Approximation of Time-Dependent, Viscoelastic Fluid Flow: SUPG Approximation

Vincent J. Ervin* William W. Miles*
Department of Mathematical Sciences
Clemson University
Clemson, South Carolina 29634-0975

Abstract. In this article we consider the numerical approximation to the time dependent viscoelasticity equations with an Oldroyd B constitutive equation. The approximation is stabilized by using a SUPG approximation for the constitutive equation. We analyse both the semi-discrete and fully discrete numerical approximations. For both discretizations we prove the existence of, and derive a priori error estimates for, the numerical approximations.

Key words. viscoelasticity, finite element method, fully discrete, SUPG

AMS Mathematics subject classifications. 65N30

1 Introduction

Accurate numerical simulations of time dependent viscoelastic flows are important to the understanding of many phenomena in non-Newtonian fluid mechanics, particularly those associated with flow instabilities. Aside from [3], previous numerical analysis in this area has been for steady state flows.

In the case of Newtonian fluid flow the assumption that the extra stress tensor is proportional to the deformation tensor allows the stress to be eliminated from the modeling equations, giving the Navier–Stokes equations. In viscoelasticity, assuming an Oldroyd B type fluid, the stress is defined by a (hyperbolic) differential constitutive equation. Very different from computational fluid dynamics simulations, in viscoelasticity because of a "slow flow" assumption, the non-linearity in the momentum equation is often neglected. The difficulty in performing accurate numerical computations arises from the hyperbolic character of the constitutive equation, which does not contain a dissipative (stabilizing) term for the stress. Care must be used in discretizing the constitutive equation to avoid the introduction of spurious oscillations into the approximation.

The first error analysis for the steady-state finite element approximation of viscoelastic fluid was presented by Baranger and Sandri [2]. In [2] a discontinuous finite element formulation was used for the discretization of the constitutive equation, with the approximation for the stress being

*This work was partially supported by the ERC Program of the National Science Foundation under Award Number ERC-9731680.

discontinuous. Motivated by implementation consideration, Najib and Sandri in [12] modified the discretization in [2] to obtain a decoupled system of two equations, showed the algorithm was convergent, and derived a priori error estimates. In [14], Sandri presented an analysis of a finite element approximation to this problem wherein the constitutive equation was discretized using a Streamline Upwind Petrov Galerkin (SUPG) method. For the constitutive equation discretized using the method of characteristics, Baranger and Machmoum in [1] analysed this approach and gave error estimates for the approximations.

For the analysis of the time dependent problem, Baranger and Wardi [3] studied a DG approximation to inertialess flow in $\mathbb{R}^2$, using similar techniques as used for the steady state problem. With the Hood-Taylor finite element (FE) pair used to approximate the velocity and pressure, and a discontinuous linear approximation for the stress they showed, under the assumption $\Delta t \leq C_1 h^{3/2}$, that the discrete H^1 and L^2 errors for the velocity and stress, respectively, were bounded by $C(\Delta t + h^{3/2})$.

In this paper we analyse the SUPG approximation to the time dependent equations in $\mathbb{R}^{\acute{d}}$, $\acute{d} = 2, 3$. For the fully discrete analysis we extend the approach used in [11] for compressible Navier-Stokes to non-Newtonian flow. For ν denoting the SUPG coefficient, and assuming Hood-Taylor FE pair approximation for the velocity, pressure, and a continuous FE approximation for the viscoelastic stress, under the assumption $\Delta t,\ \nu \leq C_1 h^{\acute{d}/2}$, we obtain that the discrete H^1 and L^2 errors for the velocity and stress, respectively, are bounded by $C(\Delta t + \nu + h^2)$.

This paper is organized as follows. A description of the modeling equations is presented in section 2. Section 3 contains a description of the mathematical notation, and several lemmas used in the analysis. The semi-discrete and fully discrete approximations are then presented and analysed in sections 4 and 5, respectively.

2 The Oldroyd B Model and the Approximating System

In this section we describe the modeling equations for viscoelastic fluid flow (see also [2]).

2.1 The Problem

Consider a fluid flowing in a bounded, connected domain $\Omega \in \mathcal{R}^{\acute{d}}$. The boundary of Ω, $\partial\Omega$, is assumed to be Lipschitzian. The vector $\mathbf{n}$ represents the outward unit normal to $\partial\Omega$. The velocity vector is denoted by $\mathbf{u}$, pressure by p, total stress by $\mathbf{T}$, and extra stress by τ. For ease of notation, we use the convention of summation on repeated indices and denote differientation with a comma. For example, $\frac{\partial \mathbf{u}}{\partial x_i}$ is written $u_{,i}$. Then for a tensor τ and a vector $\mathbf{w}$, $\nabla \cdot \tau$ denotes $\tau_{ij,j}$, and $\mathbf{w} \cdot \nabla$ denotes the operator $w_i \frac{\partial}{\partial x_i}$. The deformation tensor, $D(\mathbf{u})$, and the vorticity tensor, $W(\mathbf{u})$, are given by

$$D(\mathbf{u}) \;=\; \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right), \quad W(\mathbf{u}) \;=\; \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right).$$

The Oldroyd model can be described using an *objective derivative* [2], denoted by $\hat{\partial}\sigma/\partial t$, where

$$\frac{\hat{\partial}\sigma}{\partial t} := \frac{\partial\sigma}{\partial t} + \mathbf{u}\cdot\nabla\sigma + g_a(\sigma, \nabla\mathbf{u}), \qquad a \in [-1, 1]$$

and

$$\begin{aligned} g_a(\sigma, \nabla \mathbf{u}) &:= \sigma W(\mathbf{u}) - W(\mathbf{u})\sigma - a(D(\mathbf{u})\sigma + \sigma D(\mathbf{u})) \\ &= \frac{1-a}{2}\left(\sigma \nabla \mathbf{u} + (\nabla \mathbf{u})^T \sigma\right) - \frac{1+a}{2}\left((\nabla \mathbf{u})\sigma + \sigma(\nabla \mathbf{u})^T\right). \end{aligned}$$

Oldroyd's model for stress employs a decomposition of the extra stress into two parts: a Newtonian part and a viscoelastic part. So $\tau = \tau_N + \tau_V$. The Newtonian part is given by $\tau_N = 2(1-\alpha)D(\mathbf{u})$. The $(1-\alpha)$ represents that part of the total viscosity which is considered Newtonian. Hence $\alpha \in (0,1)$ represents the proportion of the total viscosity that is considered to be viscoelastic in nature. For example, if a polymer is immersed within a Newtonian carrier fluid, α is related to the percentage of polymer in the mix. The constitutive law is [2]

$$\tau_V + \lambda \frac{\hat{\partial} \tau_V}{\partial t} - 2\alpha D(\mathbf{u}) = 0, \tag{2.1}$$

where λ is the Weissenberg number, which is a dimensionless constant defined as the product of the relaxation time and a characteristic strain rate [4]. For notational simplicity, the subscript, V, is dropped, and below τ will be used to denote the viscoelastic component of the extra stress.

The momentum balance for the fluid is given by

$$Re\left(\frac{d\mathbf{u}}{dt}\right) = -\nabla p + \nabla \cdot (2(1-\alpha)D(\mathbf{u}) + \tau) + \mathbf{f}\,, \tag{2.2}$$

where Re is the Reynolds number, $\mathbf{f}$ the body forces acting on the fluid, and $d\mathbf{u}/dt$ is the material derivative. Recall that

$$\begin{aligned} Re &= \tfrac{LV\rho}{\mu}, & L &= \text{characteristic length scale,} \\ & & V &= \text{characteristic velocity scale,} \\ & & \rho &= \text{fluid density,} \\ & & \mu &= \text{fluid viscosity.} \end{aligned}$$

In addition to (2.1) and (2.2) we also have the incompressibility condition:

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega.$$

To fully specify the problem, appropriate boundary conditions must also be given. The simplest such condition is the homogeneous Dirichlet condition for velocity. In this case, there is no inflow boundary, and, thus, no boundary condition is required for stress. Summarizing, the modeling equations are:

$$\begin{aligned} Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) + \nabla p - 2(1-\alpha)\nabla \cdot D(\mathbf{u}) - \nabla \cdot \tau &= \mathbf{f} && \text{in } \Omega, && (2.3) \\ \tau + \lambda\left(\frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau + g_a(\tau, \nabla \mathbf{u})\right) - 2\alpha D(\mathbf{u}) &= 0 && \text{in } \Omega, && (2.4) \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, && (2.5) \\ \mathbf{u} &= 0 && \text{on } \partial\Omega, && (2.6) \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega, && (2.7) \\ \tau(0, \mathbf{x}) &= \tau_0(\mathbf{x}) && \text{in } \Omega. && (2.8) \end{aligned}$$

In [8], Guillope and Saut proved the following for the "slow-flow" model of (2.3)-(2.8) (i.e. $\mathbf{u} \cdot \nabla \mathbf{u}$ term in (2.3) is ignored):

1. local existence, in time, of a unique, regular solution, and
2. under a small data assumption on $\mathbf{f}, \mathbf{f}', \mathbf{u}_0, \tau_0$, the global existence (in time) of a unique solution for $\mathbf{u}$ and τ.

In contrast to the Navier–Stokes equations, well-posedness for general models in viscoelasticity is still not well understood. Results which are known fall into one of three types [13]:

1. for inital value problems, solutions have been shown to exist locally in time,
2. global existence (in time) of solutions if the initial conditions are small perturbations of the rest state, and
3. for steady-state problems, existence of solutions which are small perturbations of the analagous Newtonian case.

2.2 The Variational Formulation

In this section, we develop the variational formulation of (2.3)-(2.6). The following notation will be used. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ denotes the norm in H^k. The following function spaces are used in the analysis:

$$
\begin{aligned}
\text{Velocity Space} \quad &: \quad X := H_0^1(\Omega) := \Big\{\mathbf{u} \in H^1(\Omega) : \mathbf{u} = 0 \ \ on \ \ \partial\Omega\Big\}, \\
\text{Stress Space} \quad &: \quad S := \Big\{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji}; \tau_{ij} \in L^2(\Omega); 1 \le i, j \le 3\Big\} \\
&\qquad\quad \cap \Big\{\tau = (\tau_{ij}) : \mathbf{u} \cdot \nabla\tau \ \in L^2(\Omega), \forall \mathbf{u} \in X\Big\}, \\
\text{Pressure Space} \quad &: \quad Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \ dx = 0\}, \\
\text{Divergence} - \text{free Space} \quad &: \quad Z := \{v \in X : \int_\Omega q(\nabla \cdot v) \ dx = 0, \ \forall \ q \in Q\}.
\end{aligned}
$$

The variational formulation of (2.3)-(2.6) proceeds in the usual manner. Taking the inner product of (2.3), (2.4), and (2.5) with a velocity test function, a stress test function, and a pressure test function respectively, we obtain

$$
\begin{aligned}
Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}\right) - (p, \nabla \cdot \mathbf{v}) + (2(1-\alpha)D(\mathbf{u}) + \tau, D(\mathbf{v})) \quad &= \quad (\mathbf{f}, \mathbf{v}), \qquad \forall \ \mathbf{v} \in X, \quad (2.9) \\
\left(\tau + \lambda\left(\frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla\tau + g_a(\tau, \nabla\mathbf{u})\right) - 2\alpha D(\mathbf{u}), \psi\right) \quad &= \quad 0, \qquad \forall \ \psi \in S, \quad (2.10) \\
(\nabla \cdot \mathbf{u}, q) \quad &= \quad 0, \qquad \forall \ q \in Q. \quad (2.11)
\end{aligned}
$$

The space Z is the space of weakly divergence free functions. Note that the condition

$$(\nabla \cdot \mathbf{u}, q) = 0, \ \ \forall \ q \in Q, \ \ \mathbf{u} \in X,$$

is equivalent in a "distributional" sense to

$$(\mathbf{u}, \nabla q) = 0, \ \ \forall\, q \in Q, \ \mathbf{u} \in X, \tag{2.12}$$

where in (2.12), $(\cdot,\cdot)$ denotes the duality pairing between H^{-1} and H_0^1 functions. In addition, note that the velocity and pressure spaces, X and Q, satisfy the *inf-sup* condition

$$\inf_{q\in Q} \sup_{\mathbf{v}\in X} \frac{(q, \nabla\cdot\mathbf{v})}{\|q\|\,\|\mathbf{v}\|_1} \geq \beta > 0. \tag{2.13}$$

Since the inf-sup condition (2.13) holds, an equivalent variational formulation to (2.9)-(2.11) is: *Find* $(\mathbf{u}, \tau) : [0, T] \rightarrow X \times S$ *such that*

$$\begin{aligned} Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}\right) + (2(1-\alpha)D(\mathbf{u}) + \tau, D(\mathbf{v})) &= (\mathbf{f}, \mathbf{v}), \ \ \forall\, \mathbf{v} \in Z, &(2.14)\\ \left(\tau + \lambda\left(\frac{\partial\tau}{\partial t} + \mathbf{u}\cdot\nabla\tau + g_a(\tau, \nabla\mathbf{u})\right) - 2\alpha D(\mathbf{u}), \psi\right) &= 0, \qquad \forall\, \psi \in S. &(2.15)\end{aligned}$$

Before discussion the numerical approximation of (2.14),(2.15), we summarize the mathematical notation and interpolation properties used in the analysis.

3 Mathematical Notation

In this section the mathematical framework and approximation properties are summarized.

Let $\Omega \subset \mathbb{R}^{\acute{d}} (\acute{d} = 2, 3)$ be a polygonal domain and let T_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrals (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\Omega = \bigcup K; \quad K \in T_h.$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \leq h_K \leq c_2 \rho_K$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K\in T_h} h_K$. Let $P_k(A)$ denote the space of polynomials on A of degree no greater than k. Then we define the finite element spaces as follows.

$$\begin{aligned} X_h &:= \left\{\mathbf{v} \in X \cap C(\bar{\Omega})^{\acute{d}} : \mathbf{v}|_K \in P_k(K), \ \forall K \in T_h\right\}, \\ S_h &:= \left\{\sigma \in S \cap C(\bar{\Omega})^{\acute{d}\times\acute{d}} : \sigma|_K \in P_m(K), \ \forall K \in T_h\right\}, \\ Q_h &:= \left\{q \in Q \cap C(\bar{\Omega}) : q|_K \in P_q(K), \ \forall K \in T_h\right\}, \\ Z_h &:= \left\{\mathbf{v} \in X_h : (q, \nabla\cdot\mathbf{v}) = 0, \ \forall q \in Q_h\right\},\end{aligned}$$

where $C(\bar{\Omega})^{\acute{d}}$ denotes a vector valued function with $\acute{d}$ components continuous on $\bar{\Omega}$. Analogous to the continuous spaces, we assume that X_h and Q_h satisfy the discrete *inf-sup* condition

$$\inf_{q\in Q_h} \sup_{\mathbf{v}\in X_h} \frac{(q, \nabla\cdot\mathbf{v})}{\|q\|\,\|\mathbf{v}\|_1} \geq \beta > 0\,. \tag{3.1}$$

We summarize several properties of finite element spaces and Sobolev's spaces which we will use in our subsequent analysis. For $(\mathbf{u}, p) \in H^{k+1}(\Omega)^{\acute{d}} \times H^{q+1}(\Omega)$ we have (see [7]) that there exists $(\mathcal{U}, \mathcal{P}) \in Z_h \times Q_h$ such that

$$\begin{aligned} \|\mathbf{u} - \mathcal{U}\| &\leq C_I h^{k+1} |\mathbf{u}|_{W_2^{k+1}} \, , &(3.2)\\ \|\mathbf{u} - \mathcal{U}\|_{W_2^1} &\leq C_I h^{k} |\mathbf{u}|_{W_2^{k+1}} \, , &(3.3)\\ \|p - \mathcal{P}\| &\leq C_I h^{q+1} |p|_{W_2^{q+1}} \, . &(3.4) \end{aligned}$$

Let $\mathcal{T} \in S_h$ be a P_1 continuous interpolant of τ. For $\tau \in H^{m+1}(\Omega)^{\acute{d}\times\acute{d}}$ we have that

$$\begin{aligned} \|\tau - \mathcal{T}\| \; + \; h|\tau - \mathcal{T}|_{W_2^1} &\leq C_I h^{m+1} \|\tau\|_{W_2^{m+1}} \, , &(3.5)\\ \|\tau - \mathcal{T}\|_{L^4} \; + \; h|\tau - \mathcal{T}|_{W_4^1} &\leq C_I h^{m+1-\acute{d}/4} \|\tau\|_{W_2^{m+1}} \, . &(3.6) \end{aligned}$$

From [5], we have the following results.

Lemma 1 *: Let $\{T_h\}$, $0 < h \leq 1$, denote a quasi-uniform family of subdivisions of a polyhedral domain $\Omega \subset \mathrm{I\!R}^{\acute{d}}$. Let $(\hat{K}, P, N)$ be a reference finite element such that $P \subset W_p^l(\hat{K}) \cap W_q^m(\hat{K})$ where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $0 \leq m \leq l$. For $K \in T_h$, let (K, P_K, N_K) be the affine equivalent element, and $V_h = \{v : v$ is measurable and $v|_K \in P_K, \forall K \in T_h\}$. Then there exists $C = C(l, p, q)$ such that*

$$\left[\sum_{K \in T_h} \|v\|^p_{W_p^l(K)} \right]^{1/p} \leq C \, h^{m-l+\min(0\,,\,\frac{\acute{d}}{p} - \frac{\acute{d}}{q})} \left[\sum_{K \in T_h} \|v\|^q_{W_q^m(K)} \right]^{1/q} , \qquad (3.7)$$

for all $v \in V_h$.

■

Lemma 2 *: Let I_h denote the interpolant of v. Then for all $v \in W_p^m(\Omega) \cap C^r(\Omega)$ and $0 \leq s \leq$ $\min\{m, r+1\}$,*

$$\|v - I_h\|_{s,\infty} \leq C \, h^{m-s-\acute{d}/p} \, |v|_{W_p^m} \, . \qquad (3.8)$$

■

When $v(\mathbf{x}, t)$ is defined on the entire time interval $(0, T)$, we define

$$\begin{aligned} \|v\|_{\infty,k} &:= \sup_{0<t<T} \|v(\cdot, t)\|_k \, , \\ \|v\|_{0,k} &:= \left(\int_0^T \|v(\cdot, t)\|_k^2 \, dt \right)^{1/2} \, . \end{aligned}$$

For the analysis of the fully discrete approximation we use Δt to denote the step size for t so that $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$, and define

$$d_t f := \frac{f(t_n) - f(t_{n-1})}{\Delta t} \, . \qquad (3.9)$$

We also use the following additional norms:

$$\|\!|v|\!\|_{\infty,k} \;:=\; \max_{1\le n\le N} \|v^n\|_k \,,$$
$$\|\!|v|\!\|_{0,k} \;:=\; \left[\sum_{n=1}^{N} \Delta t \; \|v^n\|_k^2\right]^{1/2} .$$

4 Semi-Discrete Approximation

In this section we present the analysis of a semi–discrete approximation to (2.14),(2.15). We begin by introducing some notation specific to the semi–discrete approximation and cite some lemmas used in the analysis.

