Defect Correction Method for Viscoelastic Fluid Flows at High Weissenberg Number

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We study a defect correction method for the approximation of viscoelastic fluid flow. In the defect step, the constitutive equation is computed with an artificially reduced Weissenberg parameter for stability, and the resulting residual is corrected in the correction step. We prove the convergence of the defect correction method and derive an error estimate for the Oseen-viscoelastic model problem. The derived theoretical results are supported by numerical tests for both the Oseen-viscoelastic problem and the Johnson-Segalman model problem. Numer Methods Partial Differential Eq 21: 000–000, 2005

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1. INTRODUCTION

The numerical simulation of viscoelastic fluid flows is a challenging problem because of (i) the number of unknowns required for an accurate computation, and (ii) the hyperbolic, nonlinear character of the constitutive equation for the stress. In addition, as the Weissenberg number increases boundary layers for the stress develop, which add to the difficulty of computing accurate numerical approximations. A common difficulty for approximation algorithms is that for high Weissenberg numbers the nonlinear iteration used to compute the approximation fails to converge. Therefore, there has been considerable interest by researchers over the years in developing stable numerical algorithms for high Weissenberg number flows (see [1–3]). Critical values of the Weissenberg number beyond which approximation schemes failed to converge were investigated in [1, 4, 5,] and [6], for different problem settings.

Because of its fast convergence property, a Newton iteration scheme is often used in numerical simulation schemes for viscoelasticity. However, as the Weissenberg number increases convergence of the Newton iteration becomes unreliable and eventually, beyond a

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critical value, the iteration fails to converge. In this article we consider a defect-correction method for flow equations with Weissenberg numbers larger than the critical value for the Newton iteration.

Defect-correction methods have been used very effectively in the numerical approximation of convection dominated flow equations, e.g., Navier-Stokes equations, convection-diffusion problems. For such problems, in order to avoid spurious oscillations in the approximation, the defect step acts to regularize the differential equation. This regularization generates a lower order (in accuracy) approximation to the equations. The correction steps then recapture the accuracy lost in those regions of the domain where the solution is sufficiently smooth. (See [7–11] and the references therein.)

The motivation for using defect correction for the approximation of viscoelastic flow equations is different from that for convection dominated flow equations. For viscoelasticity the problem is not spurious oscillations in the numerical approximation, but rather the failure of the nonlinear iteration (used to compute the approximation) to converge. The idea of the defect-correction method for viscoelasticity is as follows. In the defect step the given Weissenberg number λ is replaced by artificially reduced values $\bar{\lambda}$ and $\tilde{\lambda}$, reducing the coefficients of the nonlinearities in the system. (The value of λ is reduced so that the nonlinear solver converges with the reduced value $\bar{\lambda}$ and $\tilde{\lambda}$.) Then, in the correction step, the initial approximation is improved using residual correction. (The residual correction only requires the solution of a linear system of equations.) In [12] a defect-correction method for the Johnson-Segalman model, based on a Picard iteration was analyzed. The *defect* used was chosen proportional to the mesh size and only applied to the convective term in the constitutive equation.

In this article we introduce and analyze a defect correction method for a "nearby" linear problem to the Johnson-Segalman model, the Oseen-viscoelastic model. For the Oseen-viscoelastic model we provide a detailed analysis of the defect correction method for defects applied to both the convective term and the nonlinear contribution of the extra stress. By studying the Oseen-viscoelastic equations we are able to give a complete analysis of the defect-correction method without the complication of accounting for nonlinearities in the equations. The numerical experiments indicate that the defect correction method is most effective when only the nonlinear extra stress term is "defected."

This article is organized as follows. In the next section, we present the Johnson-Segalman viscoelastic equations and introduce the Oseen-viscoelastic equations. In Section 3, we introduce finite element spaces and a discrete variational formulation for a discontinuous Galerkin (DG) approximation to the Oseen-viscoelastic equations. Existence, uniqueness, and an error estimate for the DG approximation are proven in Section 4. The defect-correction method and an error estimate for the approximation is derived in Section 5. Numerical results are then presented in Section 6. In Section 7 we apply the defect-correction method to the Johnson-Segalman equations. Concluding remarks are given in Section 8.

