DIMACS Technical Report 2001-18
May 2001

Short containers in Cayley graphs

by

Shuhong Gao[1]
Department of Mathematical Sciences
Clemson University
Clemson, SC 29634 USA
Email: sgao@math.clemson.edu

D. Frank Hsu[2,3]
Department of Computer and Information Science
113 West 60th Street, LL 813
Fordham University
New York, NY 10023 USA
Email: hsu@trill.cis.fordham.edu

[1]Supported in part by NSF under Grant DMS9970637, NSA under Grant MDA904-00-1-0048, ONR under Grant N00014-00-1-0565, and by South Carolina Committee on Higher Education.
[2]DIMACS Center, Rutgers University, 96 Frelinghuysen Road, Piscataway, NJ 08854-8018 USA
[3]Supported in part by the DIMACS NSF grant STC–91–19999 and by NJ Commission

---

ABSTRACT

The star diameter of a graph measures the minimum distance from any source node to several other target nodes in the graph. For a class of Cayley graphs from abelian groups, a good upper bound for their star diameters is given in terms of the usual diameters and the orders of elements in the generating subsets. This bound is tight for several classes of graphs including hypercubes and directed n-dimensional tori. The technique used is the so-called disjoint ordering for a system of subsets, due to Gao, Novick and Qiu (1998).

1 Introduction

A graph models a communication network for a computer system, a parallel computer, or a telephone system. A node of the graph represents a processor or a switch, and an edge corresponds to a link between two processors or switches. In several applications, it is desirable to send messages from one node to several other nodes simultaneously in the network in minimum delay time. This applies in particular to Rabin's information dispersal algorithm (IDA) [18] for efficient and accurate transmission of large files in a parallel computer or a distributed network. This motivates us studying the star diameter of a graph, which measures the minimum delay time in such transmission.

Suppose G is a graph (without self-loops and multiple edges). Let w be a positive integer. For any vertices $x, y_1, \ldots, y_w$ of G with $x \neq y_i$, $1 \leq i \leq w$, a *w-star container* from x to $y_1, \ldots, y_w$ is a collection of w (internally) node-disjoint paths from x to $y_1, \ldots, y_w$, one for each y_i. Here the vertices $y_1, \ldots, y_w$ may have repetition, thus if y_1 appears r times then the container has r disjoint paths from x to y_1. In the case that $y_1 = \cdots = y_w = y$, a w-star container is also called a w-wide container from x to y. The length of a container is the maximum length of its paths. The *w-star distance* from x to $y_1, \ldots, y_w$, denoted by $d(x; y_1, \ldots, y_w)$, is the minimum length among all the w-star containers from x to $y_1, \ldots, y_w$. When $y_1 = \cdots = y_w = y$, $d(x; y_1, \ldots, y_w)$ is simply denoted as $d_w(x, y)$. Following [10], the w-wide diameter of G, denoted by $d_w(G)$, is defined to be the maximum of $d_w(x, y)$ for all pairs of distinct vertices x and y in G. The *w-star diameter* of G, denoted by $D_w(G)$, is defined to be the maximum of $d(x; y_1, \ldots, y_w)$ for all vertices $x, y_1, \ldots, y_w$ (possibly with repetition) of G with $x \neq y_i$, $1 \leq i \leq w$. Certainly, $d_w(G) \leq D_w(G)$.

Note that $D_1(G)$ is just the usual diameter of G. Obviously, $D_1(G) \leq D_2(G) \leq \cdots \leq D_w(G) \leq \cdots$. Suppose that G has connectivity k. Then Menger's theorem implies that $D_w(G) < \infty$ iff $w \leq k$. A natural question is to quantize Menger's theorem, that is, to give a good bound on $D_k(G)$.

The above definition of w-star diameter is slightly different from that in the literature [10] where it is required that the target nodes be distinct. The benefit of our definition is that the w-star diameter bounds both the star diameter in [10] and the wide diameter $d_w(G)$, thus allows a uniform treatment for these two parameters. For containers and wide diameters, see [3, 4, 5, 11, 12, 15, 16, 17, 19, 20, 21]. In general, it seems more difficult to determine star diameters than wide diameters due to the possibly complicated configuration of the target nodes.

In this paper, we study a class of Cayley graphs that are defined over abelian groups. We give a good upper bound for their star diameters in terms of the usual diameters and the orders of the elements in the generating subsets. This bound is tight for several classes of graphs including hypercubes and directed n-dimensional tori.

The concept of star diameter applies to both directed and undirected graphs. We view undirected graphs as special cases of directed graphs where each undirected edge is just two directed edges with one in each direction.

The rest of the paper is organized as follows. In the next section, we define Cayley graphs and state our main results. In Section 3, we present the concept of disjoint ordering for a system of finite sets and the related results from Gao et al [7], which will be useful for construction of short disjoint paths later. Section 4 is the technical part of the paper where we show how to construct short containers in Cayley graphs from abelian groups via disjoint ordering of sets and thus proves our main results. We conclude in Section 5 with some comments and open problems for future studies.

2 Main Results

Let G be any group with its binary operation written multiplicatively, and let S be a subset of G not containing the identity element 1. The Cayley graph $\Gamma(G, S)$ is defined to be the (directed) graph whose vertices are the elements of G and, for $x, y \in G$, there is an edge $x \to y$ iff $x \cdot g = y$ for some $g \in S$. When S contains the inverses of all its elements, the Cayley graph $\Gamma(G, S)$ is an undirected graph.

For example, the n-dimensional hypercube H_n has a vertex set $\mathbb{Z}_2^n = \{(a_1, \ldots, a_n) : a_i = 0 \text{ or } 1\}$ and two vertices are adjacent if and only if they differ by exactly one coordinate. This is an undirected graph and can be viewed as a Cayley graph as follows. We know that $G = \mathbb{Z}_2^n$ is a group under componentwise addition modulo 2. Take S to be the set of unit vectors $(0, \ldots, 1, \ldots, 0)$ where the i-th component is 1 and zero elsewhere, $1 \le i \le n$. Then the Cayley graph $\Gamma(G, S)$ is precisely the hypercube H_n.