For $\sigma_u := \sigma \,+\, \nu h \, \mathbf{u}\cdot\nabla\sigma$ we define

$$\begin{aligned}
A(\mathbf{w},(\mathbf{u},\tau),(\mathbf{v},\psi)) \;&:=\; (\tau,\psi_w) - 2\alpha(D(\mathbf{u}),\psi_w) + 2\alpha(\tau,D(\mathbf{v})) + \alpha(1-\alpha)(\nabla\mathbf{u}\,,\nabla\mathbf{v})\,, && (4.1)\\
B(\mathbf{u},\mathbf{v},\tau,\sigma) \;&:=\; (\mathbf{u}\cdot\nabla\tau,\sigma_v) \;+\; \frac{1}{2}(\nabla\cdot\mathbf{u}\,\tau,\sigma), && (4.2)\\
c(\mathbf{w},\mathbf{u},\mathbf{v}) \;&:=\; (\mathbf{w}\cdot\nabla\mathbf{u},\mathbf{v}), && (4.3)\\
\tilde{c}(\mathbf{w},\mathbf{u},\mathbf{v}) \;&:=\; \frac{1}{2}\left(c(\mathbf{w},\mathbf{u},\mathbf{v}) - c(\mathbf{w},\mathbf{v},\mathbf{u})\right)\;. && (4.4)
\end{aligned}$$

Lemma 3 *: [10] For $\mathbf{u},\mathbf{v},\mathbf{w}\in X$, there exists a constant C_1 such that*

$$|\tilde{c}(\mathbf{u},\mathbf{v},\mathbf{w})| \le C_1\,\|\mathbf{u}\|^{1/2}\,\|\nabla\mathbf{u}\|^{1/2}\,\|\nabla\mathbf{v}\|^{1/2}\,\|\nabla\mathbf{w}\|^{1/2}\,. \tag{4.5}$$

■

Note:

(i) $\tilde{c}(\mathbf{u},\mathbf{v},\mathbf{w}) = c(\mathbf{u},\mathbf{v},\mathbf{w})$ when $\nabla\cdot\mathbf{u}=0$ in Ω, and $\mathbf{u}=0$ on $\partial\Omega$. (4.6)

(ii) $\tilde{c}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$, even when $\nabla\cdot\mathbf{u}\neq 0$. (4.7)

(iii) For $\mathbf{u}\in X$, from the Poincare–Friedrich's inequality we have that there exists a constant $C_{PF} = C(\Omega)$ such that $\|\mathbf{u}\|^2 \le C_{PF}^2\,\|\nabla\mathbf{u}\|^2$. (4.8)

The operators $A(\cdot,(\cdot,\cdot),(\cdot,\cdot)) : X\times(X\times H^1(\Omega)^{n\times n})\times(X\times H^1(\Omega)^{n\times n}) \to \mathbb{R}$, and $B(\cdot,\cdot,\cdot,\cdot) : X\times X\times H^1(\Omega)^{n\times n}\times H^1(\Omega)^{n\times n} \to \mathbb{R}$ are the same as that used in [2],[14]. When $\mathbf{u}=\mathbf{v}$ we omit the second variable in $B(\cdot,\cdot,\cdot,\cdot)$.

Lemma 4 *: We have that*

$$B(\mathbf{u},\tau,\tau) \;=\; \nu h(\mathbf{u}\cdot\nabla\tau,\mathbf{u}\cdot\nabla\tau). \tag{4.9}$$

Proof: On integrating $(\mathbf{u}\cdot\nabla\tau,\sigma)$ by parts we have:

$$B(\mathbf{u},\mathbf{v},\tau,\sigma) \;:=\; -\,(\mathbf{u}\cdot\nabla\sigma,\tau) \;+\; \nu\,h\,(\mathbf{u}\cdot\nabla\tau,\mathbf{v}\cdot\nabla\sigma) \;-\; \frac{1}{2}(\nabla\cdot\mathbf{u}\,\sigma,\tau). \tag{4.10}$$

Setting $\mathbf{v}=\mathbf{u}$, $\sigma=\tau$, and combining (4.2) and (4.10) the stated result follows.

■

Lemma 5 : *For $\mathbf{w} \in X, (\mathbf{u}, \tau) \in X \times S$, and h sufficiently small, we have*

$$A(\mathbf{w}, (\mathbf{u}, \tau), (\mathbf{u}, \tau)) \, + \, \lambda B(\mathbf{w}, \tau, \tau) \geq C_A \left(\|\tau\|^2 + \|\mathbf{u}\|_1^2 \right).$$

Proof: Using the definitions of A and B we obtain

$$\begin{aligned}
A(\mathbf{w}, (\mathbf{u}, \tau), (\mathbf{u}, \tau)) \; + \; & \lambda B(\mathbf{w}, \tau, \tau) \; = \; \|\tau\|^2 \, + \, (\tau, \nu h \, \mathbf{w} \cdot \nabla \tau) - 2\alpha(D(\mathbf{u}), \tau) - 2\alpha(D(\mathbf{u}), \nu h \, \mathbf{w} \cdot \nabla \tau) \\
& +2\alpha(\tau, D(\mathbf{u})) + \alpha(1-\alpha)\|\nabla \mathbf{u}\|^2 + \lambda \nu h \|\mathbf{w} \cdot \nabla \tau\|^2 \\
\geq \; & \|\tau\|^2 + \alpha(1-\alpha)\|\nabla \mathbf{u}\|^2 + \lambda \nu h \, \|\mathbf{w} \cdot \nabla \tau\|^2 - \frac{1}{2}\|\tau\|^2 \\
& - \frac{1}{2}\nu^2 h^2 \, \|\mathbf{w} \cdot \nabla \tau\|^2 \; - \; \frac{1}{2}\alpha(1-\alpha)\|\nabla \mathbf{u}\|^2 - \frac{\alpha \nu^2 h^2}{2(1-\alpha)}\|\mathbf{w} \cdot \nabla \tau\|^2 \\
\geq \; & \frac{1}{2}\|\tau\|^2 + \frac{\alpha(1-\alpha)}{2}\|\nabla \mathbf{u}\|^2 + \left(\lambda \nu h \; - \; \frac{\nu^2 h^2}{2} \; - \; \frac{\alpha \nu^2 h^2}{2(1-\alpha)} \right) \|\mathbf{w} \cdot \nabla \tau\|^2 \qquad (4.11) \\
\geq \; & C_A \left(\|\tau\|^2 + \|\mathbf{u}\|_1^2 \right) \; , \text{for } h \text{ sufficiently small, using (4.8).}
\end{aligned}$$

■

Now, we define the semi–discrete approximation of (2.14),(2.15) as:

Find $(\mathbf{u}_h, \tau_h) : [0, T] \rightarrow X_h \times S_h$ *such that*

$$\begin{aligned}
Re\,(\mathbf{u}_{h\,t}, \mathbf{v}) \;\; + \;\; & Re\,\tilde{c}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) \, + \, (1-\alpha)(\nabla \mathbf{u}_h, \nabla \mathbf{v}) \, + \, (\tau_h, D(\mathbf{v})) \; = \; (\mathbf{f}, \mathbf{v}), \;\; \forall \, \mathbf{v} \in Z_h, \quad (4.12) \\
\lambda\,(\tau_{h\,t}, \sigma) \;\; + \;\; & \lambda\,B(\mathbf{u}_h, \tau_h, \sigma) \, + \, \lambda\,(g_a(\tau_h, \nabla \mathbf{u}_h), \sigma_{u_h}) \, + \, (\tau_h, \sigma_{u_h}) \, - \, 2\alpha(D(\mathbf{u}_h), \sigma_{u_h}) \; = \; 0, (4.13) \\
& \forall \; \sigma \in S_h.
\end{aligned}$$

4.1 Analysis of the semi–discrete approximation

In this section, we show that, under suitable conditions, a unique solution to the discretized system exists. Fixed point theory is used to establish the desired result. The proof is established using the following four steps.

1. Define an iterative map in such a way that a fixed point of the map is a solution to (4.12),(4.13).
2. Show the map is well-defined and bounded on bounded sets.
3. Show there exists an invariant ball on which the map is a contraction.
4. Apply Schauder's fixed point theorem to establish the existence and uniqueness of the discrete approximation.

Theorem 4.1 : *Assume that the system (2.3)-(2.8) (and thus, (2.14)-(2.15)) has a solution* $(\mathbf{u}, \tau, \mathbf{p}) \in L^2(0, T; H^{k+1}) \times L^\infty(0, T; H^{m+1}) \times L^2(0, T; H^{q+1})$*. In addition assume that* $k, m \geq \acute{d}/2$*, and*

$$\|\nabla \mathbf{u}\|_\infty, \|\tau\|_\infty, \|\nabla \tau\|_\infty, \|\mathbf{u}\|_{k+1}, \|\tau\|_{m+1}, \|p\|_{q+1} \leq D_0 \;\; \textit{for all } t \in [0, T] \, . \qquad (4.14)$$

Then, for D_0 and h sufficiently small, there exists a unique solution to (4.12)-(4.13) satisfying

$$\int_0^T \left(\|\mathbf{u}-\mathbf{u}_h\|^2 + \|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2\right) dt \;\leq\; Ch^{\min\{k,m,q+1\}} , \tag{4.15}$$

$$\sup_{0\leq t\leq T} \|\tau-\tau_h\| \;\leq\; Ch^{\min\{k,m,q+1\}} . \tag{4.16}$$

Proof:

Step 1: The Iterative Map

A mapping $\xi : L^2(0,T;Z_h)\times L^\infty(0,T;S_h) \to L^2(0,T;Z_h)\times L^\infty(0,T;S_h)$ is defined via:

$(\mathbf{u}_2,\tau_2) = \xi(\mathbf{u}_1,\tau_1)$ *where* $(\mathbf{u}_2,\tau_2)$ *satisfies*

$$Re\,(\mathbf{u}_{2\,t},\mathbf{v}) \;+\; Re\,\tilde{c}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}) + (1-\alpha)(\nabla\mathbf{u}_2,\nabla\mathbf{v}) + (\tau_2,D(\mathbf{v})) \;=\; (\mathbf{f},\mathbf{v}),\;\; \forall\, \mathbf{v}\in Z_h, \tag{4.17}$$

$$\lambda\,(\tau_{2\,t},\sigma) \;+\; \lambda\,B(\mathbf{u}_1,\tau_2,\sigma) + (\tau_2,\sigma_{u_1}) - 2\alpha(D(\mathbf{u}_h),\sigma_{u_1}) \;=\; -\lambda(g_a(\tau_1,\nabla\mathbf{u}_1),\sigma_{u_1}), \quad \forall\,\sigma\in S_h. \tag{4.18}$$

Thus, given an initial guess $(\mathbf{u}_h,\tau_h)\approx(\mathbf{u}_1,\tau_1)$, solving (4.17),(4.18) for $(\mathbf{u}_2,\tau_2)$ gives a new approximation to the solution. Also, it is clear that a fixed point of (4.17),(4.18) is a solution to the approximating system (4.12),(4.13) (i.e. $\xi(\mathbf{u}_1,\tau_1)=(\mathbf{u}_1,\tau_1)$ implies that $(\mathbf{u}_1,\tau_1)$ is a solution to (4.12),(4.13)).

Step 2: Show ξ is well–defined and bounded on bounded sets

Note that (4.17)(4.18) corresponds to a first order system of ODEs for the FEM coefficients $\mathbf{c}_{\mathbf{u}_2}$ and $\mathbf{c}_{\tau_2}$ of $\mathbf{u}_2$ and τ_2, respectively. That is, (4.17)(4.18) is equivalent to

$$\begin{bmatrix} A_{11} & 0\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix}\mathbf{c}_{\mathbf{u}_2}\\ \mathbf{c}_{\tau_2}\end{bmatrix}' \;=\; \mathbf{F}(t,\mathbf{c}_{\mathbf{u}_2},\mathbf{c}_{\tau_2}) ,$$

where

$$\mathbf{F}(t,\mathbf{c}_{\mathbf{u}_2},\mathbf{c}_{\tau_2}) \;=\; \begin{bmatrix} (\mathbf{f},\mathbf{v}) - Re\,\tilde{c}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}) - (1-\alpha)(\nabla\mathbf{u}_2,\nabla\mathbf{v}) - (\tau_2,D(\mathbf{v}))\\ -\lambda(g_a(\tau_h,\nabla\mathbf{u}_h),\sigma_{u_1}) - \lambda\,B(\mathbf{u}_1,\tau_2,\sigma) - (\tau_2,\sigma_{u_1}) + 2\alpha(D(\mathbf{u}_h),\sigma_{u_1})\end{bmatrix} ,$$

and A_{11} and A_{22} are “mass” (invertible) matrices.

Note that $\mathbf{F} : [0,T]\times\mathbb{R}^{dim(\mathbf{c}_{\mathbf{u}_2})}\times\mathbb{R}^{dim(\mathbf{c}_{\tau_2})}\to\mathbb{R}^{dim(\mathbf{c}_{\mathbf{u}_2})}\times\mathbb{R}^{dim(\mathbf{c}_{\tau_2})}$ is a linear function with respect to the FEM coefficients $\mathbf{c}_{\mathbf{u}_2},\mathbf{c}_{\tau_2}$. Thus, for $f(t)$ a continuous function of t, we have that $\mathbf{F}$ is Lipschitz continuous. Then, from ODE theory (see [6]), we are guaranteed that there exists a unique local solution for $(\mathbf{c}_{\mathbf{u}_2},\mathbf{c}_{\tau_2})$, and hence for $(\mathbf{u}_2,\tau_2)$.

Next, to establish the existence of $(\mathbf{u}_2,\tau_2)$ on $[0,T]$, we show that it remains bounded in the appropriate norms on that interval.

Multiplying (4.17) through by 2α and adding the result to (4.18), $(\mathbf{u}_2,\tau_2)$ is equivalently determined via

$$\begin{aligned} 2\alpha Re\,(\mathbf{u}_{2\,t},\mathbf{v}) \;&+\; 2\alpha Re\,\tilde{c}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}) + A(\mathbf{u}_1,(\mathbf{u}_2,\tau_2),(\mathbf{v},\sigma)) + \lambda\,(\tau_{2\,t},\sigma) + \lambda\,B(\mathbf{u}_1,\tau_2,\sigma)\\ &=\; 2\alpha(\mathbf{f},\mathbf{v}) - \lambda(g_a(\tau_1,\nabla\mathbf{u}_1),\sigma_{u_1}),\;\; \forall\,(\mathbf{v},\sigma)\in Z_h\times S_h. \end{aligned} \tag{4.19}$$

Choosing $\mathbf{v} = \mathbf{u}_2$, $\sigma = \tau_2$ in (4.19), and using (4.7),(4.11), implies

$$
\begin{aligned}
\alpha Re \, \|\mathbf{u}_2\|_t^2 \,+\, \frac{\lambda}{2}\,\|\tau_2\|_t^2 \,+\, \frac{1}{2}\|\tau_2\|^2 \;\;&+\;\; \frac{\alpha(1-\alpha)}{2}\|\nabla \mathbf{u}_2\|^2 + \left(\lambda \nu h \,-\, \nu^2 h^2 \left(\frac{1}{2} \,+\, \frac{\alpha}{2(1-\alpha)}\right)\right) \|\mathbf{u}_1 \cdot \nabla \tau_2\|^2 \\
&\leq\;\; 2\alpha \|\mathbf{f}\|_{-1} \|\mathbf{u}_2\|_1 \,+\, \lambda \|g_a(\tau_1, \nabla \mathbf{u}_1)\| \left(\|\tau_2\| \,+\, \nu h \,\|\mathbf{u}_1 \cdot \nabla \tau_2\|\right) , \\
&\leq\;\; \frac{2(1 + C_{PF}^2)}{(1-\alpha)}\|\mathbf{f}\|_{-1}^2 \,+\, \frac{\alpha^2(1-\alpha)}{2}\|\nabla \mathbf{u}_2\|^2 \,+\, \lambda^2 \|g_a(\tau_1, \nabla \mathbf{u}_1)\|^2 \\
&\qquad + \frac{1}{2}\|\tau_2\|^2 \,+\, \frac{\nu^2 h^2}{2}\,\|\mathbf{u}_1 \cdot \nabla \tau_2\|^2 \,. \qquad (4.20)
\end{aligned}
$$

Thus for $c_1 = \min\{\alpha Re \, , \, \lambda/2\}$, and the restriction $\nu h \leq 2\lambda(1-\alpha)/(2-\alpha)$,

$$
\frac{d}{dt}\left(\|\mathbf{u}_2\|^2 \,+\, \|\tau_2\|^2\right) \leq \frac{2(1 + C_{PF}^2)}{c_1(1-\alpha)}\|\mathbf{f}\|_{-1}^2 \,+\, \frac{\lambda^2}{c_1}\|g_a(\tau_1, \nabla \mathbf{u}_1)\|^2 \,.
$$

Hence for $0 \leq t \leq T$,

$$
\begin{aligned}
\|\mathbf{u}_2\|^2(t) \,+\, \|\tau_2\|^2(t) \;\;&\leq\;\; \|\mathbf{u}_2\|^2(0) \,+\, \|\tau_2\|^2(0) \,+\, \frac{2(1 + C_{PF}^2)}{c_1(1-\alpha)}\|\mathbf{f}\|_{0,-1}^2 \\
&\qquad + \frac{\lambda^2 \, \acute{d}^2}{c_1}\|\tau_1\|_{\infty,\infty}^2 \,\|\nabla \mathbf{u}_1\|_{0,0}^2 \,. \qquad (4.21)
\end{aligned}
$$

By the equivalence of norm in finite dimensional spaces, (and $\mathbf{u}_2(0) = \mathbf{u}_1(0)$, $\tau_2(0) = \tau_1(0)$), we therefore have that $(\mathbf{u}_2 \, , \, \tau_2) \in L^2(0,T; Z_h) \times L^\infty(0,T; S_h)$.

Note that (4.21) also establishes that the mapping ξ is bounded on bounded sets.

Step 3: Existence of an invariant ball for ξ.