2. MODEL EQUATIONS

Let Ω be a bounded domain in $\mathbb{R}^d$ with the Lipschitz continuous boundary Γ. Consider the Johnson-Segalman problem

$$\boldsymbol{\sigma} + \lambda(\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} + \lambda g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) - 2\alpha \mathbf{D}(\mathbf{u}) = \mathbf{0} \qquad \text{in } \Omega, \tag{2.1}$$

$$-\nabla \cdot \boldsymbol{\sigma} - 2(1-\alpha)\nabla \cdot \mathbf{D}(\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{2.2}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.3}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma, \tag{2.4}$$

where $\boldsymbol{\sigma}$ denotes the polymeric stress tensor, $\mathbf{u}$ the velocity vector, p the pressure of fluid, and λ is the Weissenberg number (defined as the product of the relaxation time of the fluid and a characteristic strain rate). We assume that p has zero mean value over Ω. In (2.1) and (2.2), $\mathbf{D}(\mathbf{u}) := (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ is the rate of the strain tensor, α a number such that $0 < \alpha < 1$, which may be considered as the fraction of viscoelastic viscosity, and $\mathbf{f}$ the body force. In (2.1), $g_a(\boldsymbol{\sigma}, \nabla\mathbf{u})$ is defined by

$$g_a(\boldsymbol{\sigma}, \nabla\mathbf{u}) := \frac{1-a}{2}(\boldsymbol{\sigma}\nabla\mathbf{u} + \nabla\mathbf{u}^T\boldsymbol{\sigma}) - \frac{1+a}{2}(\nabla\mathbf{u}\boldsymbol{\sigma} + \boldsymbol{\sigma}\nabla\mathbf{u}^T) \tag{2.5}$$

for $a \in [-1, 1]$.

We use the Sobolev spaces $W^{m,p}(D)$ with norms $\|\cdot\|_{m,p,D}$ if $p < \infty$, $\|\cdot\|_{m,\infty,D}$ if $p = \infty$. We denote the Sobolev space $W^{m,2}$ by H^m with the norm $\|\cdot\|_m$. The corresponding space of vector-valued or tensor-valued functions is denoted by $\mathbf{H}^m$. If $D = \Omega$, D is omitted, i.e., $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ and $\|\cdot\| = \|\cdot\|_\Omega$.

Existence of a solution to the problem (2.1)–(2.4) has been documented by Renardy ([13]) with the small data condition: if $\mathbf{f}$ is sufficiently regular and small, the problem (2.1)–(2.4) admits a unique bounded solution $(\mathbf{u}, \boldsymbol{\sigma}, p) \in \mathbf{H}^3(\Omega) \times \mathbf{H}^2(\Omega) \times H^2(\Omega)$.

In this article, we analyze a defect-correction method for the (linear) Oseen-viscoelastic problem: Given the velocity field $\mathbf{b}(\mathbf{x})$ determine $\mathbf{u}$, $\boldsymbol{\sigma}$, p, satisfying

$$\boldsymbol{\sigma} + \lambda(\mathbf{b}\cdot\nabla)\boldsymbol{\sigma} + \lambda g_a(\boldsymbol{\sigma}, \nabla\mathbf{b}) - 2\alpha\mathbf{D}(\mathbf{u}) = \mathbf{0} \qquad \text{in } \Omega, \tag{2.6}$$

$$-\nabla\cdot\boldsymbol{\sigma} - 2(1-\alpha)\nabla\cdot\mathbf{D}(\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{2.7}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.8}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma. \tag{2.9}$$

We make the following assumption for $\mathbf{b}$:

$$\mathbf{b} \in \mathbf{H}_0^1(\Omega), \qquad \nabla\cdot\mathbf{b} = 0, \qquad \|\mathbf{b}\|_\infty \le M, \qquad \|\nabla\mathbf{b}\|_\infty \le M < \infty.$$

Next, we define the function spaces for the velocity $\mathbf{u}$, the pressure p, and the stress $\boldsymbol{\sigma}$, respectively:

$$\mathbf{X} := \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma\},$$

$$S := L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\right\},$$

$$\boldsymbol{\Sigma} := (L^2(\Omega))^{d\times d} \cap \{\boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji}, \mathbf{b}\cdot\nabla\boldsymbol{\tau} \in (L^2(\Omega))^{d\times d}\}.$$