An n-dimensional torus is a generalized hypercube. For a positive integer m, $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ denotes the ring of integers modulo m, a cyclic group of order m under addition. Let $m_1, \ldots, m_n$ be integers ≥ 2. Define

$$H(m_1, \cdots, m_n) = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n},$$

the set of all n-tuples $(a_1, \ldots, a_n)$ with $a_i \in \mathbb{Z}_{m_i}$ for $1 \le i \le n$. Note that $G = H(m_1, \cdots, m_n)$ is a group under componentwise addition. Let S be the set of unit vectors $(0, \ldots, 1, \ldots, 0)$ where the i-th component is 1 and zero elsewhere, $1 \le i \le n$. Then the Cayley graph $\Gamma(G, S)$ is called a directed n-dimensional torus. Let $S_1 = S \cup \{-S\}$. Then $\Gamma(G, S_1)$ is the undirected version of $\Gamma(G, S)$ and is simply called an n-dimensional torus. Note that an n-dimensional torus is also called a generalized hypercube or a toroidal mesh in the literature. When $m_1 = \cdots = m_n = k$, it is also called a k-ary n-cube.

The groups used in hypercube and torus graphs above are abelian. There is a large literature on Cayley graphs from other groups, see [1, 2, 8, 13, 14, 15, 19, 21] for more information. In this paper, we shall focus mainly on Cayley graphs over abelian groups.

Let G be any finite group, written multiplicatively. An ordered subset $B = \{b_1, \ldots, b_n\}$ is called *a generating basis*, or simply *a basis*, of G if each element $g \in G$ can be written as a unique product

$$g = b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}, \quad 0 \le \ell_i < e_i, 1 \le i \le n,$$

where e_i is the order of b_i (that is, e_i is the smallest positive integer such that $b_i^{e_i} = 1$). By the uniqueness, we mean that if $h = b_1^{\bar{\ell}_1} b_2^{\bar{\ell}_2} \cdots b_n^{\bar{\ell}_n}$ then $g = h$ implies that $\bar{\ell}_i \equiv \ell_i \mod e_i$ for

$1 \leq i \leq n$. If such a basis exists then G has exactly $e_1 e_2 \cdots e_n$ elements.

For example, the unit vectors form a generating basis for $\mathbb{Z}_2^n$. For another example, consider the additive group of $\mathbb{Z}_{30}$. Then the subset $\{1\}$ is a generating basis for $\mathbb{Z}_{30}$, as 1 has additive order 30 in $\mathbb{Z}_{30}$. Also, the subsets $\{4, 15\}$, $\{6, 10, 15\}$ and $\{12, 15, 20\}$ are generating bases of $\mathbb{Z}_{30}$ for its additive group. In additive notation, $\{4, 15\}$ being a basis means that each element in $\mathbb{Z}_{30}$ is of the form $15a + 4b$ where $0 \leq a < 2$ and $0 \leq b < 15$. This is due to the fact that $\mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_{15}$ by the Chinese remainder theorem and that $\{15\}$ and $\{4\}$ are bases for $\mathbb{Z}_2$ and $\mathbb{Z}_{15}$, respectively. Similarly for the other two sets, as $\mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

Theorem 2.1 *Let G be an abelian group and S is a subset of G not containing the identity. Suppose $B \subseteq S \subseteq B \cup B^{-1}$ for some generating basis B of G. Denote by k the cardinality of S and e the maximum order of elements in $S \cap B^{-1}$ (and $e = 1$ when $S \cap B^{-1}$ is empty). Then the Cayley graph $\Gamma(G, S)$ has connectivity k and has k-star diameter*

$$D_k\left(\Gamma(G, S)\right) \leq \begin{cases} d + 1, & \text{if } e \leq 2 \\ d + \lfloor (e-1)/2 \rfloor, & \text{if } e > 2. \end{cases}$$

where d is the usual diameter of $\Gamma(G, S)$.

Suppose the basis B has k elements. In the case that all elements in B have order 2, the graph $\Gamma(G, B)$ is the k-dimensional hypercube and has diameter $d = k$. In this case, the upper bound is tight as the k-star diameter is known to be $k + 1$. If all elements in B have order larger than 2 then $\Gamma(G, B)$ is a directed n-dimensional torus. We will show that the star diameter is $d + 1$, so the bound is again tight.

Corollary 2.2 (Directed n-dimensional Torus) *Let G be an abelian group with a generating basis B of n elements. Then the (directed) Cayley graph $\Gamma(G, B)$ has connectivity n and*

$$D_n\left(\Gamma(G, B)\right) = d + 1$$

where d is the diameter of $\Gamma(G, B)$.

On the other extreme, if $S = B \cup B^{-1}$ then $\Gamma(G, S)$ is undirected.

Corollary 2.3 (Undirected n-dimensional Torus) *Let G be an abelian group with a basis B with n elements and $S = B \cup B^{-1}$. Let e be the maximum order of elements in B. Suppose each element in B has order > 2 (so $e > 2$). Then the Cayley graph $\Gamma(G, S)$ has connectivity $2n$ and*

$$D_{2n}\left(\Gamma(G, S)\right) \leq d + \lfloor (e-1)/2 \rfloor$$

where d is the diameter of $\Gamma(G, S)$.

3 Disjoint ordering

The concept of disjoint ordering for a collection of subsets is introduced by Gao, Novick and Qiu [7]. We give the definition and the related results below.

A permutation of the elements of a finite set is called an *ordering*. Suppose X and Y are two sets ordered as $O_1 = (x_1, x_2, \cdots, x_k)$ and $O_2 = (y_1, y_2, \cdots, y_\ell)$ where $k = |X|$ and $\ell = |Y|$. We say that O_1 and O_2 are *disjoint* if for every $1 \leq t \leq \min(k, \ell)$

$$\{x_1, x_2, \cdots, x_t\} \neq \{y_1, y_2, \cdots, y_t\}$$

as sets, unless $t = k = \ell$. Note that X and Y may be the same set which is why we need to exclude the case $t = k = \ell$. For instance, if $X = Y = \{1, 2, 3\}$ then $(1, 2, 3)$ and $(2, 3, 1)$ are disjoint but $(1, 2, 3)$ and $(2, 1, 3)$ are not. Also, if $X = Y = \{1\}$ then the trivial ordering (1) is disjoint to itself.

A collection of finite sets is said to have a *disjoint ordering* if each set has an ordering and all the orderings are pairwise disjoint. In particular, as long as all singletons in the collection are distinct, the elements in the first position of a disjoint ordering form a system of distinct representatives. So for a disjoint ordering to exist, the conditions in Hall's matching theorem [9] must be satisfied. The converse is also true.

Theorem 3.1 (Gao et al 1998) *For any finite collection of nonempty finite sets in which all singletons are distinct, there is a disjoint ordering if and only if there is a system of distinctive representatives.*

Recall that a system of distinctive representatives (SDR) for k sets consists of k distinct elements with one from each set. A partial SDR is an SDR for a subcollection of the sets. When an SDR does not exist, one needs to add elements to the sets so that SDR and thus disjoint ordering exists. By using this technique, Gao et al [7] show how to construct short containers on hypercube graphs. In the next section, we adapt this method to Cayley graphs over abelian groups.