We begin by defining an invariant ball.
Let $R = c^* h^{\min\{k,m,q+1\}}$ for $0 < c^* < 1$, and define the ball B_h as

$$
B_h := \left\{ (\mathbf{v}, \sigma) \in L^2(0,T; Z_h) \times L^\infty(0,T; S_h) \,:\, \int_0^T \|\mathbf{u} - \mathbf{v}\|^2 + \|\nabla(\mathbf{u} - \mathbf{v})\|^2 \, dt \leq R^2 \;,\; \sup_{0 \leq t \leq T} \|\tau - \sigma\| \leq R \right\}. \qquad (4.22)
$$

The exact solution $(\mathbf{u}, p, \tau)$ of (2.9)-(2.11) satisfies

$$
\begin{aligned}
2\alpha Re \,(\mathbf{u}_t, \mathbf{v}) \;\;&+\;\; 2\alpha Re \, \tilde{c}(\mathbf{u}, \mathbf{u}, \mathbf{v}) \,+\, A(\mathbf{u}_1, (\mathbf{u}, \tau), (\mathbf{v}, \sigma)) \,+\, \lambda\,(\tau_t, \sigma) \,+\, \lambda \, B(\mathbf{u}, \mathbf{u}_1, \tau, \sigma) \\
&=\;\; 2\alpha(p, \nabla \cdot \mathbf{v}) \,+\, 2\alpha(\mathbf{f}, \mathbf{v}) - \lambda(g_a(\tau, \nabla \mathbf{u}), \sigma_{u_1}), \;\; \forall \, (\mathbf{v}, \sigma) \in Z \times S. \qquad (4.23)
\end{aligned}
$$

Subtracting (4.19) from (4.23) implies that

$$
\begin{aligned}
2\alpha Re \,((\mathbf{u} - \mathbf{u}_2)_t, \mathbf{v}) \;\;&+\;\; 2\alpha Re \, \tilde{c}(\mathbf{u}, \mathbf{u}, \mathbf{v}) \,-\, 2\alpha Re \, \tilde{c}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}) \,+\, A(\mathbf{u}_1, ((\mathbf{u} - \mathbf{u}_2), (\tau - \tau_2), (\mathbf{v}, \sigma)) \\
&+\;\; \lambda\,((\tau - \tau_2)_t, \sigma) \,+\, \lambda \, B(\mathbf{u}_1, (\tau - \tau_2), \sigma) \\
&=\;\; 2\alpha(p, \nabla \cdot \mathbf{v}) \,-\, \lambda\,((g_a(\tau, \nabla \mathbf{u}), \sigma_{u_1}) - (g_a(\tau_1, \nabla \mathbf{u}_1), \sigma_{u_1})) \\
&\qquad - \lambda B(\mathbf{u}, \mathbf{u}_1, \tau, \sigma) \,+\, \lambda B(\mathbf{u}_1, \tau, \sigma) \;,\;\; \forall \, (\mathbf{v}, \sigma) \in Z_h \times S_h. \qquad (4.24)
\end{aligned}
$$

Let

$$\Lambda := \mathbf{u} - \mathcal{U} \quad , \quad E := \mathcal{U} - \mathbf{u}_2 \tag{4.25}$$

$$\Gamma := \tau - \mathcal{T} \quad , \quad F := \mathcal{T} - \tau_2 \tag{4.26}$$

$$\text{and} \quad \epsilon_{\mathbf{u}} := \Lambda + E \;=\; \mathbf{u} - \mathbf{u}_2 \quad , \quad \epsilon_{\tau} := \Gamma + F \;=\; \tau - \tau_2 . \tag{4.27}$$

Rewriting (4.24) using these definitions, along with the choice $\sigma = F$, $\mathbf{v} = E$, we obtain

$$\begin{aligned} 2\alpha Re\,(E_t, E) \quad &+ \quad 2\alpha Re\,\tilde{c}(\mathbf{u}, \mathbf{u}, E) \;-\; 2\alpha Re\,\tilde{c}(\mathbf{u}_1, \mathbf{u}_2, E) \;+\; A(\mathbf{u}_1, (E, F), (E, F)) \\ &+ \quad \lambda\,(F_t, F) \;+\; \lambda\,B(\mathbf{u}_1, F, F) \\ &= \quad -2\alpha Re\,(\Lambda_t, E) \;-\; A(\mathbf{u}_1, (\Lambda, \Gamma), (E, F)) \;-\; \lambda\,(\Gamma_t, F) \;-\; \lambda\,B(\mathbf{u}_1, \Gamma, F) \\ &\quad\;\; + 2\alpha(p, \nabla \cdot E) \;-\; \lambda\,((g_a(\tau, \nabla \mathbf{u}), F_{u_1}) - (g_a(\tau_1, \nabla \mathbf{u}_1), F_{u_1})) \\ &\quad\;\; - \lambda B(\mathbf{u}, \mathbf{u}_1, \tau, F) \;+\; \lambda B(\mathbf{u}_1, \tau, F) \;. \end{aligned} \tag{4.28}$$

We now proceed to bound E in terms of F, $\mathbf{u}$, and $\mathbf{u}_1$.

For the $\tilde{c}$ terms we have:

$$\begin{aligned} \tilde{c}(\mathbf{u}, \mathbf{u}, E) \;-\; \tilde{c}(\mathbf{u}_1, \mathbf{u}_2, E) \quad &= \quad \tilde{c}(\mathbf{u} - \mathbf{u}_1, \mathbf{u}, E) \;+\; \tilde{c}(\mathbf{u}_1, \mathbf{u} - \mathbf{u}_2, E) \\ &= \quad \tilde{c}(\mathbf{u} - \mathbf{u}_1, \mathbf{u}, E) \;+\; \tilde{c}(\mathbf{u}_1, E + \Lambda, E) \\ &= \quad \tilde{c}(\mathbf{u} - \mathbf{u}_1, \mathbf{u}, E) \;+\; \tilde{c}(\mathbf{u}_1, \Lambda, E) \;\; (\text{ using } (4.7)\;) \;. \end{aligned} \tag{4.29}$$

We estimate the first term on the rhs of (4.29) by

$$\begin{aligned} |\tilde{c}(\mathbf{u} - \mathbf{u}_1, \mathbf{u}, E)| \quad &\leq \quad C_1\,\|\mathbf{u} - \mathbf{u}_1\|^{1/2} \|\nabla(\mathbf{u} - \mathbf{u}_1)\|^{1/2} \|\nabla \mathbf{u}\| \|\nabla E\| \quad (\text{ using } (4.5)\;) \\ &\leq \quad \epsilon_1 \|\nabla E\|^2 \;+\; \frac{C_1^2}{4\epsilon_1} \|\mathbf{u} - \mathbf{u}_1\| \|\nabla(\mathbf{u} - \mathbf{u}_1)\| \|\nabla \mathbf{u}\|^2 . \end{aligned} \tag{4.30}$$

For the second term on the rhs of (4.29)

$$\begin{aligned} |\tilde{c}(\mathbf{u}_1, \Lambda, E)| \quad &\leq \quad |-\tilde{c}((\mathbf{u} - \mathbf{u}_1), \Lambda, E)| \;+\; |\tilde{c}(\mathbf{u}, \Lambda, E)| \\ &\leq \quad C_1\,\|\mathbf{u} - \mathbf{u}_1\|^{1/2} \|\nabla(\mathbf{u} - \mathbf{u}_1)\|^{1/2} \|\nabla \Lambda\| \|\nabla E\| \;+\; C_2 \|\mathbf{u}\|_\infty \|\nabla \Lambda\| \|\nabla E\| \\ &\leq \quad \epsilon_3 \|\nabla E\|^2 \;+\; \frac{C_1^2}{4\epsilon_3} \|\mathbf{u} - \mathbf{u}_1\|_1^2 \|\nabla \Lambda\|^2 \;+\; \epsilon_4 \|\nabla E\|^2 \;+\; \frac{C_2^2}{4\epsilon_4} \|\mathbf{u}\|_\infty^2 \|\nabla \Lambda\|^2 . \end{aligned} \tag{4.31}$$

In view of the estimates (4.11) and (4.9) we proceed next to consider the terms on the rhs of equation (4.28).

$$(\Lambda_t, E) \quad \leq \quad \|\Lambda_t\| \|E\| \;\leq\; \epsilon_5 \|\nabla E\|^2 \;+\; \frac{C_{PF}^2}{4\epsilon_5} \|\Lambda_t\|^2 \;, \tag{4.32}$$

$$(\Gamma_t, F) \quad \leq \quad \|\Gamma_t\| \|F\| \;\leq\; \epsilon_6 \|F\|^2 \;+\; \frac{1}{4\epsilon_6} \|\Gamma_t\|^2 \;. \tag{4.33}$$

For the pressure term we have

$$\begin{aligned} 2\alpha\,|(p, \nabla \cdot E)| \;=\; 2\alpha\,|((p - \mathcal{P}), \nabla \cdot E)| \quad &\leq \quad 2\alpha\,\|p - \mathcal{P}\| \|\nabla \cdot E\| \\ &\leq \quad 2\alpha \acute{d}^{1/2}\,\|p - \mathcal{P}\| \|\nabla E\| \\ &\leq \quad \frac{\alpha^2 \acute{d}}{\epsilon_7}\,\|p - \mathcal{P}\|^2 \;+\; \epsilon_7 \|\nabla E\|^2 \;. \end{aligned} \tag{4.34}$$

Writing out the A term on the rhs of (4.28) we have the terms

$$A(\mathbf{u}_1, (\Lambda, \Gamma), (E, F)) \;=\; (\Gamma, F_{u_1}) \,-\, 2\alpha(D(\Lambda), F_{u_1}) \,+\, 2\alpha(\Gamma, D(E)) \,+\, \alpha(1-\alpha)(\nabla\Lambda, \nabla E)\;. \tag{4.35}$$

For the first term in A:

$$\begin{aligned}
(\Gamma, F_{u_1}) \;&=\; (\Gamma, F) \,+\, (\Gamma, \nu h\, \mathbf{u}_1 \cdot \nabla F) \\
&=\; \|\Gamma\|\,\|F\| \,+\, \|\Gamma\|\,\nu h\,\|\mathbf{u}_1 \cdot \nabla F\| \\
&=\; \epsilon_8 \|F\|^2 \,+\, \frac{1}{4\epsilon_8}\|\Gamma\|^2 \,+\, \nu^2 h^2\,\|\mathbf{u}_1 \cdot \nabla F\|^2 \,+\, \frac{1}{4}\|\Gamma\|^2\;.
\end{aligned} \tag{4.36}$$

Similarly,

$$2\alpha\,(D(\Lambda), F_{u_1}) \;\le\; \epsilon_9\|F\|^2 \,+\, \frac{\alpha^2}{\epsilon_9}\|D(\Lambda)\|^2 \,+\, \nu^2 h^2\,\|\mathbf{u}_1 \cdot \nabla F\|^2 \,+\, \alpha^2\|D(\Lambda)\|^2\;, \tag{4.37}$$

$$2\alpha\,(\Gamma, D(E)) \;\le\; \epsilon_{10}\|\nabla E\|^2 \,+\, \frac{\alpha^2}{4\epsilon_{10}}\|\Gamma\|^2\;, \tag{4.38}$$

$$\alpha(1-\alpha)\,(\nabla\Gamma, \nabla E) \;\le\; \epsilon_{11}\|\nabla E\|^2 \,+\, \frac{\alpha^2(1-\alpha)^2}{4\epsilon_{11}}\|\nabla\Gamma\|^2\;. \tag{4.39}$$

Bounding the $g_a(\cdot,\cdot)$ terms on the rhs of (4.28) is more involved. We rewrite the difference as the sum of three terms and then bound each of the terms individually.

We have that

$$\begin{aligned}
(g_a(\tau, \nabla\mathbf{u}) - g_a(\tau_1, \nabla\mathbf{u}_1)\,,\, F_{u_1}) \;&=\; (g_a(\tau - \tau_1, \nabla\mathbf{u})\,,\, F_{u_1}) \,+\, (g_a(\tau_1, \nabla(\mathbf{u} - \mathbf{u}_1)\,,\, F_{u_1}) \\
&=\; (g_a(\tau - \tau_1, \nabla\mathbf{u})\,,\, F_{u_1}) \,+\, (g_a(\tau_1 - \tau, \nabla(\mathbf{u} - \mathbf{u}_1)\,,\, F_{u_1}) \\
&\quad\;+\, (g_a(\tau, \nabla(\mathbf{u} - \mathbf{u}_1)\,,\, F_{u_1})\;.
\end{aligned} \tag{4.40}$$

For the first term on the rhs of (4.40)

$$\begin{aligned}
(g_a(\tau - \tau_1, \nabla\mathbf{u})\,,\, F_{u_1}) \;&\le\; 4\|(\tau - \tau_1)\,\nabla\mathbf{u}\|\,\|F\| \;+\; 4\|(\tau - \tau_1)\,\nabla\mathbf{u}\|\,\|\nu h\,\mathbf{u}_1 \cdot \nabla F\| \\
&\le\; 4\acute{d}\,\|\nabla\mathbf{u}\|_\infty\|(\tau - \tau_1)\|\,\|F\| \,+\, 4\acute{d}\,\|\nabla\mathbf{u}\|_\infty\|(\tau - \tau_1)\|\,\|\nu h\,\mathbf{u}_1 \cdot \nabla F\| \\
&\le\; \epsilon_{12}\|F\|^2 \,+\, \frac{4\acute{d}^2}{\epsilon_{12}}\|\nabla\mathbf{u}\|_\infty^2\|(\tau - \tau_1)\|^2 \,+\, \nu^2 h^2\|\nu h\,\mathbf{u}_1 \cdot \nabla F\|^2 \\
&\quad\; + 4\acute{d}^2\|\nabla\mathbf{u}\|_\infty^2\|(\tau - \tau_1)\|^2\;.
\end{aligned} \tag{4.41}$$

For the second term

$$\begin{aligned}
(g_a(\tau - \tau_1, \nabla(\mathbf{u} - \mathbf{u}_1))\,,\, F_{u_1}) \;&\le\; 4\|(\tau - \tau_1)\,\nabla(\mathbf{u} - \mathbf{u}_1)\|\,\|F\| \;+\; 4\|(\tau - \tau_1)\,\nabla(\mathbf{u} - \mathbf{u}_1)\|\,\|\nu h\,\mathbf{u}_1 \cdot \nabla F\| \\
&\le\; \epsilon_{13}\|F\|^2 \,+\, \frac{4}{\epsilon_{13}}\|(\tau - \tau_1)\,\nabla(\mathbf{u} - \mathbf{u}_1)\|^2 \,+\, \nu^2 h^2\|\nu h\,\mathbf{u}_1 \cdot \nabla F\|^2 \\
&\quad\; + 4\|(\tau - \tau_1)\,\nabla(\mathbf{u} - \mathbf{u}_1)\|^2\;.
\end{aligned} \tag{4.42}$$

Note that

$$\|(\tau - \tau_1)\,\nabla(\mathbf{u} - \mathbf{u}_1)\| \le \|(\tau - \tau_1)\|_{L^4}\,\|\nabla(\mathbf{u} - \mathbf{u}_1)\|_{L^4}\,,$$

and, using (3.7),

$$
\begin{aligned}
\|\tau_1 - \mathcal{T}\|_{L^4} &\leq C_I h^{-\acute{d}/4} \|\tau_1 - \mathcal{T}\| \\
&\leq C_I h^{-\acute{d}/4} \|\tau_1 - \tau\| + C_I h^{-\acute{d}/4} \|\tau - \mathcal{T}\|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\tau - \tau_1\|_{L^4} &\leq \|\tau - \mathcal{T}\|_{L^4} + \|\mathcal{T} - \tau_1\|_{L^4} \\
&\leq \|\tau - \mathcal{T}\|_{L^4} + C h^{-\acute{d}/4} \|\tau_1 - \tau\| + C h^{-\acute{d}/4} \|\tau - \mathcal{T}\| \\
&\leq 2 C_I h^{m+1-\acute{d}/4} \|\tau\|_{m+1} + C_I h^{-\acute{d}/4} \|\tau_1 - \tau\| \, . \qquad (4.43)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|\nabla(\mathbf{u} - \mathbf{u}_1)\|_{L^4} &\leq \|\nabla(\mathbf{u} - \mathcal{U})\|_{L^4} + C h^{-\acute{d}/4} \|\mathbf{u} - \mathbf{u}_1\|_1 + C h^{-\acute{d}/4} \|\mathbf{u} - \mathcal{U}\|_1 \\
&\leq 2 C_I h^{k-\acute{d}/4} \|\mathbf{u}\|_{k+1} + C_I h^{-\acute{d}/4} \|\mathbf{u} - \mathbf{u}_1\|_1 \, . \qquad (4.44)
\end{aligned}
$$

Combining (4.43),(4.44) with (4.42) yields

$$
\begin{aligned}
|(g_a(\tau - \tau_1, \nabla(\mathbf{u} - \mathbf{u}_1)) \, , \, F_{u_1})| &\leq \epsilon_{13} \|F\|^2 + \nu^2 h^2 \|\mathbf{u}_1 \cdot \nabla F\|^2 \\
&\quad + \left(\frac{4}{\epsilon_{13}} + 4 \right) \left(2 C_I h^{m+1-\acute{d}/4} \|\tau\|_{m+1} + C_I h^{-\acute{d}/4} \|\tau_1 - \tau\| \right)^2 \cdot \\
&\quad \left(2 C_I h^{k-\acute{d}/4} \|\mathbf{u}\|_{k+1} + C_I h^{-\acute{d}/4} \|\mathbf{u} - \mathbf{u}_1\|_1 \right)^2 \, . \qquad (4.45)
\end{aligned}
$$

For the third $g_a(\cdot, \cdot)$ terms on the rhs of (4.40) we have

$$
\begin{aligned}
|(g_a(\tau, \nabla(\mathbf{u} - \mathbf{u}_1)) \, , \, F_{u_1})| &\leq 4 \|\tau \, \nabla(\mathbf{u} - \mathbf{u}_1)\| \, \|F\| + 4 \|\tau \, \nabla(\mathbf{u} - \mathbf{u}_1)\| \, \|\nu h \, \mathbf{u}_1 \cdot \nabla F\| \\
&\leq 4\acute{d} \, \|\tau\|_\infty \|\nabla(\mathbf{u} - \mathbf{u}_1)\| \, \|F\| + 4\acute{d} \, \|\tau\|_\infty \|\nabla(\mathbf{u} - \mathbf{u}_1)\| \, \|\nu h \, \mathbf{u}_1 \cdot \nabla F\| \\
&\leq \epsilon_{14} \|F\| + \frac{4\acute{d}^2}{\epsilon_{14}} \|\tau\|_\infty^2 \|\nabla(\mathbf{u} - \mathbf{u}_1)\|^2 + \nu^2 h^2 \, \|\mathbf{u}_1 \cdot \nabla F\|^2 \\
&\quad + 4\acute{d}^2 \|\tau\|_\infty^2 \|\nabla(\mathbf{u} - \mathbf{u}_1)\|^2 . \qquad (4.46)
\end{aligned}
$$

