)'' &= \sum_{n=2}^{\infty} n\,(n-1)\, a_n t^{n-2} \end{aligned}$$

and these series converge on $(-R, R)$ also. So we can rewrite the ODE as

$$(1+t)\left(\sum_{n=2}^{\infty} n\,(n-1)\, a_n t^{n-2}\right) + t^2\left(\sum_{n=1}^{\infty} n\, a_n t^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0$$

and the second initial condition becomes

$$\left(a_1 + 2a_2 t + \ldots \right)_{t=0} = 3 \Longrightarrow a_1 = 3$$

We can show we can bring the outside powers of t inside the series. So the ODE becomes

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n t^{n-2} + \sum_{n=2}^{\infty} n\,(n-1)\,a_n t^{n-1} + \sum_{n=1}^{\infty} n\,a_n t^{n+1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

Now we want all powers of t inside the series to have the form t^n. So we make changes to the summation variables in the first three pieces above.

(Piece 1):
Let $k = n - 2$ implying we get

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n t^{n-2} \;=\; \sum_{k=0}^{\infty}(k+2)\,(k+1)\,a_{k+2} t^k$$

(Piece 2):
Let $k = n - 1$ implying we get

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n t^{n-1} \;=\; \sum_{k=1}^{\infty}(k+1)\,(k)\,a_{k+1} t^k$$

(Piece 3):
Let $k = n + 1$ implying we get

$$\sum_{n=1}^{\infty} n\,a_n t^{n+1} \;=\; \sum_{k=2}^{\infty}(k-1)\,a_{k-1} t^k$$

Now the summation variables are all *dummy variables* so it doesn't matter what letter we use. Switch the first two pieces back to n. We get

$$\sum_{n=0}^{\infty}(n+2)\,(n+1)\,a_{n+2} t^n + \sum_{n=1}^{\infty}(n+1)\,(n)\,a_{n+1} t^n + \sum_{n=2}^{\infty}(n-1)\,a_{n-1} t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

We can rewrite this as

$$(2a_2 + 2a_0)t^0 + (6a_3 + 2a_2 + 2a_1)t^1 + \sum_{n=2}^{\infty}\Big((n+2)\,(n+1)\,a_{n+2} + (n+1)\,(n)a_{n+1} + (n-1)a_{n-1} + 2a_n\Big)t^n = 0$$

The coefficients of each power of t must be zero. So we get

$$\begin{aligned} 2a_2 &= -2a_0 = -1 \Longrightarrow a_2 = -1 \\ 6a_3 &= -2a_2 - 2a_1 = 2 - 6 = -4 \Longrightarrow a_3 = -2/3 \end{aligned}$$

and

$$(n+2)(n+1)\, a_{n+2} + (n+1)(n)a_{n+1} + (n-1)a_{n-1} + 2a_n = 0, \quad \forall\, n \geq 2$$

The last equation is called a **recursion** equation for the coefficients.
Solveing for a_{n+1} we have

$$(n+2)(n+1)\, a_{n+2} = -(n+1)(n)a_{n+1} - 2a_n - (n-1)a_{n-1}$$

We can solve this for a_{n+2} to get

$$\begin{aligned} a_{n+2} &= -\frac{(n+1)(n)}{(n+2)(n+1)} a_{n+1} - \frac{1}{(n+2)(n+1)} 2\, a_n - \frac{(n-1)}{(n+2)(n+1)} a_{n-1} \\ &= -\frac{n}{n+2} a_{n+1} - \frac{2}{(n+2)(n+1)} a_n - \frac{n-1}{(n+2)(n+1)} a_{n-1} \end{aligned}$$

So for $n = 2$ we have

$$\begin{aligned} a_4 &= -\frac{2}{4} a_3 - \frac{2}{12} a_2 - \frac{1}{12} a_1 = -\frac{1}{2}\left(-\frac{2}{3}\right) - \frac{2}{12}\left(-1\right) - \frac{1}{12}(3) \\ &= \frac{1}{3} + \frac{1}{6} - \frac{1}{4} = \frac{1}{4} \end{aligned}$$

and so on. What is remarkable is that we can prove a theorem that tells us this series, whose coefficients are defined recursively, converges!

Homework 4

4.1 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, give an $\epsilon - N$ proof that $\sum_{n=1}^{\infty}(2a_n + 5b_n)$ also converges.

4.2 Show $\sum_{n=1}^{\infty}(-1)^n$ does not converge using the n^{th} term test.

4.3 Show $\sum_{n=1}^{\infty}(-1)^n$ has the partial sums $\{-1, 0, -1, 0, \ldots\}$. Find the $\underline{\lim}\ S_n$ and the $\overline{\lim}\ S_n$ and use that to show the series does not converge.