We shall need the following lemmas.

Lemma 3.2 *Suppose $X_1, \ldots X_w$ are subsets of a finite set S where $w \leq k = |S|$. Let $t_i \in X_i$, $1 \leq i \leq m$, be a partial SDR of maximum size. Pick any distinct elements $t_i \in S \setminus \{t_1, \ldots, t_m\}$, $m < i \leq w$. Then for any disjoint ordering of the system*

$$X_1, \ldots, X_m, X_{m+1} \cup \{t_{m+1}\}, \ldots, X_w \cup \{t_w\} \tag{1}$$

the element t_i must be the initial element in the ordering of X_i for all $m < i \leq w$.

Proof. Suppose for some $i > m$ the initial element a in the ordering of $X_i \cup \{t_i\}$ is different from t_i. Then $a \in X_i$. Note that the initial elements of the ordering form an SDR for the system (1). Particularly, $X_1, \ldots, X_m$ have representatives different from a. This means that the sets $X_1, \ldots, X_m, X_i$ have an SDR, contradicting to the maximality of m. □

Lemma 3.3 *Let $S = \{g_1, \dots, g_k\}$ be any finite set and $X_i \subseteq S$, $1 \le i \le w$. For each pair $1 \le i \le w$ and $1 \le j \le k$, there is associated with a real number e_{ij}. Suppose the system $X_1, \dots, X_w$ has an SDR. Then there is a disjoint ordering for the system satisfying the following condition:*

> *Let $g_{\sigma(i)}$ be the last element in the ordering of X_i, $1 \le i \le w$. For any pair $1 \le i < j \le w$ with $X_i = X_j$, if $e_{i\sigma(i)} \ge e_{j\sigma(i)}$ and $e_{j\sigma(j)} \ge e_{i\sigma(j)}$ then $e_{i\sigma(i)} = e_{j\sigma(i)}$ and $e_{j\sigma(j)} = e_{i\sigma(j)}$.*

Proof. By Theorem 3.1, the system X_i, $1 \le i \le w$, has a disjoint ordering, say O_i for the ordering of X_i, $1 \le i \le w$. We show how to rearrange the ordering so that the condition in the lemma is satisfied. Suppose it is violated by some pair i_0 and j_0 with $X_{i_0} = X_{j_0}$. We consider all the sets X_i's that are equal to X_{i_0}. For convenience of notation, we may assume that they are $X_1, \dots, X_m$ for some $1 < m \le w$. So $X_1 = \cdots = X_m \ne X_j$ for $m < j \le w$. Let g_{u_i} be the last element in the ordering O_i where $1 \le u_i \le k$ and $1 \le i \le m$. Take any bijection

$$\eta : \{1, \dots, m\} \to \{u_1, \dots, u_m\},$$

the latter is viewed as a multiset, that minimizes (among all the bijections) the sum $\sum_{i=1}^{m} e_{i\eta(i)}$. We claim that, for any pair $1 \le i < j \le m$, if

$$e_{i\eta(i)} \ge e_{j\eta(i)} \quad \text{and} \quad e_{j\eta(j)} \ge e_{i\eta(j)}$$

then $e_{i\eta(i)} = e_{j\eta(i)}$ and $e_{j\eta(j)} = e_{i\eta(j)}$. Suppose otherwise, namely, one of the inequalities is strict. Then

$$e_{i\eta(i)} + e_{j\eta(j)} > e_{j\eta(i)} + e_{i\eta(j)}.$$

Switching the values $\eta(i)$ and $\eta(j)$ of η would yield a bijection with a smaller sum, contradicting to the choice of η.

Now we rearrange the orderings $O_1, \dots, O_m$ as follows. Suppose $\eta(i) = u_{\tau(i)}$, for $1 \le i \le m$, where $\tau(1), \dots, \tau(m)$ is a permutation of $1, \dots, m$. This means that $\eta(i)$ is the last element in the ordering $O_{\tau(i)}$ of $X_{\tau(i)}$. To get the desired new ordering of the system, let $O_{\tau(i)}$ be the new ordering of X_i for $1 \le i \le m$, with the orderings of other sets X_i, $i > m$, unchanged. Then the condition in the lemma is satisfied for all pairs $1 \le i < j \le m$. Certainly, the new ordering for the system X_i, $1 \le i \le w$, is still disjoint and no new violating pairs are introduced. Repeat this process if the condition in the lemma is violated by any other pair among $X_{m+1}, \dots, X_k$. The condition is satisfied after finitely many steps. □

4 Short containers

Let G be a group and S a subset of it not containing the identity 1. Suppose S generates G as a group. Then the Cayley graph $\Gamma(G, S)$ is connected and the left multiplication by any element of G induces an automorphism of $\Gamma(G, S)$. Hence $\Gamma(G, S)$ is vertex symmetric. This implies in particular that, for any two vertices x and y, the set of all the paths from x to y in $\Gamma(G,S)$ is in 1-1 correspondence to that from 1 to $x^{-1}y$ with length preserved. Similarly, for any $y_1,\dots,y_w$, the star containers from x to $y_1,\dots,y_w$ are in 1-1 correspondence to those from 1 to $x^{-1}y_1,\dots,x^{-1}y_w$ with length preserved. Because of this correspondence, we discuss below how to construct short w-star containers that start at 1 only.

Let $y \in G$. Suppose y is represented as

$$y = g_1 g_2 \cdots g_\ell, \quad g_i \in S.$$

Then there is a natural induced path from 1 to y:

$$1 \bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \cdots \bullet \xrightarrow{g_\ell} \bullet\ y.$$

Note that the number ℓ of elements in y is equal to the length of the induced path. We call ℓ the length of y, denoted by $|y|$. Let $y_1 = g_1 g_2 \cdots g_\ell$ and $y_2 = h_1 h_2 \cdots h_k$ be two representations where $g_i, h_j \in S$. We say that y_1 and y_2 are *disjoint* if their induced paths are disjoint, namely,

$$g_1 \cdots g_i \neq h_1 \cdots h_j$$

as elements of G, for all $1 \le i \le \ell$ and $1 \le j \le k$, except when $i = \ell$ and $j = k$. The exception allows y_1 and y_2 being the same vertex of $\Gamma(G,S)$.

When G is abelian, one can change the order of the elements in y in any fashion, and y is still the same element of G (thus the same node of $\Gamma(G,S)$ but the induced path will likely be different. It is exactly this flexibility of reordering that allows us to construct short w-containers in $\Gamma(G,S)$. In the following, we view a representation (i.e. a product) of y as ordered and identify it with its induced path from 1 to y. It should be clear from the context whether y is viewed as an element of G (thus a node of $\Gamma(G,S)$) or a path from 1 to y.