What remains is to estimate the three B terms on the rhs of (4.28). We begin by rewriting the terms in a more convenient form.

$$
\begin{aligned}
-B(\mathbf{u}_1, \Gamma, F) - B(\mathbf{u}, \mathbf{u}_1, \tau, F) + B(\mathbf{u}_1, \tau, F) &= B(\mathbf{u}_1, \mathcal{T}, F) - B(\mathbf{u}, \mathbf{u}_1, \tau, F) \\
&= -B(\mathbf{u} - \mathbf{u}_1, \mathbf{u}_1, \mathcal{T}, F) - B(\mathbf{u}, \mathbf{u}_1, \Gamma, F) \\
&= B(\mathbf{u} - \mathbf{u}_1, \mathbf{u}_1, \Gamma, F) - B(\mathbf{u} - \mathbf{u}_1, \mathbf{u}_1, \tau, F) \\
&\quad - B(\mathbf{u}, \mathbf{u}_1, \Gamma, F) \, . \qquad (4.47)
\end{aligned}
$$

For the first B term in (4.47) we have

$$
\begin{aligned}
B(\mathbf{u} - \mathbf{u}_1, \mathbf{u}_1, \Gamma, F) &= ((\mathbf{u} - \mathbf{u}_1) \cdot \nabla \Gamma, F) + ((\mathbf{u} - \mathbf{u}_1) \cdot \nabla \Gamma, \nu h \, \mathbf{u}_1 \cdot \nabla F) + \frac{1}{2} (\nabla \cdot (\mathbf{u} - \mathbf{u}_1) \, \Gamma, F) \\
&\leq \|(\mathbf{u} - \mathbf{u}_1) \cdot \nabla \Gamma\| \|F\| + \|(\mathbf{u} - \mathbf{u}_1) \cdot \nabla \Gamma\| \|\nu h \, \mathbf{u}_1 \cdot \nabla F\|
\end{aligned}
$$

$$
\begin{aligned}
& \quad + \frac{1}{2}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_1)\,\Gamma\|\|F\| \\
\le\ & \epsilon_{15}\|F\|^2 + (\frac{1}{4\epsilon_{15}} + \frac{1}{4})\|(\mathbf{u}-\mathbf{u}_1)\cdot\nabla\Gamma\|^2 + \nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
& \quad + \epsilon_{16}\|F\|^2 + \frac{1}{16\epsilon_{16}}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_1)\Gamma\|^2\,. \qquad (4.48)
\end{aligned}
$$

For I_u the interpolant of $\mathbf{u}$ we have, using (3.7),(3.8),

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_1\|_\infty &\le \|\mathbf{u}-I_u\|_\infty + \|I_u-\mathbf{u}_1\|_\infty \\
&\le C_nh^{k+1-\acute{d}/2}\|\mathbf{u}\|_{k+1} + C_vh^{-\acute{d}/2}\|I_u-\mathbf{u}_1\| \\
&\le C_nh^{k+1-\acute{d}/2}\|\mathbf{u}\|_{k+1} + C_vh^{-\acute{d}/2}\|I_u-\mathbf{u}\| + C_vh^{-\acute{d}/2}\|\mathbf{u}-\mathbf{u}_1\| \\
&\le C_{nv}h^{k+1-\acute{d}/2}\|\mathbf{u}\|_{k+1} + C_vh^{-\acute{d}/2}\|\mathbf{u}-\mathbf{u}_1\|\,. \qquad (4.49)
\end{aligned}
$$

Using this estimate we obtain that

$$
\begin{aligned}
\|(\mathbf{u}-\mathbf{u}_1)\cdot\nabla\Gamma\| &\le \acute{d}\|\mathbf{u}-\mathbf{u}_1\|_\infty\|\nabla\Gamma\| \\
&\le \acute{d}\left(C_{nv}h^{k+1-\acute{d}/2}\|\mathbf{u}\|_{k+1} + C_vh^{-\acute{d}/2}\|\mathbf{u}-\mathbf{u}_1\|\right)\|\nabla\Gamma\|\,. \qquad (4.50)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|\nabla\cdot(\mathbf{u}-\mathbf{u}_1)\,\Gamma\| &\le \acute{d}^{3/2}\|\nabla(\mathbf{u}-\mathbf{u}_1)\|\|\Gamma\|_\infty \\
&\le C_{vi}\acute{d}^{3/2}h^{m+1-\acute{d}/2}\,\|\mathbf{u}-\mathbf{u}_1\|_1\|\tau\|_{m+1}\,. \qquad (4.51)
\end{aligned}
$$

Combining (4.48),(4.50), and (4.51) we have

$$
\begin{aligned}
B(\mathbf{u}-\mathbf{u}_1,\mathbf{u}_1,\Gamma,F) &\le (\epsilon_{15} + \epsilon_{16})\|F\|^2 + \nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + (\frac{1}{4\epsilon_{15}} + \frac{1}{4})\acute{d}^2\left(C_{nv}h^{k+1-\acute{d}/2}\|\mathbf{u}\|_{k+1} + C_vh^{-\acute{d}/2}\|\mathbf{u}-\mathbf{u}_1\|\right)^2\|\nabla\Gamma\|^2 \\
&\quad + \frac{1}{16\epsilon_{16}}\left(C_{vi}\acute{d}^{3/2}h^{m+1-\acute{d}/2}\,\|\mathbf{u}-\mathbf{u}_1\|_1\|\tau\|_{m+1}\right)^2\,. \qquad (4.52)
\end{aligned}
$$

For the second B term on the rhs of (4.47)

$$
\begin{aligned}
B(\mathbf{u}-\mathbf{u}_1,\mathbf{u}_1,\tau,F) &= ((\mathbf{u}-\mathbf{u}_1)\cdot\nabla\tau,F) + ((\mathbf{u}-\mathbf{u}_1)\cdot\nabla\tau,\nu h\,\mathbf{u}_1\cdot\nabla F) + \frac{1}{2}(\nabla\cdot(\mathbf{u}-\mathbf{u}_1)\,\tau,F) \\
&\le \|(\mathbf{u}-\mathbf{u}_1)\cdot\nabla\tau\|\,\|F\| + \|(\mathbf{u}-\mathbf{u}_1)\cdot\nabla\tau\|\,\|\nu h\,\mathbf{u}_1\cdot\nabla F\| + \frac{1}{2}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_1)\,\tau\|\,\|F\| \\
&\le \epsilon_{17}\|F\|^2 + \frac{1}{4\epsilon_{17}}\|(\mathbf{u}-\mathbf{u}_1)\cdot\nabla\tau\|^2 + \nu^2h^2\,\|\mathbf{u}_1\cdot\nabla F\|^2 + \frac{1}{4}\|(\mathbf{u}-\mathbf{u}_1)\cdot\nabla\tau\|^2 \\
&\quad + \epsilon_{18}\|F\|^2 + \frac{1}{16\epsilon_{18}}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_1)\,\tau\|^2 \\
&\le (\epsilon_{17} + \epsilon_{18})\|F\|^2 + \nu^2h^2\,\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \acute{d}^3\left(\frac{1}{4\epsilon_{17}} + \frac{1}{4}\right)\|\nabla\tau\|_\infty^2\,\|\mathbf{u}-\mathbf{u}_1\|^2 + \frac{\acute{d}^3}{16\epsilon_{18}}\|\tau\|_\infty^2\|\nabla(\mathbf{u}-\mathbf{u}_1)\|^2\,. \qquad (4.53)
\end{aligned}
$$

For the third B term on the rhs of (4.47)

$$
\begin{aligned}
B(\mathbf{u}, \mathbf{u}_1, \Gamma, F) &= (\mathbf{u}\cdot\nabla\Gamma, F) + (\mathbf{u}\cdot\nabla\Gamma, \nu h\, \mathbf{u}_1\cdot\nabla F) + \frac{1}{2}(\nabla\cdot\mathbf{u}\,\Gamma, F) \\
&\le \|\mathbf{u}\cdot\nabla\Gamma\|\,\|F\| + \|\mathbf{u}\cdot\nabla\Gamma\|\,\nu h\,\|\mathbf{u}_1\cdot\nabla F\| + \frac{1}{2}\|\nabla\cdot\mathbf{u}\,\Gamma\|\,\|F\| \\
&\le (\epsilon_{19} + \epsilon_{20})\|F\|^2 + \nu^2 h^2\,\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \acute{d}\left(\frac{1}{4\epsilon_{19}} + \frac{1}{4}\right)\|\mathbf{u}\|_\infty^2\|\nabla\Gamma\|^2 + \frac{\acute{d}^2}{16\epsilon_{20}}\|\nabla\mathbf{u}\|_\infty^2\|\Gamma\|^2 . \qquad (4.54)
\end{aligned}
$$

Returning to (4.28) and putting everything back together:

$$
\begin{aligned}
\alpha Re\,\frac{d}{dt}\|E\|^2 \;&+\; \frac{\lambda}{2}\frac{d}{dt}\|F\|^2 + \frac{1}{2}\|F\|^2 + \frac{\alpha(1-\alpha)}{2}\|\nabla E\|^2 \\
&\left(\nu h - \frac{1}{2}\nu^2h^2 - \frac{\alpha\nu^2h^2}{2(1-\alpha)}\right)\|\mathbf{u}_1\cdot\nabla F\|^2 - 2\alpha(\epsilon_1+\epsilon_3+\epsilon_4)\|\nabla E\|^2 \\
&- 2\alpha\frac{C_1^2}{4\epsilon_1}\|\nabla\mathbf{u}\|^2\|\mathbf{u}-\mathbf{u}_1\|_1^2 - 2\alpha\frac{C_1^2}{4\epsilon_3}\|\mathbf{u}-\mathbf{u}_1\|_1^2\|\nabla\Lambda\|^2 - 2\alpha\frac{C_2^2}{4\epsilon_4}\|\mathbf{u}\|_\infty^2\|\nabla\Lambda\|^2 \\
&\le\; 2\alpha Re\,\frac{C_{PF}^2}{4\epsilon_5}\|\Lambda_t\|^2 + 2\alpha\epsilon_5\|\nabla E\|^2 + \frac{1}{4\epsilon_6}\|\Gamma\|^2 + \epsilon_6\|F\|^2 \\
&\quad + \epsilon_8\|F\|^2 + \frac{1}{4\epsilon_8}\|\Gamma\|^2 + \frac{1}{4}\|\Gamma\|^2 + \nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \frac{1}{4}\|\Gamma\|^2 + \nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 + \frac{\alpha^2}{2\epsilon_9}\|\nabla\Lambda\|^2 + \epsilon_9\|F\|^2 + \frac{\alpha^2}{2}\|\nabla\Lambda\|^2 + \nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \frac{\alpha^2}{4\epsilon_{10}}\|\Gamma\|^2 + \epsilon_{10}\|\nabla E\|^2 + \frac{\alpha^2(1-\alpha)^2}{4\epsilon_{11}}\|\nabla\Gamma\|^2 + \epsilon_{11}\|\nabla E\|^2 \\
&\quad + \frac{\lambda}{4\epsilon_6}\|\Gamma_t\|^2 + \lambda\epsilon_6\|F\|^2 + + \frac{\alpha^2}{\epsilon_7}\acute{d}\|p-\mathcal{P}\|^2 + \epsilon_7\|\nabla E\|^2 \\
&\quad + \lambda\frac{4\acute{d}^2}{\epsilon_{12}}\|\nabla\mathbf{u}\|_\infty^2\|\tau-\tau_1\|^2 + \lambda 4\acute{d}^2\,\|\nabla\mathbf{u}\|_\infty^2\|\tau-\tau_1\|^2 + \lambda\epsilon_{12}\|F\|^2 + \lambda\nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \lambda\left(\frac{4}{\epsilon_{13}} + 4\right)\left(2C_I h^{m+1-\acute{d}/4}\|\tau\|_{m+1} + C_I h^{-\acute{d}/4}\|\tau_1-\tau\|\right)^2 \cdot \\
&\qquad\qquad \left(2C_I h^{k-\acute{d}/4}\|\mathbf{u}\|_{k+1} + C_I h^{-\acute{d}/4}\|\mathbf{u}-\mathbf{u}_1\|_1\right)^2 \\
&\quad + \lambda\epsilon_{13}\|F\|^2 + \lambda\nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 + \lambda 4\acute{d}^2\left(\frac{1}{\epsilon_{14}} + 1\right)\|\tau\|_\infty^2\|\mathbf{u}-\mathbf{u}_1\|^2 + \lambda\epsilon_{14}\|F\|^2 \\
&\quad + \lambda\nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \lambda(\frac{1}{4\epsilon_{15}} + \frac{1}{4})\acute{d}^2\left(C_{nv}h^{k+1-\acute{d}/2}\|\mathbf{u}\|_{k+1} + C_v h^{-\acute{d}/2}\|\mathbf{u}-\mathbf{u}_1\|\right)^2\|\nabla\Gamma\|^2 \\
&\quad + \lambda\frac{1}{16\,\epsilon_{16}}\left(C_{vi}\acute{d}^{3/2}h^{m+1-\acute{d}/2}\,\|\mathbf{u}-\mathbf{u}_1\|_1\|\tau\|_{m+1}\right)^2 + \lambda\,(\epsilon_{15} + \epsilon_{16})\|F\|^2 \\
&\quad + \lambda\,\nu^2h^2\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \lambda\,(\epsilon_{17} + \epsilon_{18})\|F\|^2 + \lambda\,\nu^2h^2\,\|\mathbf{u}_1\cdot\nabla F\|^2 \\
&\quad + \lambda\,\acute{d}^3\left(\frac{1}{4\epsilon_{17}} + \frac{1}{4}\right)\|\nabla\tau\|_\infty\,\|\mathbf{u}-\mathbf{u}_1\| + \lambda\,\frac{\acute{d}^3}{16\epsilon_{18}}\|\tau\|_\infty\|\nabla(\mathbf{u}-\mathbf{u}_1)\|^2
\end{aligned}
$$

$$+\lambda\,(\epsilon_{19}\,+\,\epsilon_{20})\|F\|^2\,+\,\lambda\,\nu^2h^2\,\|\mathbf{u}_1\cdot\nabla F\|^2$$
$$+\,\lambda\,\acute{d}\left(\frac{1}{4\epsilon_{19}}+\frac{1}{4}\right)\|\mathbf{u}\|_\infty^2\|\nabla\Gamma\|^2\,+\,\lambda\,\frac{\acute{d}^2}{16\epsilon_{20}}\|\nabla\mathbf{u}\|_\infty^2\|\Gamma\|^2\,. \tag{4.55}$$

Now, rewriting (4.55) with all the E and F terms on the LHS and the RHS terms written in terms "controlled" by the *ball*, terms controlled by *interpolation approximation*, and terms controlled by both the *ball* and *interpolation approximation*, we have:

$$\begin{aligned}
\alpha Re\,\frac{d}{dt}\|E\|^2 \;&+\; \frac{\lambda}{2}\frac{d}{dt}\|F\|^2 \;+\; \left(\frac{\alpha(1-\alpha)}{2}-2\alpha(\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)-(\epsilon_7+\epsilon_{10}+\epsilon_{11})\right)\|\nabla E\|^2 \\
&+\left(\frac{1}{2}-(\epsilon_6+\epsilon_8+\epsilon_9)-\lambda(\epsilon_6+\epsilon_{12}+\epsilon_{13}+\epsilon_{14}+\epsilon_{15}+\epsilon_{16}+\epsilon_{17}+\epsilon_{18}+\epsilon_{19}+\epsilon_{20})\right)\|F\|^2 \\
&+\left(\nu h-\nu^2h^2(\frac{7}{2}-\frac{\alpha}{2(1-\alpha)}+6\lambda)\right)\|\mathbf{u}_1\cdot\nabla F\|^2 \\
\le\;& \|\mathbf{u}-\mathbf{u}_1\|^2\left\{\lambda 4\acute{d}^2\left(\frac{1}{\epsilon_{14}}+1\right)\|\tau\|_\infty^2+\lambda\acute{d}^3\left(\frac{1}{4\epsilon_{17}}+\frac{1}{4}\right)\|\nabla\tau\|_\infty^2\right\} \\
&+\|\tau-\tau_1\|^2\left\{\lambda\frac{4\acute{d}^2}{\epsilon_{12}}\|\nabla\mathbf{u}\|_\infty^2\,+\,4\lambda\acute{d}^2\|\nabla\mathbf{u}\|_\infty^2\right\} \\
&+\|\mathbf{u}-\mathbf{u}_1\|_1^2\left\{2\alpha\frac{C_1^2}{4\epsilon_1}\|\nabla\mathbf{u}\|^2+\lambda\frac{\acute{d}^3}{16\epsilon_{18}}\|\tau\|_\infty^2\right\} \\
&+\|\mathbf{u}-\mathbf{u}_1\|_1^2\|\tau-\tau_1\|^2\left\{\lambda\left(\frac{4}{\epsilon_{13}}+4\right)C_I^2 4h^{-\acute{d}}\right\} \\
&+\|\Gamma\|^2\left\{\frac{1}{4}+\frac{1}{4\epsilon_6}+\frac{\alpha^2}{4\epsilon_{10}}+\lambda\frac{\acute{d}^2}{16\epsilon_{20}}\|\nabla\mathbf{u}\|_\infty^2\right\}\;+\;\|\Gamma_t\|^2\left\{\frac{\lambda}{4\epsilon_6}\right\} \\
&+\|\nabla\Gamma\|^2\left\{\frac{\alpha^2(1-\alpha)^2}{4\epsilon_{11}}\,+\,2C_{nv}^2h^{2k+2-\acute{d}}\|\mathbf{u}\|_{k+1}^2\,+\,\lambda\acute{d}\left(\frac{1}{4\epsilon_{19}}+\frac{1}{4}\right)\|\mathbf{u}\|_\infty^2\right\} \\
&+\|\Lambda_t\|^2\left\{2\alpha Re\frac{C_{PF}^2}{4\epsilon_5}\right\}\,+\,\|\nabla\Lambda\|^2\left\{2\alpha\frac{C_2^2}{4\epsilon_4}\|\mathbf{u}\|_\infty^2\,+\,\frac{\alpha^2}{2\epsilon_9}\,+\,\frac{\alpha^2}{2}\right\} \\
&+\|p-\mathcal{P}\|^2\left\{\acute{d}\frac{\alpha^2}{\epsilon_7}\right\}\,+\,\lambda\left(\frac{4}{\epsilon_{13}}+4\right)C_I^2 64h^{2m+2k+2-\acute{d}}\|\tau\|_{m+1}^2\|\mathbf{u}\|_{k+1}^2 \\
&+\|\mathbf{u}-\mathbf{u}_1\|^2\left\{2\lambda\left(\frac{1}{4\epsilon_{15}}+\frac{1}{4}\right)\acute{d}^2C_v^2h^{-\acute{d}}\|\nabla\Gamma\|^2\right\} \\
&+\|\mathbf{u}-\mathbf{u}_1\|_1^2\left\{2\alpha\frac{C_1^2}{4\epsilon_3}\|\nabla\Lambda\|^2\right\} \\
&+\|\tau-\tau_1\|^2\left\{\lambda\left(\frac{4}{\epsilon_{13}}+4\right)C_I^2 16h^{2k-\acute{d}}\|\mathbf{u}\|_{k+1}^2\right\} \\
&+\|\mathbf{u}-\mathbf{u}_1\|_1^2\left\{\lambda\left(\frac{4}{\epsilon_{13}}+4\right)C_I^2 16h^{2m+2-\acute{d}}\|\tau\|_{m+1}^2\,+\,\frac{1}{16\epsilon_{16}}C_{vi}^2\acute{d}^3h^{2m+2-\acute{d}}\|\tau\|_{m+1}^2\right\}.
\end{aligned} \tag{4.56}$$

With our assumptions that $0<\alpha<1$, and $\lambda>0$, we can choose values for the ϵ_i's, and νh sufficiently small such that the left hand side of (4.56) is bounded below by

$$\alpha Re\,\frac{d}{dt}\|E\|^2\,+\,\frac{\lambda}{2}\frac{d}{dt}\|F\|^2\,+\,\frac{1}{4}\|F\|^2\,+\,\frac{\alpha(1-\alpha)}{4}\|\nabla E\|^2\,+\,\frac{\nu h}{2}\|\mathbf{u}_1\cdot\nabla F\|^2\,. \tag{4.57}$$

Let D_i, $i = 1, \ldots, 6$ denote constants dependent upon $\mathbf{u}$, p, τ, their derivatives and T. (Recall the definition of c^*, R in (4.22), and D_0 in theorem 4.1.) As usual C_j, $j = 4, \ldots, 10$, denote constants independent of the solution $\mathbf{u}$, p, τ and the mesh parameter h.