We also introduce the weak divergence free space

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{X} : \int_\Omega q \nabla \cdot \mathbf{v} \, d\Omega = 0 \ \forall q \in L_0^2(\Omega) \right\}.$$

The corresponding variational formulation of (2.6)–(2.9) is then given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \lambda((\mathbf{b} \cdot \nabla)\boldsymbol{\sigma}, \boldsymbol{\tau}) + \lambda(g_a(\boldsymbol{\sigma}, \nabla \mathbf{b}), \boldsymbol{\tau}) - 2\alpha(\mathbf{D}(\mathbf{u}), \boldsymbol{\tau}) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \tag{2.10}$$

$$(\boldsymbol{\sigma}, \mathbf{D}(\mathbf{v})) + 2(1 - \alpha)(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.11}$$

$$(q, \nabla \cdot \mathbf{u}) = 0 \qquad \forall q \in L_0^2(\Omega). \tag{2.12}$$

Using the weak divergence free space $\mathbf{V}$, the variational formulation (2.10)–(2.12) is equivalent to

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \lambda((\mathbf{b} \cdot \nabla)\boldsymbol{\sigma}, \boldsymbol{\tau}) + \lambda(g_a(\boldsymbol{\sigma}, \nabla \mathbf{b}), \boldsymbol{\tau}) - 2\alpha(\mathbf{D}(\mathbf{u}), \boldsymbol{\tau}) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \tag{2.13}$$

$$(\boldsymbol{\sigma}, \mathbf{D}(\mathbf{v})) + 2(1 - \alpha)(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{2.14}$$

Existence and uniqueness of the solution to (2.10)–(2.12) is presented in [14].

3. FINITE ELEMENT APPROXIMATION

Let T_h denote a triangulation of Ω such that $\bar{\Omega} = \{\cup K : K \in T_h\}$. Assume that there exist positive constants c_1, c_2 such that

$$c_1 h \le h_K \le c_2 \rho_K,$$

where h_K is the diameter of K, ρ_K is the diameter of the greatest ball included in K, and $h = \max_{K \in T_h} h_K$.

Let $P_k(K)$ denote the space of polynomials of degree less than or equal to k on $K \in T_h$. We define finite element spaces for the approximate of $(\mathbf{u}, p)$:

$$\mathbf{X}^h := \{\mathbf{v} \in \mathbf{X} \cap (C^0(\bar{\Omega}))^d : \mathbf{v}|_K \in P_2(K)^d, \forall K \in T_h\},$$
$$S^h := \{q \in S \cap C^0(\bar{\Omega}) : q|_K \in P_1(K), \ \forall \, K \in T_h\},$$
$$\mathbf{V}^h := \{\mathbf{v} \in \mathbf{X}^h : (q, \nabla \cdot \mathbf{v}) = 0, \ \forall \, q \in S^h\}.$$

The polymeric stress $\boldsymbol{\sigma}$ is approximated in the discontinuous finite element space of piecewise linears:

$$\boldsymbol{\Sigma}^h := \{\boldsymbol{\tau} \in \boldsymbol{\Sigma} : \boldsymbol{\tau}|_K \in P_1(K)^{d \times d}, \forall K \in T_h\}.$$

It is well known that the Taylor-Hood pair $(\mathbf{X}^h, S^h)$ satisfies the inf-sup (or LBB) condition [15]:

$$\inf_{0\neq q^h\in S^h}\ \sup_{0\neq \mathbf{v}^h\in \mathbf{X}^h} \frac{(q^h, \nabla\cdot\mathbf{v}^h)}{\|\mathbf{v}^h\|_1\|q^h\|_0} \geq C, \tag{3.1}$$

where C is a positive constant independent of h.