We assume from now on that G is abelian and $B \subseteq S \subseteq B \cup B^{-1}$ for some basis B of G. For convenience of discussion, we fix that

$$B = \{b_1,\dots,b_r\} \quad \text{and} \quad S = \{b_1, b_1^{-1},\dots,b_s,b_s^{-1},b_{s+1},\dots,b_r\} \tag{2}$$

where $b_i \neq b_i^{-1}$ for $1 \le i \le s$, and for $s < i \le r$, either $b_i = b_i^{-1}$ or $b_i^{-1} \notin S$. Denote by e_i the order of b_i for $1 \le i \le r$.

Since B is a basis of G, any $y \in G$ can be written uniquely as $y = b_1^{\ell_1} \cdots b_r^{\ell_r}$ where $0 \le \ell_i < e_i$ for $1 \le i \le r$. When $b_i^{-1} \in S$, we may replace $b_i^{\ell_i}$ by $b_i^{-(e_i-\ell_i)}$, which yields a shorter path if $e_i - \ell_i < \ell_i$. So y is better written in the form

$$y = b_1^{\ell_1} \cdots b_r^{\ell_r} \tag{3}$$

where

$$-\frac{e_i}{2} < \ell_i \le \frac{e_i}{2}, \qquad \text{if } 1 \le i \le s \tag{4}$$

$$0 \le \ell_i < e_i, \qquad \text{if } s < i \le r. \tag{5}$$

It is straightforward to check that this representation of y is unique, that is, different values of the ℓ_i's in (4) and (5) give different y's in (3) as elements of G.

Lemma 4.1 *Suppose that y is written in the form (3)–(5). Then the distance from* 1 *to y in $\Gamma(G,S)$ is $d(1,y)=\sum_{i=1}^{r}|\ell_i|$.*

Proof. Certainly, the induced path of y has length $\sum_{i=1}^{r}|\ell_i|$. Suppose that P is any path from 1 to y in $\Gamma(G,S)$. We need to show that $|P|\geq\sum_{i=1}^{r}|\ell_i|$. The path P corresponds to writing y as a product of elements in S. Since G is abelian, we may reordering the elements in the product and write y in the following form

$$\begin{aligned} y &= b_1^{u_1}(b_1^{-1})^{v_1}\cdots b_s^{u_s}(b_s^{-1})^{v_s}b_{s+1}^{u_{s+1}}\cdots b_r^{u_r} \\ &= b_1^{u_1-v_1}\cdots b_s^{u_s-v_s}b_{s+1}^{u_{s+1}}\cdots b_r^{u_r} \end{aligned}$$

where u_i and v_i are nonnegative integers counting for the numbers of times of b_i and b_i^{-1} used in forming the edges of P. Reducing the exponents of b_i modulo e_i appropriately, we can write y as

$$y=b_1^{\bar{\ell}_1}\cdots b_r^{\bar{\ell}_r}$$

where $\bar{\ell}_i$ satisfy (4) and (5). The length $\sum_{i=1}^{r}|\bar{\ell}_i|$ is never larger than $|P|=u_1+v_1+\cdots+u_s+v_s+u_{s+1}+\cdots+u_r$. By the uniqueness of the representation of y in (3)–(5), we have that $\bar{\ell}_i=\ell_i$ for $1\leq i\leq r$. Therefore $|P|\geq\sum_{i=1}^{r}|\ell_i|$ as desired. □.

Corollary 4.2 *Let S be as in (2). The diameter of $\Gamma(G,S)$ is*

$$d=\sum_{i=1}^{s}\lfloor e_i/2\rfloor+\sum_{i=s+1}^{r}(e_i-1).$$

Proof. Since G is vertex symmetric, we just need to compare $d(1,y)$ for $y\in G$. The corollary follows from Lemma 4.1. □

A representation $y=\prod_{i=1}^{t}g_i^{\ell_i}$, where $g_i\in S$ and $\ell_i\geq 0$, is called *minimal* if $\sum_{i=1}^{t}\ell_i$ is equal to the distance from 1 to y in $\Gamma(G,S)$. By Lemma 4.1, the representation of y in (3)–(5) is a minimal representation by rewriting $b_i^{\ell_i}=(b_i^{-1})^{-\ell_i}$ if $\ell_i<0$. Thus we also call (3)–(5) a minimal representation of y. Note that minimal representation may not be unique. For instance, if b_1 has order 2ℓ for some $\ell>1$ and if $b_1,b_1^{-1}\in S$ then $b_1^{\ell}=(b_1^{-1})^{\ell}$ are both minimal but $b_1\neq b_1^{-1}$. In any case, a representation $y=\prod_{i=}^{t}g_i^{\ell_i}$, where $g_i\in S$ and $\ell_i\geq 0$, is minimal iff the following two conditions are satisfied:

(a) $0\leq\ell_i\leq\bar{e}_i-1$ where $\bar{e}_i$ denotes the order of g_i, $1\leq i\leq t$; and

(b) $g_1,\ldots,g_t$ are distinct elements in S, and if both g_i and g_i^{-1} are in S then $\ell_i\leq\bar{e}_i/2$ and only one of g_i,g_i^{-1} appears in the list $g_1,\ldots,g_t$.

A minimal representation $y=\prod_{i=1}^{t}g_i^{\ell_i}$ is called *canonical* with respect to the basis B if the following condition is satisfied:

(c) if $g_i\notin B$ then $\ell_i<\bar{e}_i/2$, $1\leq i\leq t$.

The minimal representation y in (3)–(5) is certainly canonical. By the proof of Lemma 4.1, any canonical minimal representation can be obtained from (3) by permuting the elements b_i's. So canonical minimal representation is unique up to order.