Using (4.57) and integrating (4.56) we obtain

$$\begin{aligned} \|E\|^2(t) \ + \ \|F\|^2(t) \ + \ \int_0^t \|\nabla E\|^2(s)\,ds \ \ \leq \ \ & R^2 C_4 D_0 \\ & + R^4 C_5 h^{-\acute{d}} \\ & + D_1 h^{2m+2} \ + \ D_2 h^{2m+2} \\ & + C_6 D_0 h^{2m} \ + \ D_3 h^{2k+2m+2-\acute{d}} \\ & + D_4 h^{2k+2} \ + \ C_7 D_0 h^{2k} \\ & + C_8 D_0 h^{2q+2} \ + \ D_5 h^{2k+2m+2-\acute{d}} \\ & + R^2 C_9 D_0 h^{2m-\acute{d}} \\ & + R^2 D_6 h^{2k} \\ & + R^2 C_6 D_0 h^{2k-\acute{d}} \\ & + R^2 C_7 D_0 h^{2m+2-\acute{d}} \ . \end{aligned} \tag{4.58}$$

Now, in view of (4.27), we have that for h, D_0, and c^* sufficiently small

$$\begin{aligned} \|\tau - \tau_2\|^2(t) \ & \leq \ 2\|F\|^2(t) \ + \ 2\|\Gamma\|^2(t) \\ & \leq \ cR^2 + C_{10} D_0 \left(h^{2m} + h^{2k}\right) + 2D_0 h^{2m+2} \\ & \leq \ \tilde{c} R^2 \ , \end{aligned} \tag{4.59}$$

where $0 < \tilde{c} < 1$. Similarly, for h sufficiently small

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_2\|^2(t) \ & \leq \ 2\|E\|^2(t) \ + \ 2\|\Lambda\|^2(t) \\ & \leq \ cR^2 + C_{10} D_0 \left(h^{2m} + h^{2k}\right) + 2D_0 h^{2k+2} \end{aligned} \tag{4.60}$$

$$\text{hence} \quad \int_0^T \|\mathbf{u} - \mathbf{u}_2\|^2(t)\,dt \ \leq \ \frac{\tilde{c}}{2} R^2 \ . \tag{4.61}$$

Also, for h sufficiently small

$$\begin{aligned} \int_0^T \|\nabla(\mathbf{u} - \mathbf{u}_2)\|^2(t)\,dt \ & \leq \ 2\int_0^T \|\nabla E\|^2(t)\,dt \ + \ 2\int_0^T \|\nabla \Lambda\|^2(t)\,dt \\ & \leq \ c1R^2 \ + \ 2D_0 T h^{2k} \\ & \leq \ \frac{\tilde{c}}{2} R^2 \ . \end{aligned} \tag{4.62}$$

Combining (4.59)–4.62) we have that for h sufficiently small that ξ is a strict contraction on the *ball* defined in (4.22).

Step 4: A direct application of Schauder's fixed point theorem now establishes the uniqueness of the approximation and the stated error estimates.

■

5 Fully-Discrete Approximation

In this section we analyse a fully discrete approximation to (2.14),(2.15).

We assume that the fluid flow satisfies the following properties:

$$\|\mathbf{u}\|_\infty\,,\ \|\tau\|_\infty\,,\ \|\nabla\mathbf{u}\|_\infty\,,\ \|\nabla\tau\|_\infty\ \le M, \tag{5.1}$$

for all $t \in [0,T]$.

Note that it follows from (5.1) and inverse estimates that

$$\|\mathcal{U}^n\|_\infty\,, \|\nabla\mathcal{U}^n\|_\infty \le \tilde{M} \approx M\ . \tag{5.2}$$

Below, for simplicity, we take $\tilde{M} = M$.

To simplify the notation, the following definition is used in the analysis.
Definitions:

$$b(\mathbf{u},\tau,\psi)\ :=\ (\mathbf{u}\cdot\nabla\tau,\psi)\,. \tag{5.3}$$

To obtain the fully discretized approximation, the time derivatives are replaced by backward differences and the nonlinear terms are lagged. As we are assuming "slow flow", i.e. $Re \equiv O(1)$, we use a conforming finite element method to discretize the momentum equation. For the constitutive equation for stress, we use a streamline upwind Petrov-Galerkin (SUPG) discretization to control the production of spurious oscillations in the approximation. The discrete approximating system of equations is then:

Approximating System
For $n = 1,2,\ldots,N$, find $\mathbf{u}_h^n \in Z_h, \tau_\mathbf{h}^\mathbf{n} \in S_h$ such that

$$Re\,(d_t\mathbf{u}_h^n,\mathbf{v}) + Re\ c\left(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,\mathbf{v}\right) + (1-\alpha)\,(\nabla\mathbf{u}_h^n,\nabla\mathbf{v}) + (\tau_h^n, D(\mathbf{v})) = (\mathbf{f}^n,\mathbf{v})\,,\ \mathbf{v}\in Z_h \tag{5.4}$$

$$\frac{1}{\lambda}\,(\tau_h^n,\tilde{\sigma}) + (d_t\tau_h^n,\sigma) + b\left(\mathbf{u}_h^{n-1},\tau_h^n,\tilde{\sigma}\right) - \overline{\lambda}\,(D(\mathbf{u}_h^n),\tilde{\sigma}) = -\left(g_a(\tau_h^{n-1},\nabla\mathbf{u}_h^{n-1}),\tilde{\sigma}\right)\,,\ \sigma\in S_h \tag{5.5}$$

where $\tilde{\sigma} := \sigma + \nu\sigma_u^n$, $\sigma_u^n := \mathbf{u}_h^{n-1}\cdot\nabla\sigma$, ν is a small positive constant, and $\overline{\lambda} = \lambda/(2\alpha)$.

The parameter $\nu > 0$ is used to supress the production of spurious oscillations in the approximation. Note that for $\nu = 0$ the discretization of the constitutive equation is a conforming Galerkin method. The goal in choosing ν is to keep it as small as possible, but large enough to control the generation of catastrophic spurious oscillations in the approximate stress.

To ensure computability of the algorithm, we begin by showing that (5.4)-(5.5) is uniquely solvable for $\mathbf{u}_h$ and τ_h at each time step n. We use the following induction hypothesis.

$$(IH1) \qquad \left\|\mathbf{u}_h^{n-1}\right\|_\infty, \left\|\tau_h^{n-1}\right\|_\infty \le K.$$

Lemma 6 *Assume (IH1) is true. For sufficiently small step size Δt, there exists a unique solution $(\mathbf{u}_h^n,\tau_h^n) \in Z_h \times S_h$ satisfying (5.4)-(5.5).*

Proof: For notational simplicity, in this proof we drop the subscript h from the variables. Choosing $\mathbf{v} = \mathbf{u}_h^n, \sigma = \tau_h^n$, multiplying (5.4) by $\overline{\lambda}$ and adding to (5.5) we obtain

$$a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n) = \overline{\lambda}\,(\mathbf{f}^n, \mathbf{u}^n) + \overline{\lambda}\frac{Re}{\Delta t}\left(\mathbf{u}^{n-1}, \mathbf{u}^n\right) - \left(g_a\left(\tau^{n-1}, \nabla \mathbf{u}^{n-1}\right), \tilde{\tau}^n\right) + \frac{1}{\Delta t}\left(\tau^{n-1}, \tau^n\right), \quad (5.6)$$

where the bilinear form $a(\mathbf{u}, \tau; \mathbf{v}, \sigma)$ is defined as:

$$\begin{aligned} a(\mathbf{u}, \tau; \mathbf{v}, \sigma) \;:=\; & \overline{\lambda}\frac{Re}{\Delta t}(\mathbf{u}, \mathbf{v}) + \overline{\lambda}\, Re\; c(\mathbf{u}^{n-1}, \mathbf{u}, \mathbf{v}) + \overline{\lambda}(1-\alpha)(\nabla \mathbf{u}, \nabla \mathbf{v}) + \frac{1}{\lambda}(\tau, \tilde{\sigma}) + \frac{1}{\Delta t}(\tau, \sigma) \\ & + b\left(\mathbf{u}^{n-1}, \tau, \sigma\right) + b\left(\mathbf{u}^{n-1}, \tau, \nu \mathbf{u}^{n-1} \cdot \nabla \sigma\right) - \overline{\lambda}\left(D(\mathbf{u}), \nu \mathbf{u}^{n-1} \cdot \nabla \sigma\right). \end{aligned}$$

We now estimate the terms in $a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n)$. We have

$$\begin{aligned} \left|c(\mathbf{u}^{n-1}, \mathbf{u}, \mathbf{u})\right| = \left|\left(\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}, \mathbf{u}\right)\right| &\leq \acute{d}^{\frac{1}{2}} \left\|\mathbf{u}^{n-1}\right\|_{\infty} \|\nabla \mathbf{u}\|\, \|\mathbf{u}\| \\ &\leq \epsilon_1 \|\nabla \mathbf{u}\|^2 + \frac{\acute{d}K^2}{4\epsilon_1} \|\mathbf{u}\|^2, \\ \left|b(\mathbf{u}^{n-1}, \tau, \tau)\right| = \left|\left(\mathbf{u}^{n-1} \cdot \nabla \tau, \tau\right)\right| &\leq \left\|\mathbf{u}^{n-1} \cdot \nabla \tau\right\| \|\tau\| \\ &\leq \epsilon_2 \left\|\mathbf{u}^{n-1} \cdot \nabla \tau\right\|^2 + \frac{1}{4\epsilon_2}\|\tau\|^2. \\ b(\mathbf{u}^{n-1}, \tau, \nu \mathbf{u}^{n-1} \cdot \nabla \tau) &= \nu \left\|\mathbf{u}^{n-1} \cdot \nabla \tau\right\|^2, \\ \left|\left(D(\mathbf{u}), \nu \mathbf{u}^{n-1} \cdot \nabla \tau\right)\right| &\leq \|D(\mathbf{u})\| \left\|\nu \mathbf{u}^{n-1} \cdot \nabla \tau\right\| \\ &\leq \epsilon_3 \|D(\mathbf{u})\|^2 + \frac{\nu^2}{4\epsilon_3}\left\|\mathbf{u}^{n-1} \cdot \nabla \tau\right\|^2 \\ &\leq \epsilon_3 \|\nabla \mathbf{u}\|^2 + \frac{\nu^2}{4\epsilon_3}\left\|\mathbf{u}^{n-1} \cdot \nabla \tau\right\|^2. \end{aligned}$$

Applying these inequalities to the bilinear form $a(\cdot, \cdot; \cdot, \cdot)$ yields

$$\begin{aligned} a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n) \;\geq\; & \overline{\lambda}\, Re\left(\frac{1}{\Delta t} - \frac{\acute{d}K^2}{4\epsilon_1}\right)\|\mathbf{u}^n\|^2 + \overline{\lambda}\left((1-\alpha) - Re\;\epsilon_1 - \epsilon_3\right)\|\nabla \mathbf{u}\|^2 \\ & + \left(\frac{1}{\lambda} + \frac{1}{\Delta t} - \frac{1}{4\epsilon_2}\right)\|\tau^n\|^2 + \left(\nu - \epsilon_2 - \frac{\nu^2}{4\epsilon_3}\right)\left\|\mathbf{u}^{n-1} \cdot \nabla \tau^n\right\|^2. \end{aligned}$$

Choosing $\epsilon_1 = \frac{(1-\alpha)}{4\,Re}, \epsilon_2 = \frac{\nu}{3}, \epsilon_3 = \frac{(1-\alpha)}{4}, \nu \leq \frac{2(1-\alpha)}{3}$, and $\Delta t \leq \min\left\{\frac{1-\alpha}{Re\;\acute{d}K^2}, \nu\right\}$, it follows that the bilinear form $a(\cdot, \cdot; \cdot, \cdot)$ is positive. Hence, (5.6) has at most one solution. Since (5.6) is a finite dimensional linear system, the uniqueness of the solution implies the existence of the solution.

■

The discrete Gronwall's lemma plays an important role in the following analysis.

Lemma 7 (Discrete Gronwall's Lemma) *[9] Let Δt, H, and a_n, b_n, c_n, γ_n, (for integers $n \geq 0$), be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \;\leq\; \Delta t \sum_{n=0}^{l} \gamma_n\, a_n \;+\; \Delta t \sum_{n=0}^{l} c_n \;+\; H \quad \textit{for } l \geq 0.$$

Suppose that $\Delta t\, \gamma_n < 1$, for all n, and set $\sigma_n = (1 - \Delta t\, \gamma_n)^{-1}$. Then,

$$a_l + \Delta t \sum_{n=0}^{l} b_n \le \exp\left(\Delta t \sum_{n=0}^{l} \sigma_n \gamma_n\right) \left\{\Delta t \sum_{n=0}^{l} c_n + H\right\} \quad \text{for } l \ge 0 . \tag{5.7}$$

■

5.1 Analysis of the fully–discrete approximation

In this section we analyze the error between the finite element approximation given by (5.4),(5.5) and the true solution. A priori error estimates for the approximation are in theorem 5.2.

Theorem 5.2 *Assume that the system (2.3)-(2.8) (and thus, (2.14)-(2.15)) has a solution $(\mathbf{u}, \tau, \mathbf{p}) \in C^2(0,T; H^{k+1}) \times C^2(0,T; H^{m+1}) \times C(0,T; H^{q+1})$. In addition assume that $\Delta t,\ \nu \le c\, h^{\acute{d}/2}$, and*

$$\|\mathbf{u}\|_\infty, \|\nabla \mathbf{u}\|_\infty, \|\tau\|_\infty, \|\nabla \tau\|_\infty \le M \quad \text{for all } t \in [0,T] . \tag{5.8}$$

Then, the finite element approximation (5.4)-(5.5) is convergent to the solution of (2.14)-(2.15) on the interval $(0,T)$ as Δt, $h \to 0$. In addition, the approximation $(\mathbf{u}_h, \tau_h)$ satisfies the following error estimates:

$$\begin{aligned} |||\mathbf{u}_h - \mathbf{u}|||_{\infty,0} + |||\tau_h - \tau|||_{\infty,0} &\le \mathbf{F}(\Delta t, \nu, h) &(5.9)\\ |||\mathbf{u}_h - \mathbf{u}|||_{0,1} + |||\tau_h - \tau|||_{0,0} &\le \mathbf{F}(\Delta t, \nu, h) &(5.10)\end{aligned}$$

where

$$\begin{aligned} \mathbf{F}(\Delta t, \nu, h) \;=\; & C\, \left(h^k\, |||\mathbf{u}|||_{0,k+1} + h^{k+1}\, |||\mathbf{u}_t|||_{0,k+1}\right) + C\, \left(h^m\, |||\tau|||_{0,m+1} + h^{m+1}\, |||\tau_t|||_{0,m+1}\right) \\ & + C\, h^{q+1}\, |||p|||_{0,q+1} + C\left(h^{k+1}\, |||\mathbf{u}|||_{\infty,k+1} + h^{m+1}\, |||\tau|||_{\infty,m+1}\right) \\ & + C\, |\Delta t| \left(\|\mathbf{u}_t\|_{0,1} + \|\mathbf{u}_{tt}\|_{0,0} + \|\tau_t\|_{0,1} + \|\tau_{tt}\|_{0,0}\right) \\ & + C\, \nu \left(|||\tau_t|||_{0,1} + |||\tau_t|||_{\infty,0}\right) . \end{aligned}$$

In order to establish the estimates (5.9)-(5.10), we begin by introducing the following notation. Let $\mathbf{u}^n = \mathbf{u}(t_n), \tau^n = \tau(t_n)$ represent the solution of (2.14)-(2.15), and $\mathbf{u}_h^n, \tau_h^n$ denote the solution of (5.4)-(5.5).

Define $\mathbf{\Lambda}^n, \mathbf{E}^n, \mathbf{\Gamma}^n, \mathbf{F}^n, \epsilon_u, \epsilon_\tau$ as

$$\begin{aligned} \mathbf{\Lambda}^n &= \mathbf{u}^n - \mathcal{U}^n, & \mathbf{E}^n &= \mathcal{U}^n - \mathbf{u}_h^n, \\ \mathbf{\Gamma}^n &= \tau^n - \mathcal{T}^n, & \mathbf{F}^n &= \mathcal{T}^n - \tau_h^n, \\ \epsilon_u &= \mathbf{u} - \mathbf{u}_h^n, & \epsilon_\tau &= \tau - \tau_h^n . \end{aligned}$$

The proof of theorem 5.2 is established in three steps.