Below we introduce some notation used in [16] in order to analyze an approximate solution obtained using the discontinuous Galerkin method. Let

$$\Gamma^h = \{\cap\,\partial K, K \in T_h\}\backslash\Gamma,$$

$$\partial K_{\mathbf{b}}^- := \{\mathbf{x}\in\partial K, \mathbf{b}\cdot\mathbf{n} < 0\},$$

where ∂K is the boundary of K and $\mathbf{n}$ is outward unit normal, and

$$\boldsymbol{\tau}_{\mathbf{b}}^{\pm}(\mathbf{x}) := \lim_{\epsilon\to 0^{\pm}} \boldsymbol{\tau}(\mathbf{x}+\epsilon\mathbf{b}(\mathbf{x})).$$

We also define for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \Pi_{K\in T_h}(L^2(K))^{d\times d}$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_h := \sum_{K\in T_h} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K,$$

$$\langle \boldsymbol{\sigma}^{\pm}, \boldsymbol{\tau}^{\pm}\rangle_{h,\mathbf{b}} := \sum_{K\in T_h}\int_{\partial K_{\mathbf{b}}^-} (\boldsymbol{\sigma}_{\mathbf{b}}^{\pm} : \boldsymbol{\tau}_{\mathbf{b}}^{\pm})|\mathbf{n}\cdot\mathbf{b}|\, ds,$$

$$\langle\, \langle \boldsymbol{\sigma}^{\pm}\rangle\, \rangle_{h,\mathbf{b}} := \langle \boldsymbol{\sigma}^{\pm}, \boldsymbol{\sigma}^{\pm}\rangle_{h,\mathbf{b}}^{1/2},$$

$$\|\boldsymbol{\tau}\|_{0,\Gamma^h} := \left(\sum_{K\in T_h} \|\boldsymbol{\tau}\|_{0,\partial K}^2\right)^{1/2},$$

and for $\boldsymbol{\xi} \in \Pi_{K\in T_h}(W^{m,2}(K))^{d\times d}$, $m < \infty$,

$$\|\boldsymbol{\xi}\|_{m,h} := \left(\sum_{K\in T_h} \|\boldsymbol{\xi}\|_{m,K}^2\right)^{1/2}.$$

We introduce the operator B^h on $\boldsymbol{\Sigma}^h \times \boldsymbol{\Sigma}^h$ defined by

$$B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) := ((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h)_h + \langle \boldsymbol{\sigma}^{h^+} - \boldsymbol{\sigma}^{h^+}, \boldsymbol{\tau}^{h^+}\rangle_{h,\mathbf{b}}. \tag{3.2}$$

Using integration by parts, B^h may be written as

$$B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) = -((\mathbf{b}\cdot\nabla)\boldsymbol{\tau}^h, \boldsymbol{\sigma}^h)_h + \langle \boldsymbol{\sigma}^{h^-}, \boldsymbol{\tau}^{h^-} - \boldsymbol{\tau}^{h^+}\rangle_{h,\mathbf{b}}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\sigma}^h) = \frac{1}{2}\langle \boldsymbol{\sigma}^{h^+} - \boldsymbol{\sigma}^{h^-}, \boldsymbol{\sigma}^{h^+} - \boldsymbol{\sigma}^{h^-}\rangle_{h,\mathbf{b}} = \frac{1}{2}\langle\, \langle \boldsymbol{\sigma}^{h^+} - \boldsymbol{\sigma}^{h^-} \rangle\, \rangle_{h,\mathbf{b}}^2 \geq 0. \tag{3.4}$$

The discontinuous Galerkin finite element approximation of (2.10)–(2.12) is then as follows: Find $\mathbf{u}^h \in \mathbf{X}^h$, $p^h \in S^h$, $\boldsymbol{\sigma}^h \in \boldsymbol{\Sigma}^h$ such that

$$(\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + \lambda(g_a(\boldsymbol{\sigma}^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}^h), \boldsymbol{\tau}^h) = 0 \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h, \tag{3.5}$$

$$(\boldsymbol{\sigma}^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}^h), \mathbf{D}(\mathbf{v}^h)) - (p^h, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{3.6}$$

$$(q^h, \nabla \cdot \mathbf{u}^h) = 0 \qquad \forall q^h \in S^h. \tag{3.7}$$

Notice that, in view of (3.1), (3.5)–(3.7) is equivalent to: Find $\mathbf{u}^h \in \mathbf{V}^h$ and $\boldsymbol{\sigma}^h \in \boldsymbol{\Sigma}^h$ such that

$$(\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + \lambda(g_a(\boldsymbol{\sigma}^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}^h), \boldsymbol{\tau}^h) = 0 \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h, \tag{3.8}$$

$$(\boldsymbol{\sigma}^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}^h), \mathbf{D}(\mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathbf{V}^h. \tag{3.9}$$