We next define the supports of elements in G. For any element $y \in G$, write y in a canonical minimal representation $y = \prod_{i=}^{t} g_i^{\ell_i}$ where $g_i \in S$ and $\ell_i \geq 0$. The *support* of y is defined to be

$$\mathrm{Supp}(y) = \{g_i : \ell_i > 0, 1 \leq i \leq t\}.$$

which is a subset of S. For example, assuming that b_1 has order 5 and $b_1, b_1^{-1} \in S$, we have $\mathrm{Supp}(b_1^2) = \{b_1\}$ but $\mathrm{Supp}(b_1^3) = \{b_1^{-1}\}$, as $b_1^3 = (b_1^{-1})^2$. Also, $\mathrm{Supp}(b_2 b_1 b_2^{-1}) = \mathrm{Supp}(b_1)$. Certainly, if y is of the form (3)–(5) then

$$\mathrm{Supp}(y) = \{b_i : \ell_i > 0\} \cup \{b_i^{-1} : \ell_i < 0\}.$$

Lemma 4.3 *Let $x = g_1^{u_1} \cdots g_s^{u_s}$ and $y = h_1^{v_1} \cdots h_t^{v_t}$ be two canonical minimal representations with Supp$(x) = \{g_1, \dots, g_s\}$ and Supp$(y) = \{h_1, \dots, h_t\}$. Suppose that $g_1 \neq h_1$ and the ordering $(g_1, \dots, g_s)$ of Supp(x) is disjoint from the ordering $(h_1, \dots, h_t)$ of Supp(y). Then the induced paths of x and y are internally node disjoint whenever the condition in Lemma 3.3 is satisfied, namely, if Supp$(x) =$ Supp(y), supposing that $g_s = h_m$ for some $m < s$ and $h_t = g_n$ for some $n < t$, then $u_s \geq v_m$ and $v_t \geq u_n$ imply that $u_s = v_m$ and $v_t = u_n$.*

Proof. A node, other than 1, on the induced path of x is of the form

$$x_1 = g_1^{u_1} \cdots g_{i-1}^{u_{i-1}} g_i^u \quad \text{for some } 1 \leq i \leq s \text{ and } 1 \leq u \leq u_i.$$

Similarly a node, other than 1, on the induced path of y is of the form

$$y_1 = h_1^{v_1} \cdots h_{j-1}^{v_{j-1}} g_j^v \quad \text{for some } 1 \leq j \leq t \text{ and } 1 \leq v \leq v_j.$$

Then x_1 and y_1 are both canonical minimal representation with

$$\mathrm{Supp}(x_1) = \{g_1, \dots, g_i\} \text{ and } \mathrm{Supp}(y_1) = \{h_1, \dots, h_j\}.$$

Suppose that $x_1 = y_1$. Since canonical minimal representation is unique up to order, we have $\mathrm{Supp}(x_1) = \mathrm{Supp}(y_1)$ and the exponents of the g's and h's must be equal accordingly. Hence $\{g_1, \dots, g_i\} = \{h_1, \dots, h_j\}$ and thus $i = j$. But $(g_1, \dots, g_s)$ is disjoint from $(h_1, \dots, h_t)$, it follows that $i = s$ and $j = t$. So $i = j = s = t$. Since $g_1 \neq h_1$, we have $s = t > 1$. But $\{g_1, \dots, g_{t-1}\} \neq \{h_1, \dots, h_{t-1}\}$, we see that $g_t \neq h_t$. Thus $g_s = h_m$ for some $m < t$ and $h_t = g_n$ for some $n < t$. Comparing their exponents g_s and h_t in x_1 and y_1, we have $v_m = u \leq u_s$ and $u_n = v \leq v_t$. If x_1 or y_1 is an internal node, then one of the inequalities is strict. This is impossible by the condition (ii). Therefore, the induced paths of x and y have no common internal node. □

We define a partial ordering on the elements of G, which is needed in the proof of the next theorem. Let $y_1, y_2 \in G$. Represent them in canonical minimal form, say

$$y_1 = g_1^{u_1} \cdots g_t^{u_t}, \quad y_2 = g_1^{v_1} \cdots g_t^{v_t}$$

where $g_i \in S$, $u_i \geq 0$ and $v_i \geq 0$. We say that $y_1 \prec y_2$ if $u_i \leq v_i$ for $1 \leq i \leq t$. We note that if $y_1 \prec y_2$ and $y_1 \neq y_2$ then $|y_1| < |y_2|$.

Theorem 4.4 *Let B and S as in (2) where B is a generating basis of G. Let $x, y_1, \ldots, y_w$ be any vertices of $\Gamma(G,S)$ with $x \neq y_i$, $1 \leq i \leq w$. Suppose that d_i is the distance from x to y_i, $1 \leq i \leq w$. Then there is a container from x to $y_1, \ldots, y_w$ with the path from x to y_i having length at most $d_i + \bar{e}$ where $\bar{e} = \max\{e_1, \ldots, e_r\}$.*

Proof. Since $\Gamma(G,S)$ is vertex symmetric, we may assume that $x = 1$, the identity of G. Write y_i in the form (3)–(5):

$$y_i = b_1^{e_{i1}} b_2^{e_{i2}} \cdots b_r^{e_{ir}}, \quad 1 \leq i \leq w.$$

Then, by Lemma 4.1, $d_i = |y_i| = \sum_{j=1}^{r} |e_{ij}|$. Let $X_i = \mathrm{Supp}(y_i)$. The system of subsets X_i, $1 \leq i \leq w$, has a partial SDR of maximum size, say m. Without loss of generality, we may assume that $t_1 \in X_1, \ldots, t_m \in X_m$ is such a maximum partial SDR. We may assume that the following is satisfied:

(A) There is no $j > m$ and $i \leq m$ such that $y_j \prec y_i$ with $y_j \neq y_i$ and the system

$$X_1, \ldots, X_{i-1}, X_j, X_{i+1}, \ldots, X_m$$

has an SDR of size m.

If this condition is not satisfied, we can replace X_i by X_j and we still have a maximal SDR for the original system. Repeat this process until there is no such j. The process has to stop as the total size of the y_i's where X_i have representatives decreases by at least one with each replacement.

Let $S_0 = S \setminus \{t_1, \ldots, t_m\}$. Since $t_i \in X_i$, $1 \leq i \leq m$, form a maximal partial SDR, we have

$$S_0 \cap X_j = \emptyset, \quad m < j \leq w. \tag{6}$$

We want to add the elements in S_0 to X_j one in each for $m < j \leq w$. Since complication arises when $S_0^{-1} \cap X_j \neq \emptyset$, we need to be careful. Here $S_0^{-1} = \{t^{-1} : t \in S_0\}$. Define

$$Z_j = S_0^{-1} \cap X_j, \quad m < j \leq w.$$

If there are empty sets among them, just discard them. Among all the maximal partial SDR's for the system Z_j, $m < j \leq w$, we take one that maximizes the total sum of the lengths of the y_j's where Z_j have representatives. For convenience of notation, we assume that

$$t_\ell^{-1} \in Z_\ell \subseteq X_\ell, \quad m_0 < \ell \leq w$$

is such a maximal SDR where $m_0 \geq m$. We claim that the following condition is satisfied:

(B) There is no pair j and ℓ with $m < j \leq m_0$ and $m_0 < \ell \leq w$ such that

$$t_\ell^{-1} \in X_j \quad \text{but} \quad y_\ell \prec y_j, \quad y_\ell \neq y_j.$$

If (B) is not satisfied for some j, ℓ, we can always let t_ℓ^{-1} to represent Z_j instead of Z_ℓ. Then the total length of the y_ℓ's with representatives increases by at least one, contradicting to the choice of the t_ℓ's.