1. Prove a lemma, assuming two induction hypotheses.

2. Show that the induction hypotheses are true.

3. Prove the error estimates given in (5.9),(5.10).

Step 1. We prove the following lemma.

Lemma 8 *Under the induction hypothesis* $(IH1)$ *and the additional assumption*

$$(IH2) \qquad\qquad \sum_{n=1}^{l-1} \Delta t \left\| \nabla E^n \right\|_\infty \leq 1 \ ,$$

we have that

$$\left\| \mathbf{E}^l \right\|^2 + \left\| \mathbf{F}^l \right\|^2 \leq G(\Delta t, h, \nu), \tag{5.11}$$

where

$$\begin{aligned} G(\Delta t, h, \nu) \ &= \ C \ \left(h^{2k} \left\| |\mathbf{u}| \right\|_{0,k+1}^2 + h^{2k+2} \left\| |\mathbf{u}_t| \right\|_{0,k+1}^2 \right) + C \ \left(h^{2m} \left\| |\tau| \right\|_{0,m+1}^2 + h^{2m+2} \left\| |\tau_t| \right\|_{0,m+1}^2 \right) \\ &+ C \ h^{2q+2} \left\| |p| \right\|_{0,q+1}^2 + C \left| \Delta t \right|^2 \left(\left\| |\mathbf{u}_t| \right\|_{0,1}^2 + \left\| |\mathbf{u}_{tt}| \right\|_{0,0}^2 + \left\| |\tau_t| \right\|_{0,1}^2 + \left\| |\tau_{tt}| \right\|_{0,0}^2 \right) \\ &+ C \ \nu^2 \left(\left\| |\tau_t| \right\|_{0,1}^2 + \left\| |\tau_t| \right\|_{\infty,0}^2 \right) . \end{aligned}$$

Proof of lemma 8: From (2.14)-(2.15), it is clear that the true solution $(\mathbf{u}, \tau)$ satisfies

$$\begin{aligned} Re \ (d_t \mathbf{u}^n, \mathbf{v}) \ &+ \ Re \ c\left(\mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{v}\right) + (1-\alpha) \left(\nabla \mathbf{u}^n, \nabla \mathbf{v}\right) + \left(\tau^n, D(\mathbf{v})\right) \\ &= (\mathbf{f}^n, \mathbf{v}) + (p^n, \nabla \cdot \mathbf{v}) + R_1(\mathbf{v}), \ \forall \ \mathbf{v} \in Z_h, \end{aligned} \tag{5.12}$$

$$\begin{aligned} (d_t \tau^n, \sigma) \ &+ \ b\left(\mathbf{u}_h^{n-1}, \tau^n, \tilde{\sigma}\right) - \hat{\lambda} \left(D(\mathbf{u}^n), \tilde{\sigma}\right) + \frac{1}{\lambda} \left(\tau^n, \tilde{\sigma}\right) \\ &= - \left(g_a \left(\tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1}\right), \tilde{\sigma} \right) + R_2(\sigma), \ \forall \sigma \in S_h, \end{aligned} \tag{5.13}$$

where

$$R_1(\mathbf{v}) := Re \ (d_t \mathbf{u}^n, \mathbf{v}) - Re \ (\mathbf{u}_t^n, \mathbf{v}) + Re \ c(\mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{v}) - Re \ c(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}),$$

and

$$\begin{aligned} R_2(\sigma) \ \ := \ \ &(d_t \tau^n, \sigma) - (\tau_t^n, \sigma) - \nu \left(\tau_t^n, \mathbf{u}_h^{n-1} \cdot \nabla \sigma \right) + b(\mathbf{u}_h^{n-1}, \tau^n, \tilde{\sigma}) \\ &- b(\mathbf{u}^n, \tau^n, \tilde{\sigma}) + \left(g_a \left(\tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1} \right), \tilde{\sigma} \right) - \left(g_a \left(\tau^n, \nabla \mathbf{u}^n \right), \tilde{\sigma} \right) . \end{aligned}$$

Subtracting (5.4)-(5.5) from (5.12)-(5.13) we obtain the following equations for ϵ_u and ϵ_τ:

$$\begin{aligned} Re \ (d_t \epsilon_u, \mathbf{v}) + Re \ c(\mathbf{u}_h^{n-1}, \epsilon_u, \mathbf{v}) \ &+ \ (1-\alpha) \left(\nabla \epsilon_u, \nabla \mathbf{v}\right) + \left(\epsilon_\tau, D(\mathbf{v})\right) \\ &= (p^n, \nabla \cdot \mathbf{v}) + R_1(\mathbf{v}), \ \ \forall \ \mathbf{v} \ \in \ Z_h, \end{aligned} \tag{5.14}$$

$$(d_t \epsilon_\tau, \sigma) + b(\mathbf{u}_h^{n-1}, \epsilon_\tau, \tilde{\sigma}) - \hat{\lambda} \left(D(\epsilon_u), \tilde{\sigma}\right) \ + \ \frac{1}{\lambda} \left(\epsilon_\tau, \tilde{\sigma}\right) = R_2(\sigma), \ \ \forall \ \sigma \ \in \ S_h. \tag{5.15}$$

Substituting $\epsilon_u = \mathbf{E}^n + \mathbf{\Lambda}^n,\ \epsilon_\tau = \mathbf{F}^n + \mathbf{\Gamma}^n,\ \mathbf{v} = \mathbf{E}^n,\ \sigma = \mathbf{F}^n$ into (5.14)-(5.15), we obtain

$$\begin{aligned} Re\ (d_t\mathbf{E}^n, \mathbf{E}^n) + Re\ c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) + (1-\alpha)\left(\nabla\mathbf{E}^n, \nabla\mathbf{E}^n\right) + \left(\mathbf{F}^n, D(\mathbf{E}^n)\right) &= \mathcal{F}_1(\mathbf{E}^n), && (5.16)\\ (d_t\mathbf{F}^n, \mathbf{F}^n) + b(\mathbf{u}_h^{n-1}, \mathbf{F}^n, \tilde{\mathbf{F}}^n) - \hat{\lambda}\left(D(\mathbf{E}^n), \tilde{\mathbf{F}}^n\right) + \frac{1}{\lambda}\left(\mathbf{F}^n, \tilde{\mathbf{F}}^n\right) &= \mathcal{F}_2(\mathbf{F}^n), && (5.17) \end{aligned}$$

where,

$$\begin{aligned} \mathcal{F}_1(\mathbf{E}^n) &= (p^n, \nabla\cdot\mathbf{E}^n) + R_1(\mathbf{E}^n) - Re\ (d_t\mathbf{\Lambda}^n, \mathbf{E}^n) - Re\ c(\mathbf{u}_h^{n-1}, \mathbf{\Lambda}^n, \mathbf{E}^n)\\ &\quad -(1-\alpha)\left(\nabla\mathbf{\Lambda}^n, \nabla\mathbf{E}^n\right) - \left(\mathbf{\Gamma}^n, D(\mathbf{E}^n)\right),\\ \mathcal{F}_2(\mathbf{F}^n) &= R_2(\mathbf{F}^n) - (d_t\mathbf{\Gamma}^n, \mathbf{F}^n) - b(\mathbf{u}_h^{n-1}, \mathbf{\Gamma}^n, \tilde{\mathbf{F}}^n) + \hat{\lambda}\left(D(\mathbf{\Lambda}^n), \tilde{\mathbf{F}}^n\right) - \frac{1}{\lambda}\left(\mathbf{\Gamma}^n, \tilde{\mathbf{F}}^n\right). \end{aligned}$$

Multiplying (5.16) by $\hat{\lambda}$ and adding to (5.17) we obtain the single equation

$$\begin{aligned} Re\ \hat{\lambda}\,(d_t\mathbf{E}^n, \mathbf{E}^n) &+ Re\ \hat{\lambda}\,c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) + (1-\alpha)\hat{\lambda}\left(\nabla\mathbf{E}^n, \nabla\mathbf{E}^n\right) + (d_t\mathbf{F}^n, \mathbf{F}^n)\\ &+ b(\mathbf{u}_h^{n-1}, \mathbf{F}^n, \tilde{\mathbf{F}}^n) - \hat{\lambda}\left(D(\mathbf{E}^n), \nu\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right) + \frac{1}{\lambda}\left(\mathbf{F}^n, \tilde{\mathbf{F}}^n\right)\\ &= \hat{\lambda}\mathcal{F}_1(\mathbf{E}^n) + \mathcal{F}_2(\mathbf{F}^n). \end{aligned} \tag{5.18}$$

Note that

$$\begin{aligned} (d_t\mathbf{E}^n, \mathbf{E}^n) &= \frac{1}{\Delta t}\left[(\mathbf{E}^n, \mathbf{E}^n) - \left(\mathbf{E}^{n-1}, \mathbf{E}^n\right)\right]\\ &\geq \frac{1}{\Delta t}\left[\|\mathbf{E}^n\|^2 - \|\mathbf{E}^n\|\left\|\mathbf{E}^{n-1}\right\|\right]\\ &\geq \frac{1}{2\Delta t}\left[\|\mathbf{E}^n\|^2 - \left\|\mathbf{E}^{n-1}\right\|^2\right], \end{aligned}$$

and similarly, $(d_t\mathbf{F}^n, \mathbf{F}^n) \geq \frac{1}{2\Delta t}\left[\|\mathbf{F}^n\|^2 - \left\|\mathbf{F}^{n-1}\right\|^2\right]$. Thus, we have

$$\begin{aligned} &\frac{Re\ \hat{\lambda}}{2\Delta t}\left[\|\mathbf{E}^n\|^2 - \left\|\mathbf{E}^{n-1}\right\|^2\right] + \frac{1}{2\Delta t}\left[\|\mathbf{F}^n\|^2 - \left\|\mathbf{F}^{n-1}\right\|^2\right] + (1-\alpha)\hat{\lambda}\,\|\nabla\mathbf{E}^n\|^2 + \nu\left\|\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right\|^2\\ &\quad + \frac{1}{\lambda}\|\mathbf{F}^n\|^2 \leq -Re\ \hat{\lambda}\,c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) - b(\mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) + \hat{\lambda}\left(D(\mathbf{E}^n), \nu\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right)\\ &\quad - \frac{1}{\lambda}\left(\mathbf{F}^n, \nu\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right) + \hat{\lambda}\mathcal{F}_1(\mathbf{E}^n) + \mathcal{F}_2(\mathbf{F}^n). \end{aligned} \tag{5.19}$$

Multiplying (5.19) by Δt and summing from $n = 1$ to l yields:

$$\begin{aligned} \frac{Re\ \hat{\lambda}}{2}\left[\left\|\mathbf{E}^l\right\|^2 - \left\|\mathbf{E}^0\right\|^2\right] &+ \frac{1}{2}\left[\left\|\mathbf{F}^l\right\|^2 - \left\|\mathbf{F}^0\right\|^2\right] + (1-\alpha)\hat{\lambda}\sum_{n=1}^{l}\Delta t\,\|\nabla\mathbf{E}^n\|^2 + \nu\sum_{n=1}^{l}\Delta t\left\|\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right\|^2\\ + \frac{1}{\lambda}\sum_{n=1}^{l}\Delta t\,\|\mathbf{F}^n\|^2 &\leq \Delta t\sum_{n=1}^{l}\Big[-Re\ \hat{\lambda}\,c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) - b(\mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n)\\ &\quad + \hat{\lambda}\left(D(\mathbf{E}^n), \nu\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right) - \frac{1}{\lambda}\left(\mathbf{F}^n, \nu\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right)\Big]\\ &\quad + \hat{\lambda}\Delta t\sum_{n=1}^{l}\mathcal{F}_1(\mathbf{E}^n) + \Delta t\sum_{n=1}^{l}\mathcal{F}_2(\mathbf{F}^n). \end{aligned} \tag{5.20}$$

We now estimate each term on the right hand side of (5.20). For $c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n)$ we have that

$$\begin{aligned} \left| c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) \right| &\leq \left| \left(\mathbf{u}_h^{n-1} \cdot \nabla \mathbf{E}^n, \mathbf{E}^n \right) \right| \\ &\leq \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{E}^n \right\| \|\mathbf{E}^n\| \\ &\leq \left\| \mathbf{u}_h^{n-1} \right\|_\infty \acute{d}^{\frac{1}{2}} \|\nabla \mathbf{E}^n\| \|\mathbf{E}^n\| \\ &\leq \epsilon_1 \|\nabla \mathbf{E}^n\|^2 + \frac{\acute{d} K^2}{4\epsilon_1} \|\mathbf{E}^n\|^2 , \text{ using (IH1).} \end{aligned} \tag{5.21}$$

Note that for $\mathbf{v} = 0$ on $\partial\Omega$, applying Green's theorem we have

$$b(\mathbf{v}, \tau, \sigma) = -b(\mathbf{v}, \sigma, \tau) - (\nabla \cdot \mathbf{v}\, \tau, \sigma) , \tag{5.22}$$

$$\Rightarrow b(\mathbf{v}, \tau, \tau) = -\frac{1}{2} (\nabla \cdot \mathbf{v}\, \tau, \tau) . \tag{5.23}$$

Using (5.23),

$$\begin{aligned} \left| b(\mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) \right| &= \frac{1}{2} \left| \left(\nabla \cdot \mathbf{u}_h^{n-1}\, \mathbf{F}^n, \mathbf{F}^n \right) \right| \\ &= \frac{1}{2} \left| \left(\nabla \cdot (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1})\, \mathbf{F}^n, \mathbf{F}^n \right) + \left(\nabla \cdot \mathcal{U}^{n-1}\, \mathbf{F}^n, \mathbf{F}^n \right) \right| \\ &\leq \frac{1}{2} \left\| \nabla \cdot \mathbf{E}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 + \frac{1}{2} \left\| \nabla \cdot \mathcal{U}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 \\ &\leq \frac{1}{2} \left\| \nabla \cdot \mathbf{E}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 + \frac{1}{2} M \|\mathbf{F}^n\|^2 , \text{using (5.2).} \end{aligned}$$

Next,

$$\begin{aligned} \left| \left(D(\mathbf{E}^n), \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right) \right| &\leq \|D(\mathbf{E}^n)\| \left\| \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right\| \\ &\leq \|\nabla \mathbf{E}^n\| \left\| \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right\| \\ &\leq \epsilon_2 \|\nabla \mathbf{E}^n\|^2 + \frac{\nu^2}{4\epsilon_2} \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right\|^2 . \end{aligned}$$

Also,

$$\begin{aligned} \left| \left(\mathbf{F}^n, \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right) \right| &= \nu \left| \left(\mathbf{F}^n, \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right) \right| \\ &\leq \nu \|\mathbf{F}^n\| \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right\| \\ &\leq \|\mathbf{F}^n\|^2 + \frac{\nu^2}{4} \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right\|^2 . \end{aligned}$$

Thus, for the first summation on the right hand side of (5.20), we have

$$\begin{aligned} \Delta t \ &\sum_{n=1}^{l} \left[-Re\, \hat{\lambda}\, c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) - b(\mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) + \hat{\lambda} \left(D(\mathbf{E}^n), \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right) - \frac{1}{\lambda} \left(\mathbf{F}^n, \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right) \right] \\ &\leq \Delta t \sum_{n=1}^{l} (Re\, \hat{\lambda} \epsilon_1 + \hat{\lambda} \epsilon_2) \|\nabla \mathbf{E}^n\|^2 + \Delta t \sum_{n=1}^{l} \frac{Re\, \hat{\lambda} \acute{d} K^2}{4\epsilon_1} \|\mathbf{E}^n\|^2 + \Delta t \sum_{n=1}^{l} \left(\frac{\hat{\lambda} \nu^2}{4\epsilon_2} + \frac{\nu^2}{\lambda 4 \epsilon_3} \right) \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n \right\|^2 \\ &\quad + \Delta t \sum_{n=1}^{l} \left(\frac{1}{2} M + \frac{1}{2} \left\| \nabla \cdot \mathbf{E}^{n-1} \right\|_\infty + \frac{\epsilon_3}{\lambda} \right) \|\mathbf{F}^n\|^2 . \end{aligned} \tag{5.24}$$

Next we consider $\mathcal{F}_1(\mathbf{E}^n)$.

$$\begin{aligned}
|(p^n, \nabla\cdot\mathbf{E}^n)| &= |(p^n - \mathcal{P}^n, \nabla\cdot\mathbf{E}^n)| \\
&\leq \|p^n - \mathcal{P}^n\| \, \acute{d}^{\frac{1}{2}} \, \|\nabla\mathbf{E}^n\| \\
&\leq \epsilon_4 \|\nabla\mathbf{E}^n\|^2 + \frac{\acute{d}}{4\epsilon_4} \|p^n - \mathcal{P}^n\|^2 . \qquad (5.25) \\
|(d_t\mathbf{\Lambda}^n, \mathbf{E}^n)| &\leq \|\mathbf{E}^n\| \, \|d_t\mathbf{\Lambda}^n\| \\
&\leq \|\mathbf{E}^n\|^2 + \frac{1}{4} \|d_t\mathbf{\Lambda}^n\|^2 . \\
\left|c(\mathbf{u}_h^{n-1}, \mathbf{\Lambda}^n, \mathbf{E}^n)\right| &\leq \|\mathbf{E}^n\| \left\|\mathbf{u}_h^{n-1} \cdot \nabla\mathbf{\Lambda}^n\right\| \\
&\leq \|\mathbf{E}^n\| \left\|\mathbf{u}_h^{n-1}\right\|_\infty \acute{d}^{\frac{1}{2}} \, \|\nabla\mathbf{\Lambda}^n\| \\
&\leq \|\mathbf{E}^n\|^2 + \frac{K^2\acute{d}}{4} \|\nabla\mathbf{\Lambda}^n\|^2 , \qquad \text{using } (IH1). \qquad (5.26) \\
|(\nabla\mathbf{\Lambda}^n, \nabla\mathbf{E}^n)| &\leq \|\nabla\mathbf{E}^n\| \, \|\nabla\mathbf{\Lambda}^n\| \\
&\leq \epsilon_5 \|\nabla\mathbf{E}^n\|^2 + \frac{1}{4\epsilon_5} \|\nabla\mathbf{\Lambda}^n\|^2 . \qquad (5.27) \\
|(\mathbf{\Gamma}^n, D(\mathbf{E}^n))| &\leq \|D(\mathbf{E}^n)\| \, \|\mathbf{\Gamma}^n\| \\
&\leq \|\nabla\mathbf{E}^n\| \, \|\mathbf{\Gamma}^n\| \\
&\leq \epsilon_6 \|\nabla\mathbf{E}^n\|^2 + \frac{1}{4\epsilon_6} \|\mathbf{\Gamma}^n\|^2 . \qquad (5.28)
\end{aligned}$$