In the following finite element analysis we use the bilinear form A defined on $\boldsymbol{\Sigma} \times \mathbf{X}$ by

$$\begin{aligned} A((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) := (\boldsymbol{\sigma}, \boldsymbol{\tau}) + \lambda(g_a(\boldsymbol{\sigma}, \nabla \mathbf{b}), \boldsymbol{\tau}) - 2\alpha(\mathbf{D}(\mathbf{u}), \boldsymbol{\tau}) + 2\alpha(\boldsymbol{\sigma}, \mathbf{D}(\mathbf{v})) \\ + 4\alpha(1-\alpha)(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})). \end{aligned} \tag{3.10}$$

Using the bilinear form A defined by (3.10), (3.8)–(3.9) can equivalently be written as

$$A((\boldsymbol{\sigma}^h, \mathbf{u}^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) = 2\alpha(\mathbf{f}, \mathbf{v}^h) \qquad \forall (\boldsymbol{\tau}^h, \mathbf{v}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{V}^h. \tag{3.11}$$

We finish this section with inverse inequalities and some approximation properties (see [17] or [15]), which will be used in proofs throughout this article. If $\tilde{\boldsymbol{\sigma}}^h \in \boldsymbol{\Sigma}^h$ is the orthogonal projection of $\boldsymbol{\sigma}$ on T^h in $\boldsymbol{\Sigma}$, $\tilde{p}^h \in S^h$ the orthogonal projection of $p \in S$, and $\tilde{\mathbf{u}}^h \in \mathbf{V}^h$ is defined as the interpolant of $\mathbf{u}$ in $\mathbf{V}$, then we have the following standard results. For $\mathbf{u} \in \mathbf{H}^3(\Omega)$, $p \in H^2(\Omega)$, and $\boldsymbol{\sigma} \in \mathbf{H}^2(\Omega)$:

$$\|\nabla(\mathbf{u} - \tilde{\mathbf{u}}^h)\|_0 \leq Ch^2\|\mathbf{u}\|_3, \tag{3.12}$$

$$\|p - \tilde{p}^h\| \leq Ch^2\|p\|_2, \tag{3.13}$$

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}^h\|_0 + h\|\nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}^h)\|_0 \leq Ch^2\|\boldsymbol{\sigma}\|_2, \tag{3.14}$$

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}^h\|_{0,\Gamma^h} \leq Ch^{3/2}\|\boldsymbol{\sigma}\|_2. \tag{3.15}$$

We will also use the inverse estimate (see [17] or [15]):

$$\|\nabla \boldsymbol{\sigma}^h\|_{0,h} \leq Ch^{-1}\|\boldsymbol{\sigma}^h\|_0 \qquad \text{for } \boldsymbol{\sigma}^h \in \boldsymbol{\Sigma}^h, \tag{3.16}$$

and the local inverse inequality ([18]),

$$\|\boldsymbol{\sigma}^h\|_{0,\partial K}^2 \leq C\frac{1}{h_K}\|\boldsymbol{\sigma}^h\|_{0,K}^2 \qquad \text{for } \boldsymbol{\sigma}^h \in \boldsymbol{\Sigma}^h. \tag{3.17}$$

4. EXISTENCE OF A FINITE ELEMENT SOLUTION

Theorem 4.1. *For M satisfying $1 - 2\lambda M\acute{d} > 0$, and $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, there exists a unique solution $(\boldsymbol{\sigma}^h, \mathbf{u}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{X}^h$ of* (3.5)–(3.7).