Furthermore, we show that the representatives for Z_ℓ's can be chosen so that the following condition is satisfied:

(C) For any pair $m_0 < i < j \le w$ with

$$\{t_i^{-1}, t_j^{-1}\} \subseteq X_i \cap X_j,$$

let u_i, u_j, v_i, v_j be the exponents of t_i^{-1}, t_j^{-1} in the expression of y_i and y_j, namely,

$$y_i = \cdots (t_i^{-1})^{u_i}(t_j^{-1})^{v_i}, \quad y_j = \cdots (t_i^{-1})^{u_j}(t_j^{-1})^{v_j}.$$

Then $u_i \le u_j$ and $v_j \le v_i$ imply that $u_i = u_j$ and $v_i = v_j$.

When (C) is not satisfied, we can switch the representatives so that t_j^{-1} represents Z_i and t_i^{-1} represents Z_j. The total sum of the exponents of the representatives increases by at least one. Repeat this process if necessary. Then (C) must be satisfied by the resulted SDR.

Hence we have $t_\ell \in S_0$ with $t_\ell^{-1} \in X_\ell$, $m_0 < \ell \le w$. By the maximality of the SDR for the system Z_j's, we have

$$Z_j \subseteq \{t_{m_0+1}^{-1}, \ldots, t_w^{-1}\}, \quad m < j \le m_0.$$

Thus

$$\text{for every } t \in S_0 \setminus \{t_{m_0+1}, \ldots, t_w\}, \quad t^{-1} \notin X_j, \text{ for all } m < j \le m_0. \tag{7}$$

Finally, pick distinct $t_j \in S_0 \setminus \{t_{m_0+1}, \ldots, t_w\}$, $m < j \le m_0$. By (6) and (7), we have w distinct elements $t_i \in S$, $1 \le i \le w$, satisfying the following:

$$
\begin{aligned}
&t_i \in X_i, && \text{if } 1 \le i \le m && (8)\\
&t_i \notin X_j, && \text{if } m < i, j \le w && (9)\\
&t_i^{-1} \notin X_j, && \text{if } m < i, j \le m_0 && (10)\\
&t_i^{-1} \in X_i, && \text{if } m_0 < i \le w. && (11)
\end{aligned}
$$

Also, the conditions (A), (B), and (C) are satisfied.

Now we are ready to construct the container required by the theorem. Suppose that

$$y_i = \tilde{y}_i (t_i^{-1})^{u_i}, \quad m_0 < i \le w \tag{12}$$

where $\tilde{y}_i$ is in canonical minimal form and does not contains any power of t_i. Also, let e_i be the order of t_i for $1 \le i \le w$. We modify the expressions of y_i's as follows. Define

$$
\begin{aligned}
&\bar{y}_i = y_i && \epsilon_i = 1 && \text{if } 1 \le i \le m && (13)\\
&\bar{y}_i = t_i y_i && \epsilon_i = t_i^{-1} && \text{if } m < i \le m_0 \text{ and } t_i^{-1} \in S && (14)\\
&\bar{y}_i = t_i y_i && \epsilon_i = t_i^{e_i - 1} && \text{if } m < i \le m_0 \text{ and } t_i^{-1} \notin S && (15)\\
&\bar{y}_i = t_i \tilde{y}_i && \epsilon_i = t_i^{e_i - u_i - 1} && \text{if } m_0 < i \le w. && (16)
\end{aligned}
$$

Certainly, the $\bar{y}_i$'s are in canonical minimal form and

$$y_i = \bar{y}_i \epsilon_i, \quad 1 \le i \le w.$$

Let $\mathrm{Supp}(\bar{y}_i) = \bar{X}_i$, $1 \le i \le w$. Then

$$\begin{array}{ll} \bar{X}_i = X_i & \text{if } 1 \le i \le m \\ \bar{X}_i = X_i \cup \{t_i\} & \text{if } m < i \le m_0 \\ \bar{X}_i = (X_i \setminus \{t_i^{-1}\}) \cup \{t_i\} & \text{if } m_0 < i \le w. \end{array}$$

Note that $t_1, \ldots, t_w$ form an SDR for the system $\bar{X}_1, \ldots, \bar{X}_w$ and each element in $\bar{X}_i$ has a postive exponent in $\bar{y}_i$, $1 \le i \le w$. By Theorem 3.1, there is a disjoint ordering and the disjoint ordering can be chosen so that the exponents of the last elements in the ordering satisfy the condition in Lemma 3.3.

We rewrite the product $\bar{y}_i$ according to the ordering of $\bar{X}_i$, $1 \le i \le w$. For instance, if $\bar{y}_i = b_1^{\ell_1} b_2^{\ell_2} b_3^{\ell_3}$ and $\bar{X}_i = \{b_1, b_2, b_3\}$ is ordered as (b_2, b_3, b_1) then $\bar{y}_i$ is rewritten as $b_2^{\ell_2} b_3^{\ell_3} b_1^{\ell_1}$. By Lemma 4.3, the resulted representations of $\bar{y}_i$, $1 \le i \le w$, are pairwise disjoint, so the induced paths are pairwise disjoint. For convenience of notation, the new $\bar{y}_i$ is still denoted by $\bar{y}_i$, $1 \le i \le w$. By appending ϵ_i to $\bar{y}_i$, we have a path $P_i = \bar{y}_i \epsilon_i$ from 1 to y_i, $1 \le i \le w$. Obviously, the length of P_i is

$$|\bar{y}_i| + |\epsilon_i| \le d_i + \bar{e}$$

for $1 \le i \le w$.

It remains to show that the paths P_i, $1 \le i \le w$, are pairwise (internally) node disjoint. We only need to prove that the end node of $\bar{y}_i$ and the nodes introduced by ϵ_i do not become an internal node of any other path. Let z be any node on P_i, other than 1. Then

$$\mathrm{Supp}(z) \subseteq \begin{cases} X_i, & \text{if } 1 \le i \le m \\ X_i \cup \{t_i\}, & \text{if } m < i \le w. \end{cases}$$

Let $a_1, \ldots, a_w$ be the initial elements in the disjoint orderings of $\bar{X}_1, \ldots, \bar{X}_w$ used above. Then $a_1, \ldots, a_w$ are distinct and, by Lemma 3.2, $a_i = t_i$ for $m < i \le w$. Since a_i is the first node after 1 on P_i, we have

$$a_i \in \mathrm{Supp}(z), \qquad \text{if } 1 \le m, \tag{17}$$

$$t_i \in \mathrm{Supp}(z), \qquad \text{if } m < i \le m_0 \text{ and } z \ne y_i \tag{18}$$

$$t_i \text{ or } t_i^{-1} \in \mathrm{Supp}(z), \qquad \text{if } m_0 < i \le w. \tag{19}$$

And, in the last case, $t_i^{-1} \in \mathrm{Supp}(z)$ only if z is of the form $z = \tilde{y}_i t_i^u$ for some $u \ge e_i/2 - 1$.