For the $R_1(\mathbf{E}^n)$ terms we have:

$$\begin{aligned}
|(d_t\mathbf{u}^n, \mathbf{E}^n) - (\mathbf{u}_t^n, \mathbf{E}^n)| &\leq \|\mathbf{E}^n\|^2 + \frac{1}{4} \|d_t\mathbf{u}^n - \mathbf{u}_t^n\|^2 \qquad (5.29) \\
\left|c(\mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{E}^n) - c(\mathbf{u}^n, \mathbf{u}^n, \mathbf{E}^n)\right| &= \Big|c(\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}, \mathbf{u}^n, \mathbf{E}^n) + c(\mathcal{U}^{n-1} - \mathbf{u}^{n-1}, \mathbf{u}^n, \mathbf{E}^n) \\
&\quad + c(\mathbf{u}^{n-1} - \mathbf{u}^n, \mathbf{u}^n, \mathbf{E}^n)\Big| \\
&\leq \left\|\mathbf{E}^{n-1} \cdot \nabla\mathbf{u}^n\right\| \|\mathbf{E}^n\| + \left\|\mathbf{\Lambda}^{n-1} \cdot \nabla\mathbf{u}^n\right\| \|\mathbf{E}^n\| + \left\|(\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla\mathbf{u}^n\right\| \|\mathbf{E}^n\| \\
&\leq \acute{d}M \left\|\mathbf{E}^{n-1}\right\| \|\mathbf{E}^n\| + \acute{d}M \left\|\mathbf{\Lambda}^{n-1}\right\| \|\mathbf{E}^n\| + \acute{d}M \left\|(\mathbf{u}^n - \mathbf{u}^{n-1})\right\| \|\mathbf{E}^n\| \\
&\leq \frac{\acute{d}M}{2} \left\|\mathbf{E}^{n-1}\right\|^2 + \left(\frac{\acute{d}M}{2} + 2\right) \|\mathbf{E}^n\|^2 + \frac{\acute{d}^2M^2}{4} \left\|\mathbf{\Lambda}^{n-1}\right\|^2 \\
&\quad + \frac{\acute{d}^2M^2}{4} \Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_t\|^2 \, dt. \qquad (5.30)
\end{aligned}$$

Combining (5.25)-(5.30) we have the following estimate for $\mathcal{F}_1(\mathbf{E}^n)$:

$$\begin{aligned}
\left|\hat{\lambda}\mathcal{F}_1(\mathbf{E}^n)\right| &\leq \hat{\lambda}(\epsilon_4 + \epsilon_5 + \epsilon_6) \|\nabla\mathbf{E}^n\|^2 + \hat{\lambda} \, Re \left(\frac{\acute{d}M}{2} + 5\right) \|\mathbf{E}^n\|^2 \\
&\quad + \hat{\lambda} \, Re \frac{\acute{d}M}{2} \left\|\mathbf{E}^{n-1}\right\|^2 + \hat{\lambda} \frac{\acute{d}}{4\epsilon_4} \|(p^n - \mathcal{P}^n)\|^2 + \hat{\lambda} \, Re \frac{\acute{d}^2M^2}{4} \left\|\mathbf{\Lambda}^{n-1}\right\|^2 \\
&\quad + \hat{\lambda} \left(\frac{Re \, K^2\acute{d}}{4} + \frac{(1-\alpha)}{4\epsilon_5}\right) \|\nabla\mathbf{\Lambda}^n\|^2 + Re \frac{1}{4} \|d_t\mathbf{\Lambda}^n\|^2 + \hat{\lambda} \frac{1}{4\epsilon_6} \|\mathbf{\Gamma}^n\|^2
\end{aligned}$$

$$+\hat{\lambda}\ Re\frac{1}{4}\left\|d_t\mathbf{u}^n-\mathbf{u}_t^n\right\|^2+\hat{\lambda}\ Re\frac{\acute{d}^2M^2}{4}\Delta t\int_{t_{n-1}}^{t_n}\left\|\mathbf{u}_t\right\|^2\ dt. \tag{5.31}$$

Next we consider the terms in $\mathcal{F}_2(\mathbf{F}^n)$.

$$\begin{aligned}
\left|(d_t\mathbf{\Gamma}^n,\mathbf{F}^n)\right| &\leq \left\|\mathbf{F}^n\right\|^2+\frac{1}{4}\left\|d_t\mathbf{\Gamma}^n\right\|^2. &(5.32)\\
\left|b(\mathbf{u}_h^{n-1},\mathbf{\Gamma}^n,\tilde{\mathbf{F}}^n)\right| &= \left|b(\mathbf{u}_h^{n-1},\mathbf{\Gamma}^n,\mathbf{F}^n)+b(\mathbf{u}_h^{n-1},\mathbf{\Gamma}^n,\nu\mathbf{F}_u^n)\right|\\
&\leq \left\|\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{\Gamma}^n\right\|\left\|\mathbf{F}^n\right\|+\left\|\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{\Gamma}^n\right\|\left\|\nu\mathbf{F}_u^n\right\|\\
&\leq \acute{d}^{\frac{1}{2}}\left\|\mathbf{u}_h^{n-1}\right\|_\infty\left\|\nabla\mathbf{\Gamma}^n\right\|\left\|\mathbf{F}^n\right\|+\acute{d}^{\frac{1}{2}}\left\|\mathbf{u}_h^{n-1}\right\|_\infty\left\|\nabla\mathbf{\Gamma}^n\right\|\left\|\nu\mathbf{F}_u^n\right\|\\
&\leq \left\|\mathbf{F}^n\right\|^2+\nu^2\left\|\mathbf{F}_u^n\right\|^2+\frac{\acute{d}K^2}{2}\left\|\nabla\mathbf{\Gamma}^n\right\|^2. &(5.33)\\
\left|\left(D(\mathbf{\Lambda}^n),\tilde{\mathbf{F}}^n\right)\right| &= \left|(D(\mathbf{\Lambda}^n),\mathbf{F}^n)+(D(\mathbf{\Lambda}^n),\nu\mathbf{F}_u^n)\right|.\\
&\leq \left\|\mathbf{F}^n\right\|^2+\nu^2\left\|\mathbf{F}_u^n\right\|^2+\frac{1}{2}\left\|\nabla\mathbf{\Lambda}^n\right\|^2. &(5.34)\\
\left|\left(\mathbf{\Gamma}^n,\tilde{\mathbf{F}}^n\right)\right| &= \left|(\mathbf{\Gamma}^n,\mathbf{F}^n)+\nu\left(\mathbf{\Gamma}^n,\nu\mathbf{F}_u^n\right)\right|\\
&\leq \left\|\mathbf{F}^n\right\|^2+\nu^2\left\|\mathbf{F}_u^n\right\|^2+\frac{1}{2}\left\|\mathbf{\Gamma}^n\right\|^2. &(5.35)
\end{aligned}$$

For the terms making up $R_2(\mathbf{F}^n)$ we have:

$$\begin{aligned}
\left|(d_t\tau^n,\mathbf{F}^n)-(\tau_t^n,\mathbf{F}^n)\right| &\leq \left\|\mathbf{F}^n\right\|^2+\frac{1}{4}\left\|d_t\tau^n-\tau_t^n\right\|^2. &(5.36)\\
\left|(\tau_t^n,\nu\mathbf{F}_u^n)\right| &= \left|\left(\tau_t^n,\nu\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{F}^n\right)\right|\\
&= \left|b(\nu\mathbf{u}_h^{n-1},\mathbf{F}^n,\tau_t^n)\right|\\
&\leq \left|b(\nu\mathbf{u}_h^{n-1},\tau_t^n,\mathbf{F}^n)\right|+\left|\left(\nabla\cdot\mathbf{u}_h^{n-1}\,\nu\mathbf{F}^n,\tau_t^n\right)\right|\quad(\text{ using (5.22) })\\
&\leq \nu\left\|\mathbf{u}_h^{n-1}\cdot\nabla\tau_t^n\right\|\left\|\mathbf{F}^n\right\|+\left|\left(\nabla\cdot(\mathbf{u}_h^{n-1}-\mathcal{U}^{n-1})\,\nu\mathbf{F}^n,\tau_t^n\right)\right|\\
&\quad+\left|\left(\nabla\cdot\mathcal{U}^{n-1}\,\nu\mathbf{F}^n,\tau_t^n\right)\right|\\
&\leq \nu\left\|\mathbf{u}_h^{n-1}\right\|_\infty\acute{d}^{\frac{1}{2}}\left\|\nabla\tau_t^n\right\|\left\|\mathbf{F}^n\right\|+\nu\left\|\nabla\cdot(\mathbf{u}_h^{n-1}-\mathcal{U}^{n-1})\right\|_\infty\left\|\mathbf{F}^n\right\|\left\|\tau_t^n\right\|\\
&\quad+\left\|\nabla\cdot\mathcal{U}^{n-1}\right\|_\infty\nu\left\|\mathbf{F}^n\right\|\left\|\tau_t^n\right\|\\
&\leq \left(2+\left\|\nabla\mathbf{E}^{n-1}\right\|_\infty\right)\left\|\mathbf{F}^n\right\|^2+\frac{\nu^2}{4}\acute{d}^2\left(M^2+\left\|\nabla\mathbf{E}^{n-1}\right\|_\infty\right)\left\|\tau_t^n\right\|^2\\
&\quad+\frac{\nu^2}{4}K^2\acute{d}\left\|\nabla\tau_t^n\right\|^2,\quad(\text{ using (5.2) and (IH1) }). &(5.37)
\end{aligned}$$

$$\begin{aligned}
\left|b(\mathbf{u}_h^{n-1},\tau^n,\tilde{\mathbf{F}}^n)-b(\mathbf{u}^n,\tau^n,\tilde{\mathbf{F}}^n)\right| &= \left|\left((\mathbf{u}_h^{n-1}-\mathbf{u}^n)\cdot\nabla\tau^n,\tilde{\mathbf{F}}^n\right)\right|\\
&\leq \left\|(\mathbf{u}_h^{n-1}-\mathbf{u}^n)\cdot\nabla\tau^n\right\|\left\|\tilde{\mathbf{F}}^n\right\|\\
&\leq \frac{1}{2}\left\|\tilde{\mathbf{F}}^n\right\|^2+\frac{1}{2}\acute{d}^3\left\|\nabla\tau^n\right\|_\infty^2\left\|\mathbf{u}_h^{n-1}-\mathbf{u}^n\right\|^2\\
&\leq \left\|\mathbf{F}^n\right\|^2+\nu^2\left\|\mathbf{F}_u^n\right\|^2+\frac{1}{2}\acute{d}^3M^2\left\|-\mathbf{E}^{n-1}-\mathbf{\Lambda}^{n-1}+\mathbf{u}^{n-1}-\mathbf{u}^n\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2 + \frac{3}{2}\acute{d}^3 M^2 \left\|\mathbf{E}^{n-1}\right\|^2 + \frac{3}{2}\acute{d}^3 M^2 \left\|\mathbf{\Lambda}^{n-1}\right\|^2 \\
&\quad + \frac{3}{2}\acute{d}^3 M^2 \Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_t\|^2 \, dt. \qquad (5.38)
\end{aligned}
$$

In order to estimate the g_a terms in $\mathcal{F}_2(\cdot)$ note that

$$
\begin{aligned}
g_a\left(\tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1}\right) - g_a\left(\tau^n, \nabla \mathbf{u}^n\right) &= g_a\left(\tau_h^{n-1}, \nabla(\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1})\right) + g_a\left(\tau_h^{n-1}, \nabla(\mathcal{U}^{n-1} - \mathbf{u}^{n-1})\right) \\
&\quad + g_a\left(\tau_h^{n-1}, \nabla(\mathbf{u}^{n-1} - \mathbf{u}^n)\right) + g_a\left(\tau_h^{n-1} - \mathcal{T}^{n-1}, \nabla \mathbf{u}^n\right) \\
&\quad + g_a\left(\mathcal{T}^{n-1} - \tau^{n-1}, \nabla \mathbf{u}^n\right) + g_a\left(\tau^{n-1} - \tau^n, \nabla \mathbf{u}^n\right) \\
&= -g_a\left(\tau_h^{n-1}, \nabla \mathbf{E}^{n-1}\right) - g_a\left(\tau_h^{n-1}, \nabla \mathbf{\Lambda}^{n-1}\right) - g_a\left(\tau_h^{n-1}, \nabla(\mathbf{u}^n - \mathbf{u}^{n-1})\right) \\
&\quad - g_a\left(\mathbf{F}^{n-1}, \nabla \mathbf{u}^n\right) - g_a\left(\mathbf{\Gamma}^{n-1}, \nabla \mathbf{u}^n\right) - g_a\left(\tau^n - \tau^{n-1}, \nabla \mathbf{u}^n\right). \qquad (5.39)
\end{aligned}
$$

Bounding each of the terms on the right hand side of (5.39) we obtain

$$
\begin{aligned}
\left|\left(g_a\left(\tau_h^{n-1}, \nabla \mathbf{E}^{n-1}\right), \tilde{\mathbf{F}}^n\right)\right| &\leq \left\|g_a\left(\tau_h^{n-1}, \nabla \mathbf{E}^{n-1}\right)\right\| \left\|\tilde{\mathbf{F}}^n\right\| \\
&\leq 4\acute{d} \left\|\tau_h^{n-1}\right\|_\infty \left\|\nabla \mathbf{E}^{n-1}\right\| \left\|\tilde{\mathbf{F}}^n\right\| \\
&\leq \epsilon_7 \left\|\nabla \mathbf{E}^{n-1}\right\|^2 + \frac{8\acute{d}^2 K^2}{\epsilon_7} \|\mathbf{F}^n\|^2 + \frac{8\acute{d}^2 K^2}{\epsilon_7} \nu^2 \|\mathbf{F}_u^n\|^2, \qquad (5.40) \\
\left|\left(g_a\left(\tau_h^{n-1}, \nabla \mathbf{\Lambda}^{n-1}\right), \tilde{\mathbf{F}}^n\right)\right| &\leq 8\acute{d}^2 K^2 \left\|\nabla \mathbf{\Lambda}^{n-1}\right\|^2 + \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2, \qquad (5.41) \\
\left|\left(g_a\left(\tau_h^{n-1}, \nabla(\mathbf{u}^n - \mathbf{u}^{n-1})\right), \tilde{\mathbf{F}}^n\right)\right| &\leq 8\acute{d}^2 K^2 \Delta t \int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_t\|^2 \, dt + \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2, \qquad (5.42) \\
\left|\left(g_a\left(\mathbf{F}^{n-1}, \nabla \mathbf{u}^n\right), \tilde{\mathbf{F}}^n\right)\right| &\leq 8\acute{d}^2 M^2 \left\|\mathbf{F}^{n-1}\right\|^2 + \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2, \qquad (5.43) \\
\left|\left(g_a\left(\mathbf{\Gamma}^{n-1}, \nabla \mathbf{u}^n\right), \tilde{\mathbf{F}}^n\right)\right| &\leq 8\acute{d}^2 M^2 \left\|\mathbf{\Gamma}^{n-1}\right\|^2 + \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2, \qquad (5.44) \\
\left|\left(g_a\left(\tau^n - \tau^{n-1}, \nabla \mathbf{u}^n\right), \tilde{\mathbf{F}}^n\right)\right| &\leq 8\acute{d}^2 M^2 \Delta t \int_{t_{n-1}}^{t_n} \|\tau_t\|^2 \, dt + \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2. \qquad (5.45)
\end{aligned}
$$

Combining the estimates in (5.32)-(5.38), (5.40)-(5.45), we obtain the following estimate for $\mathcal{F}_2(\mathbf{F}^n)$:

$$
\begin{aligned}
|\mathcal{F}_2(\mathbf{F}^n)| &\leq \epsilon_7 \left\|\nabla \mathbf{E}^{n-1}\right\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2 \left(7 + \frac{8\acute{d}^2 K^2}{\epsilon_7} + \hat{\lambda} + \frac{1}{\lambda}\right) \\
&\quad + \|\mathbf{F}^n\|^2 \left(11 + \frac{8\acute{d}^2 K^2}{\epsilon_7} + \left\|\nabla \mathbf{E}^{n-1}\right\|_\infty + \hat{\lambda} + \frac{1}{\lambda}\right) \\
&\quad + \left\|\mathbf{E}^{n-1}\right\|^2 \left(\frac{3}{2}\acute{d}^3 M^2\right) + \left\|\mathbf{F}^{n-1}\right\|^2 \left(8\acute{d}^2 M^2\right) \\
&\quad + \|\nabla \mathbf{\Lambda}^n\|^2 \left(\frac{\hat{\lambda}}{2}\right) + \|\nabla \mathbf{\Gamma}^n\|^2 \left(\frac{\acute{d} K^2}{2}\right) + \|\mathbf{\Gamma}^n\|^2 \left(\frac{1}{2\lambda}\right) + \|d_t \mathbf{\Gamma}^n\|^2 \left(\frac{1}{4}\right) \\
&\quad + \left\|\nabla \mathbf{\Lambda}^{n-1}\right\|^2 \left(8\acute{d}^2 K^2\right) + \left\|\mathbf{\Lambda}^{n-1}\right\|^2 \left(\frac{3}{2}\acute{d}^3 M^2\right) + \left\|\mathbf{\Gamma}^{n-1}\right\|^2 \left(8\acute{d}^2 M^2\right) \\
&\quad + \frac{1}{4} \|d_t \tau^n - \tau_t^n\|^2 + \frac{\nu^2}{4}\acute{d}^2 \left(M^2 + \left\|\nabla \mathbf{E}^{n-1}\right\|_\infty\right) \|\tau_t^n\|^2 + \frac{\nu^2}{4} K^2 \acute{d} \|\nabla \tau_t^n\|^2
\end{aligned}
$$

$$
+\frac{3}{2}\acute{d}^3 M^2 \Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_t\|^2 \, dt + 8\acute{d}^2 M^2 \Delta t \int_{t_{n-1}}^{t_n} \|\tau_t\|^2 \, dt + 8\acute{d}^2 K^2 \Delta t \int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_t\|^2 dt \tag{5.46}
$$