Suppose that z is a common node, other than 1, of P_i and P_j for some $1 \le i < j \le w$. We show that $z = y_i = y_j$, i.e., z is the last node of both P_i and P_j. This done in six cases according to the values of i and j.

Case 1: $1 \le i < j \le m$. Nothing to prove.

Case 2: $m < i < j \leq m_0$. Since $t_i \notin X_j$, we have $t_i \notin \mathrm{Supp}(z)$. By (18), $z = y_i$. Similarly, we also have $z = y_j$.
Case 3: $m_0 < i < j \leq w$. Since $t_i \notin X_j$ and $t_i \neq t_j$, we have $t_i \notin \mathrm{Supp}(z) \subseteq X_j \cup \{t_j\}$. By (19), $t_i^{-1} \in \mathrm{Supp}(z)$. Similarly, $t_j^{-1} \in \mathrm{Supp}(z)$. So z must be of the form

$$z = t_i \tilde{y}_i t_i^{v_i} = t_j \tilde{y}_j t_j^{v_j}$$

with $e_i/2 - 1 \leq v_i \leq e_i - u_i - 1$ and $e_j/2 - 1 \leq v_j \leq e_j - u_j - 1$. The minimal representation of z is of the form

$$z = \tilde{y}_i (t_i^{-1})^{e_i - v_i - 1} = \tilde{y}_j (t_j^{-1})^{e_j - v_j - 1}.$$

Hence t_i^{-1} appears in $\tilde{y}_j$, say with exponent c_j, and t_j^{-1} appears in $\tilde{y}_i$, say with exponent c_i. We have

$$e_i - v_i - 1 = c_j, \quad e_j - v_j - 1 = c_i. \tag{20}$$

As $v_i \leq e_i - u_i - 1$ and $v_j \leq e_j - u_j - 1$, we have $c_j \geq u_i$ and $c_i \geq u_j$. By (C), this implies that $c_j = u_i$ and $c_i = u_j$. It follows from (20) that

$$v_i = e_i - u_i - 1 \quad \text{and} \quad v_j = e_j - u_j - 1.$$

Thus $y_i = z = y_j$.
Case 4: $1 \leq i \leq m$ **and** $m < j \leq m_0$. As $t_i \neq t_j$ and $t_i \in \mathrm{Supp}(z) \subseteq X_j \cup \{t_j\}$, we have $t_i \in X_j$. If z is an internal node of P_j then $t_j \in \mathrm{Supp}(z) \subseteq X_i$, hence we have an SDR

$$t_1 \in X_1, \ldots, t_{i-1} \in X_{i-1}, t_j \in X_i, t_{i+1} \in X_{i+1}, \ldots, t_m \in X_m, t_i \in X_j$$

of size $m+1$, contradicting to the maximality of m. So z must be the end node of P_j, i.e., $z = y_j$. As $z = y_j$ is a node on P_i, we have $y_j \prec y_i$ and $a_i \in \mathrm{Supp}(z) = \mathrm{Supp}(y_j) = X_j$. Hence the system $X_1, \ldots, X_{i-1}, X_j, X_{i+1}, \ldots, X_m$ has an SDR. By the condition (A), it follows that $y_i = y_j$.
Case 5: $1 \leq i \leq m$ **and** $m_0 < j \leq w$. Since $a_i \neq a_j = t_j$ and $a_i \in \mathrm{supp}(z)$, by (19), we have $a_i \in X_j$. If $t_j \in \mathrm{Supp}(z)$ then $t_j \in X_i$ by (17), and so the system $X_1, \ldots, X_m, X_j$ has an SDR of size $m+1$, contradicting to the maximality of m. Hence $t_j^{-1} \in \mathrm{Supp}(z)$. As a node on P_j, z must be of the form

$$z = t_j \tilde{y}_j t_j^{v} = \tilde{y}_j (t_j^{-1})^{e_j - v - 1}$$

for some v satisfying $e_j/2 - 1 \leq v \leq e_j - u_j - 1$. Since z is node on P_i, we have $z \prec y_i$. As $v \leq e_j - u_j - 1$, we have $u_j \leq e_j - v - 1$ and so

$$y_j = \tilde{y}_j (t_j^{-1})^{u_j} \prec \tilde{y}_j (t_j^{-1})^{e_j - v - 1} = z \prec y_i.$$

This means that $y_j \prec y_i$ and $t_j^{-1} \in X_i$. By the condition (B), it follows that $y_j = y_i$. But $y_j \prec z \prec y_i$, we have $y_j = z = y_i$.

Case 6: $m < i \le m_0$ **and** $m_0 < j \le w$. In this case, we have

$$\mathrm{Supp}(z) \subseteq X_i \cup \{t_i\}, \quad \text{and} \quad t_j \text{ or } t_j^{-1} \in \mathrm{Supp}(z) \subseteq X_j \cup \{t_j\}.$$

Since $t_j \notin X_i$ and $t_j \neq t_i$, we see that $t_j \notin X_i \cup \{t_i\}$, so $t_j \notin \mathrm{Supp}(z)$. Hence $t_j^{-1} \in \mathrm{Supp}(z)$. It follows that z, as a node on P_j, must be of the form,

$$z = t_j \tilde{y}_j t_j^v = \tilde{y}_j (t_j^{-1})^{e_j - v - 1}$$

for some v satisfying $e_j/2 - 1 \le v \le e_j - u_j - 1$. Thus $\mathrm{Supp}(z) = X_j$. Since $i, j > m$, we have $t_i \notin X_j$ and so $t_i \notin \mathrm{Supp}(z)$. By (18), we must have $z = y_i$. As $v \le e_j - u_j - 1$, we have $u_j \le e_j - v - 1$, hence

$$y_j = \tilde{y}_j (t_j^{-1})^{u_j} \prec \tilde{y}_j (t_j^{-1})^{e_j - v - 1} = z = y_i.$$

Note that Z_j has the representative t_j^{-1} but Z_i does not. By the condition (B), we have $z = y_i = y_j$. This concludes the proof of the theorem. □

Proof of Theorem 2.1: The diameter d of $\Gamma(G, S)$ is determined by Corollary 4.2. Now use Theorem 4.4, but examine the lengths of the paths P_i's more carefully. Certainly, for $1 \le i \le m$, $|P_i| = |y_i| \le d$. For $m < i \le m_0$, $X_i = \mathrm{Supp}(y_i)$ does not contain t_i and t_i^{-1}. If $t_i^{-1} \in S$ then $|y_i| \le d - \lfloor e_i/2 \rfloor \le d - 1$, so $|P_i| \le |y_i| + 2 \le d + 1$. If $t_i^{-1} \notin S$ then $|y_i| \le d - (e_i - 1)$, so $|P_i| \le |y_i| + e_i \le d + 1$. Hence $|P_i| \le d + 1$ for $m < i \le m_0$. If $e = 1$ or 2, which means that $s = 0$ in (2), then the proof is finished, as the next case will not happen.

Assume that $e \ge 3$, thus $m_0 < w$. For $m_0 < i \le w$, $t_i^{-1} \in S$ and $|\tilde{y}_i| \le d - \lfloor e_i/2 \rfloor$ as $\tilde{y}_i$ does not contain t_i and t_i^{-1}. As $u_i \ge 1$, we have

$$|P_i| = |\tilde{y}_i| + e_i - u_i \le d - \lfloor e_i/2 \rfloor + e_i - 1 \le d + e_i - \lfloor e_i/2 \rfloor - 1 = d + \lfloor (e_i - 1)/2 \rfloor,$$

which is at most $d + \lfloor (e-1)/2 \rfloor$. This completes the proof. □

5 Comments and open questions

For the class of Cayley graphs we discussed, it remains to completely determine the true star diameters. For hypercubes and directed torus, we know that their w-star diameters are equal to their w-wide diameters. A curious question is: for which class of graphs does this phenomenon hold?

Our bound on star diameters is based on explicit construction of short containers. The main property we used is the commutativity of the group operation. It may be possible that our method could be extended to many other Cayley graphs over abelian groups.

For the class of graphs we discussed, their connectivity is just the cardinality of the generating set (which is assumed to generate the group), and their wide diameter is also easy to determine. For general Cayley graphs, however, the first obstacle is to determine its connectivity which may be much smaller than the cardinality of the generating set. The problem of deciding whether a given Cayley graph is connected itself seems already hard, since testing primitivity of elements in a finite field is just a special instance (where G is cyclic and S has only one element). Interestingly, if a Cayley graph $\Gamma(G, S)$ is given connected then its connectivity (or fault tolerance) can be determined efficiently (i.e. in time polynomial in $|S|$ and $\log |G|$). We will leave the details to a forthcoming paper [6].

Note that, for general Cayley graphs, finding the usual diameter is already NP-hard. But it may not be unreasonable to ask for a good upper bound for the star and wide diameters in term of the usual diameter. For the class of graphs we discussed, the star and wide diameters are at most $2d$ where d is the usual diameter. We wonder whether this is true for all Cayley graphs.

References

[1] S. B. Akers and B. Krishnamurthy, "A group-theoretic model for symmetric interconnection networks," *IEEE Trans. Comput.* **38** (1989), no. 4, 555–566.

[2] C.-C. Chen and J. Chen, "Optimal parallel routing in star networks," *IEEE Trans. Comput.* **46** (1997), no. 12, 1293–1303.

[3] D.-R. Duh and G.-H. Chen, "On the Rabin number problem," *Networks* **30** (1997), no. 3, 219–230.

[4] D.-R. Duh, G.-H. Chen and D. F. Hsu, "Combinatorial properties of generalized hypercube graphs," *Inform. Process. Lett.* **57** (1996), no. 1, 41–45.

[5] R. J. Faudree, R. J. Gould and L. M. Lesniak, "Generalized degrees and Menger path systems," *Discrete Appl. Math.* **37/38** (1992), 179–191.

[6] S. Gao, "Computing connectivity of Cayley graphs," in preparation.

[7] S. Gao, B. Novick and K. Qiu, "From Hall's matching theorem to optimal routing on hypercubes,", *Journal of Combinatorial Theory (B)*, **74** (1998), 291–301.

[8] Q.-P. Gu and S. Peng, "Node-to-set disjoint paths problem in star graphs," *Inform. Process. Lett.* **62** (1997), no. 4, 201–207.

[9] P. Hall, "On representatives of subsets," *J. London Math. Soc.*, vol. 10, 1935, pp 26–30.

[10] D. F. Hsu, "On Container Width and Length in Graphs, Groups, and Networks," *IEICE Transactions, on Fundamentals of Electronics, Communications and Computer Sciences*, vol. E77-A, No.4, April 1994, pp 668–680.

[11] D. F. Hsu and T. Luczak, "On the k-diameter of k-regular k-connected graphs," *Discrete Math.* **133** (1994), no. 1-3, 291–296.

[12] Y. Ishigami, "The wide-diameter of the n-dimensional toroidal mesh," *Networks*, **27** (1996), no. 4, 257–266.

[13] P. Kulasinghe and S. Bettayeb, "Vertex-disjoint paths in Cayley color graphs," *Comput. Artificial Intelligence*, **16** (1997), no. 6, 583–597.

[14] S. Lakshmivarahan, J. S. Jwo and S. K. Dhall, "Symmetry in interconnection networks based on Cayley graphs of permutation groups: a survey," *Parallel Comput.* **19** (1993), no. 4, 361–407.

[15] S. Latifi, "On the fault-diameter of the star graph," *Inform. Process. Lett.* **46** (1993), no. 3, 143–150.

[16] Q. Li, D. Sotteau and J. Xu, "2-diameter of de Bruijn graphs," *Networks* **28** (1996), no. 1, 7–14.

[17] L. Pyber and Z. Tuza, "Menger-type theorems with restrictions on path lengths," *Discrete Math.* **120** (1993), no. 1-3, 161–174.

[18] M. Rabin, "Efficient dispersal of information for security, load balancing, and fault tolerance," *Journal of the Association of Computing Machinery*, vol. 36, no. 2, 1989, pp 335–348.

[19] Y. Rouskov and P. K. Srimani, "Fault diameter of star graphs," *Inform. Process. Lett.* **48** (1993), no. 5, 243–251.

[20] Y. Saad and M.H. Schultz, "Topological Properties of Hypercubes," *IEEE Transaction on Computers*, Vol. 37, No. 7, July 1988, pp. 867-872.

[21] S. Sur and P. K. Srimani, "Topological properties of star graphs," *Comput. Math. Appl.* **25** (1993), no. 12, 87–98.