With the following choices: $\epsilon_1 = \frac{(1-\alpha)}{12 \, Re \, \hat{\lambda}}, \epsilon_2 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \frac{(1-\alpha)}{12\hat{\lambda}}, \mathbf{u}_h^0 = \mathcal{U}^0 (\Rightarrow \mathbf{E}^0 = 0), \tau_h^0 = \mathcal{T}^0 (\Rightarrow \mathbf{F}^0 = 0)$, substituting (5.24), (5.31), (5.46) into (5.20) yields

$$
\begin{aligned}
\frac{Re \, \hat{\lambda}}{2} \left\| \mathbf{E}^l \right\|^2 \quad &+ \quad \frac{1}{2} \left\| \mathbf{F}^l \right\|^2 + \frac{(1-\alpha)}{2} \hat{\lambda} \sum_{n=1}^{l} \Delta t \, \|\nabla \mathbf{E}^n\|^2 + \left[\nu - \nu^2 \left(\frac{3\hat{\lambda}^2 + 96\acute{d}^2 K^2 \hat{\lambda}}{(1-\alpha)} + 7 + \hat{\lambda} + \frac{5}{4\lambda} \right) \right] \sum_{n=1}^{l} \Delta t \, \|\mathbf{F}_u^n\|^2 \\
&\leq \quad C_1 \sum_{n=1}^{l} \Delta t \, \|\mathbf{E}^n\|^2 + C_2 \sum_{n=1}^{l} \Delta t \, \|\mathbf{F}^n\|^2 + C_3 \sum_{n=1}^{l} \Delta t \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 + C_4 \sum_{n=1}^{l} \Delta t \, \|\mathbf{\Lambda}^n\|^2 \\
&\quad + C_5 \sum_{n=1}^{l} \Delta t \, \|\nabla \mathbf{\Lambda}^n\|^2 + \frac{1}{4} \sum_{n=1}^{l} \Delta t \, \|d_t \mathbf{\Lambda}^n\|^2 + C_6 \sum_{n=1}^{l} \Delta t \, \|\mathbf{\Gamma}^n\|^2 + Re \frac{\hat{\lambda}}{4} \sum_{n=1}^{l} \Delta t \, \|d_t \mathbf{u}^n - \mathbf{u}_t^n\|^2 \\
&\quad + \left(\frac{\acute{d} K^2}{2} \right) \sum_{n=1}^{l} \Delta t \, \|\nabla \mathbf{\Gamma}^n\|^2 + \frac{1}{4} \sum_{n=1}^{l} \Delta t \, \|d_t \mathbf{\Gamma}^n\|^2 + \frac{1}{4} \sum_{n=1}^{l} \Delta t \, \|d_t \tau^n - \tau_t^n\|^2 \\
&\quad + \frac{\nu^2}{4} \sum_{n=1}^{l} \Delta t \acute{d}^2 \left(M^2 + \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \right) \|\tau_t^n\|^2 + \sum_{n=1}^{l} \Delta t \, \|p^n - \mathcal{P}^n\|^2 \\
&\quad + |\Delta t|^2 \, \acute{d} \left(Re \, \acute{d} M^2 \frac{\hat{\lambda}}{4} \|\mathbf{u}_t\|_{0,0}^2 + \frac{3}{2} \acute{d}^2 M^2 \|\mathbf{u}_t\|_{0,0}^2 + 8\acute{d} M^2 \|\tau_t\|_{0,0}^2 + 8\acute{d} K^2 \|\mathbf{u}_t\|_{0,1}^2 \right) \\
&\quad + \frac{\nu^2}{4} K^2 \acute{d} \, \|\nabla \tau_t\|_{0,0}^2 \, .
\end{aligned} \tag{5.47}
$$

We now apply the interpolation properties of the approximating spaces to estimate the terms on the right hand side of (5.47). Using elements of order k for velocity, elements of order m for stress, and elements of order q for pressure, we have

$$
\begin{aligned}
\sum_{n=1}^{l} \Delta t \, \|\nabla \mathbf{\Lambda}^n\|^2 + \sum_{n=1}^{l} \Delta t \, \|\nabla \mathbf{\Gamma}^n\|^2 \quad &\leq \quad C \left(h^{2k} \sum_{n=1}^{l} \Delta t \, \|\mathbf{u}^n\|_{k+1}^2 + h^{2m} \sum_{n=1}^{l} \Delta t \, \|\tau^n\|_{m+1}^2 \right) \\
&\leq \quad C \left(h^{2k} \, \||\mathbf{u}\||_{0,k+1}^2 + h^{2m} \, \||\tau\||_{0,m+1}^2 \right),
\end{aligned} \tag{5.48}
$$

$$
\begin{aligned}
\sum_{n=1}^{l} \Delta t \, \|\mathbf{\Lambda}^n\|^2 \quad &+ \quad \sum_{n=1}^{l} \Delta t \, \|\mathbf{\Gamma}^n\|^2 + \sum_{n=1}^{l} \Delta t \, \|p - \mathcal{P}^n\|^2 \\
&\leq \quad C \left(h^{2k+2} \sum_{n=1}^{l} \Delta t \, \|\mathbf{u}^n\|_{k+1}^2 + h^{2m+2} \sum_{n=1}^{l} \Delta t \, \|\tau^n\|_{m+1}^2 + h^{2q+2} \sum_{n=1}^{l} \Delta t \, \|p^n\|_{q+1}^2 \right) \\
&\leq \quad C \left(h^{2k+2} \, \||\mathbf{u}\||_{0,k+1}^2 + h^{2m+2} \, \||\tau\||_{0,m+1}^2 + h^{2q+2} \, \||p\||_{0,q+1}^2 \right),
\end{aligned} \tag{5.49}
$$

$$
\begin{aligned}
\sum_{n=1}^{l} \Delta t \, \|d_t \mathbf{\Lambda}^n\|^2 \quad &= \quad \sum_{n=1}^{l} \Delta t \left\| \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} 1 \frac{\partial \Lambda}{\partial t} dt \right\|^2 \\
&\leq \quad \sum_{n=1}^{l} \Delta t \left(\frac{1}{\Delta t} \right)^2 \int_\Omega \left(\int_{t_{n-1}}^{t_n} 1 \, dt \right) \left(\int_{t_{n-1}}^{t_n} \left(\frac{\partial \Lambda}{\partial t} \right)^2 \, dt \right) d\mathbf{x} \\
&\leq \quad C h^{2k+2} \, \||\mathbf{u}_t\||_{0,k+1}^2 \, ,
\end{aligned} \tag{5.50}
$$

and similarly,

$$\sum_{n=1}^{l} \Delta t \, \|d_t \mathbf{\Gamma}^n\|^2 \leq C h^{2m+2} \, \|\tau_t\|^2_{0,m+1} \, . \tag{5.51}$$

Note that $d_t \mathbf{u}^n - \mathbf{u}_t^n$ may be expressed as

$$d_t \mathbf{u}^n - \mathbf{u}_t^n = \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}(\cdot, t)(t_{n-1} - t) \, dt.$$

Also,

$$\begin{aligned}
\left(\frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}(\cdot, t)(t_{n-1} - t) \, dt \right)^2 \quad & \leq \quad \frac{1}{4 \, |\Delta t|^2} \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}^2(\cdot, t) \, dt \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2 \, dt \\
& = \quad \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}^2(\cdot, t) \, dt.
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
\sum_{n=1}^{l} \Delta t \, \|d_t \mathbf{u}^n - \mathbf{u}_t^n\|^2 \quad & \leq \quad \sum_{n=1}^{l} \Delta t \int_{\Omega} \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}^2(\cdot, t) \, dt \, d\mathbf{x} \\
& = \quad \frac{1}{12} \, |\Delta t|^2 \, \|\mathbf{u}_{tt}\|^2_{0,0} \, .
\end{aligned} \tag{5.52}$$

Similarly, for $d_t \tau^n - \tau_t^n$ we have

$$\sum_{n=1}^{l} \Delta t \, \|d_t \tau^n - \tau_t^n\|^2 \quad \leq \quad \frac{1}{12} \, |\Delta t|^2 \, \|\tau_{tt}\|^2_{0,0} \, . \tag{5.53}$$

In view of (5.48)-(5.53), our induction hypotheses (IH1),(IH2), and with ν chosen such that

$$\nu \leq \frac{1}{2} \left(\frac{3\hat{\lambda}^2 + 96\hat{d}^2 K^2 \hat{\lambda}}{(1 - \alpha)} + 7 + \hat{\lambda} + \frac{5}{4\lambda} \right)^{-1} , \tag{5.54}$$

from (5.47) we obtain

$$\begin{aligned}
\frac{Re \, \hat{\lambda}}{2} \left\| \mathbf{E}^l \right\|^2 \quad & + \quad \frac{1}{2} \left\| \mathbf{F}^l \right\|^2 + \frac{(1 - \alpha)}{2} \hat{\lambda} \sum_{n=1}^{l} \Delta t \, \|\nabla \mathbf{E}^n\|^2 + \frac{\nu}{2} \sum_{n=1}^{l} \Delta t \, \|\mathbf{F}_u^n\|^2 \\
& \leq \quad C \sum_{n=1}^{l} \Delta t \left(\|\mathbf{E}^n\|^2 + \|\mathbf{F}^n\|^2 \right) + C \sum_{n=1}^{l} \Delta t \left\| \nabla \mathbf{E}^{n-1} \right\|_{\infty} \|\mathbf{F}^n\|^2 + C\nu^2 \left(\|\!|\tau_t|\!\|^2_{0,1} + \|\!|\tau_t|\!\|^2_{\infty,0} \right) \\
& \quad\; + C \, |\Delta t|^2 \left(\|\mathbf{u}_t\|^2_{0,1} + \|\mathbf{u}_{tt}\|^2_{0,0} + \|\tau_t\|^2_{0,0} + \|\tau_{tt}\|^2_{0,0} \right) + C h^{2k+2} \, \|\!|\mathbf{u}|\!\|^2_{0,k+1} \\
& \quad\; + C h^{2m+2} \, \|\!|\tau|\!\|^2_{0,m+1} + C h^{2q+2} \, \|\!|p|\!\|^2_{0,q+1} + C h^{2k} \, \|\!|\mathbf{u}|\!\|^2_{0,k+1} + C h^{2k+2} \, \|\mathbf{u}_t\|^2_{0,k+1} \\
& \quad\; + C h^{2m} \, \|\!|\tau|\!\|^2_{0,m+1} + C h^{2m+2} \, \|\tau_t\|^2_{0,m+1} \, ,
\end{aligned} \tag{5.55}$$

where the $C's$ denote constants independent of $l, \Delta t, h, \nu$. Applying Gronwall's lemma and (IH2) to (5.55), the estimate given in (5.11) follows.

■

Step 2. We show that the induction hypotheses, $(IH1)$ and $(IH2)$, are true.
Verification of (IH1)
Assume that $(IH1)$ holds true for $n = 1, 2, \ldots, l-1$. By interpolation properties, inverse estimates and (5.11), we have that

$$\begin{aligned}
\left\|\mathbf{u}_h^l\right\|_\infty &\leq \left\|\mathbf{u}_h^l - \mathbf{u}^l\right\|_\infty + \left\|\mathbf{u}^l\right\|_\infty \\
&\leq \left\|\mathbf{E}^l\right\|_\infty + \left\|\Lambda^l\right\|_\infty + M \\
&\leq Ch^{-\frac{\acute{d}}{2}}\left\|\mathbf{E}^l\right\|_0 + Ch^{-\frac{\acute{d}}{2}}\left\|\Lambda^l\right\|_0 + M \\
&\leq C\left(|\Delta t|\, h^{-\frac{\acute{d}}{2}} + \nu h^{-\frac{\acute{d}}{2}} + h^{k-\frac{\acute{d}}{2}} + h^{m-\frac{\acute{d}}{2}} + h^{q+1-\frac{\acute{d}}{2}} + h^{k+1-\frac{\acute{d}}{2}}\right) + M. \qquad (5.56)
\end{aligned}$$

Note that the expression $C\left(|\Delta t|\, h^{-\frac{\acute{d}}{2}} + \nu h^{-\frac{\acute{d}}{2}} + h^{k-\frac{\acute{d}}{2}} + h^{m-\frac{\acute{d}}{2}} + h^{q+1-\frac{\acute{d}}{2}} + h^{k+1-\frac{\acute{d}}{2}}\right)$ is independent of l. Hence, if we set $k, m \geq \frac{\acute{d}}{2}, q \geq \frac{\acute{d}}{2} - 1$, and choose $h, \Delta t, \nu$ such that

$$h^{k-\frac{\acute{d}}{2}}, h^{m-\frac{\acute{d}}{2}}, h^{q+1-\frac{\acute{d}}{2}} \leq \frac{1}{C}, \quad \Delta t, \nu \leq \frac{h^{\frac{\acute{d}}{2}}}{C}, \qquad (5.57)$$

then from (5.56)

$$\left\|\mathbf{u}_h^l\right\|_\infty \leq M + 6.$$

Similarly it follows that $\left\|\tau_h^l\right\|_\infty \leq M + 6$.

■

Verification of $(IH2)$
Assume that $(IH2)$ is true for $n = 1, 2, \ldots, l-1$. Equations (5.11) and (5.55) imply

$$\sum_{n=1}^{l} \Delta t \left\|\nabla \mathbf{E}^n\right\|_0^2 \leq C\left(h^{2k} + h^{2m} + h^{2q+2} + |\Delta t|^2 + \nu^2\right). \qquad (5.58)$$

Applying the inverse estimate and using the inequality

$$\sum_{n=1}^{l} a_n \leq \sqrt{l}\left(\sum_{n=1}^{l} a_n^2\right)^{\frac{1}{2}},$$

from (5.58) we obtain

$$\begin{aligned}
\sum_{n=1}^{l} \Delta t \left\|\nabla \mathbf{E}^n\right\|_\infty &\leq Ch^{-\frac{\acute{d}}{2}} \sum_{n=1}^{l} \Delta t \left\|\nabla \mathbf{E}^n\right\| \\
&\leq Ch^{-\frac{\acute{d}}{2}} \sqrt{\Delta t}\, \sqrt{l}\left(\sum_{n=1}^{l} \Delta t \left\|\nabla \mathbf{E}^n\right\|^2\right)^{\frac{1}{2}} \\
&\leq \tilde{C}\left(\Delta t\, h^{-\frac{\acute{d}}{2}} + \nu h^{-\frac{\acute{d}}{2}} + h^{k-\frac{\acute{d}}{2}} + h^{m-\frac{\acute{d}}{2}} + h^{q+1-\frac{\acute{d}}{2}}\right),
\end{aligned}$$

where $\tilde{C} = C\sqrt{T}$ is a constant independent of $l, h, \Delta t$, and ν. Hence when

$$\nu, \Delta t \leq \frac{h^{\frac{\acute{d}}{2}}}{5\tilde{C}}, \qquad (5.59)$$

and

$$h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \le \frac{1}{5\tilde{C}},$$

$(IH2)$ holds.

$\blacksquare$

Step 3. We derive the error estimates in (5.9) and (5.10).
Proof of the Theorem 5.2.
Using estimates (5.11) and (approximation properties), we have

$$\begin{aligned} |||\mathbf{u}-\mathbf{u}_h|||^2_{\infty,0} + |||\tau-\tau_h|||^2_{\infty,0} &\le |||\mathbf{E}|||^2_{\infty,0} + |||\Lambda|||^2_{\infty,0} + |||\mathbf{F}|||^2_{\infty,0} + |||\Gamma|||^2_{\infty,0} \\ &\le G(\Delta t, h, \nu) + C\left(h^{2k+2}\,|||\mathbf{u}|||^2_{\infty,k+1} + h^{2m+2}\,|||\tau|||^2_{\infty,m+1}\right). \end{aligned}$$

Note the restrictions on ν from (5.54), (5.57), (5.59), and on Δt from (3.1), (5.57), (5.59). Hence, we obtain the stated estimate (5.9).

To establish (5.10), from (5.11), (5.55) we have

$$|||\nabla\mathbf{E}|||^2_{0,0} + \Delta t\,|||\mathbf{F}_u|||^2_{0,0} \le C(T+1)G(\Delta t, h, \nu) \tag{5.60}$$

and

$$|||\mathbf{E}|||^2_{0,0} + |||\mathbf{F}|||^2_{0,0} \le T\,G(\Delta t, h, \nu). \tag{5.61}$$

Hence

$$|||\mathbf{E}|||^2_{1,0} + |||\mathbf{F}|||^2_{0,0} \le \tilde{C}G(\Delta t, h, \nu). \tag{5.62}$$

$\blacksquare$

We conclude this analysis with some comments on the sensitivity of the error bounds to the physical parameters in the modeling equations. From (5.47) we note that the constants C_1, C_2, C_3, involve the terms $K^2, M^2, Re, \bar{\lambda}(=\lambda/2\alpha), \lambda^{-1}$. Thus, in view of the exponential multiplicative factor in the discrete Gronwall's lemma, we have that the generic constants C in (5.9),(5.10),(5.11), depend exponentially on these terms.

References

[1] J. Baranger and A. Machmoum, *Existence of approximate solutions and error bounds for viscoelastic fluid flow: Characteristics method*, Comput. Methods. Appl. Mech. Engrg., **148**, (1997), 39–52.

[2] J. Baranger and D. Sandri, *Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds. I. Discontinuous constraints*, Numer. Math., **63**, (1992), 13–27.

[3] J. Baranger and S. Wardi, *Numerical Analysis of a FEM for a transient viscoelastic flow*, Comput. Methods Appl. Mech. Engrg., **125**, (1995), 171–185.

[4] R.B. Bird, R.C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids*, John Wiley and Sons, Inc., 1987.

[5] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag Publishing, 1994.

[6] C. Corduneanu, *Principals of Differential and Integral Equations*, Chelsea Publishing, 1977.

[7] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer-Verlag Berlin Heidelberg, 1986.

[8] C. Guillope and J.C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Anal. Theory Methods Appl., **15**, (1990), 849–869.

[9] J.G. Heywood and R. Rannacher *Finite element approximation of the nonstationary Navier–Stokes problem. Part IV: Error analysis for second-order time discretization*, SIAM J. Numer. Anal., **2**, (1990), 353–384.

[10] W. Layton and L. Tobiska, *A two–level method with backtracking for the Navier–Stokes equations*, SIAM J. Numer. Anal., **35**, (1998), 2035–2054.

[11] B. Liu, *The analysis of a finite element method with streamline diffusion for the compressible Navier-Stokes equations*, SIAM J. Numer. Anal., **1**, (2000), 1–16.

[12] K. Najib and D. Sandri, *On a decoupled algorithm for solving a finite element problem for the approximation of viscoelastic fluid flow*, Numer. Math., **72**, (1995), 223–238.

[13] M. Renardy, *Mathematical Analysis of Viscoelastic Flows*, SIAM, Philadelphia, 2000.

[14] D. Sandri, *Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds. Continuous approximation of the stress*, SIAM J. Numer. Anal., **31**, (1994), 362–377.