Proof. Note that (3.8)–(3.9) is equivalent to

$$A((\boldsymbol{\sigma}^h, \mathbf{u}^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) = F((\boldsymbol{\tau}^h, \mathbf{v}^h)) \qquad \forall(\boldsymbol{\tau}^h, \mathbf{v}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{V}^h, \tag{4.1}$$

where $F(\cdot) : \boldsymbol{\Sigma}^h \times \mathbf{V}^h \to R$ is a functional defined by

$$F((\boldsymbol{\tau}^h, \mathbf{v}^h)) := 2\alpha(\mathbf{f}, \mathbf{v}^h).$$

It is straightforward to show that F is bounded, since

$$\left|F((\boldsymbol{\tau}^h, \mathbf{v}^h))\right| \leq 2\alpha\|\mathbf{f}\|_{-1}\|\mathbf{v}^h\|_1 \leq 2\alpha\|\mathbf{f}\|_{-1}\|(\boldsymbol{\tau}^h, \mathbf{v}^h)\|_{\boldsymbol{\Sigma}\times\mathbf{X}}, \tag{4.2}$$

where $\|(\boldsymbol{\tau}^h, \mathbf{v}^h)\|_{\boldsymbol{\Sigma}\times\mathbf{x}}$ is defined as $(\|\boldsymbol{\tau}^h\|_0^2 + \|\mathbf{v}^h\|_1^2)^{1/2}$.

We will show the operator in (4.1) is continuous and coercive on $\boldsymbol{\Sigma}^h \times \mathbf{V}^h$, if $1 - 2\lambda Md > 0$. Using (3.16) and (3.17),

$$\begin{aligned} B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) = ((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h)_h + \langle \boldsymbol{\sigma}^{h^+} - \boldsymbol{\sigma}^{h^-}, \boldsymbol{\tau}^{h^+}\rangle_{h,\mathbf{b}} \\ \leq Md\|\nabla\boldsymbol{\sigma}^h\|_{0,h}\|\boldsymbol{\tau}^h\|_0 + C_1\|\mathbf{b}\|_\infty(h^{-1/2}\|\boldsymbol{\sigma}^h\|_0)(h^{-1/2}\|\boldsymbol{\tau}^h\|_0) \\ \leq C_2Mdh^{-1}\|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0 + C_1Mh^{-1}\|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0, \\ (g_a(\boldsymbol{\sigma}^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) \leq 2Md\|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0. \end{aligned} \tag{4.3}$$

Hence,

$$\begin{aligned} A((\boldsymbol{\sigma}^h, \mathbf{u}^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) \leq \|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0 + 2Md\lambda\|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0 + 2\alpha\|\mathbf{D}(\mathbf{u}^h)\|_0\|\boldsymbol{\tau}^h\|_0 \\ + 2\alpha\|\boldsymbol{\sigma}^h\|_0\|\mathbf{D}(\mathbf{v}^h)\|_0 + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}^h)\|_0\|\mathbf{D}(\mathbf{v}^h)\|_0 + \lambda M(C_2d + C_1)h^{-1}\|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0 \\ \leq C\|(\boldsymbol{\sigma}^h, \mathbf{u}^h)\|_{\boldsymbol{\Sigma}\times\mathbf{X}}\|(\boldsymbol{\tau}^h, \mathbf{v}^h)\|_{\boldsymbol{\Sigma}\times\mathbf{X}}. \end{aligned} \tag{4.4}$$

Also, using (3.4) and (4.3),

$$\begin{aligned} A((\boldsymbol{\sigma}^h, \mathbf{u}^h), (\boldsymbol{\sigma}^h, \mathbf{u}^h)) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}^h, \boldsymbol{\sigma}^h) = \|\boldsymbol{\sigma}^h\|_0^2 + \lambda(g_a(\boldsymbol{\sigma}^h, \nabla\mathbf{b}), \boldsymbol{\sigma}^h) + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}^h)\|_0^2 \\ + \frac{\lambda}{2}\langle\, \langle \boldsymbol{\sigma}^{h^+} - \boldsymbol{\sigma}^{h^-}\rangle\, \rangle_{h,\mathbf{b}}^2 \geq \|\boldsymbol{\sigma}^h\|_0^2 - 2\lambda Md\|\boldsymbol{\sigma}^h\|_0^2 + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}^h)\|_0^2 \geq C\|(\boldsymbol{\sigma}^h, \mathbf{u}^h)\|_{\boldsymbol{\Sigma}\times\mathbf{X}}^2, \end{aligned} \tag{4.5}$$

if $1 - 2\lambda Md > 0$. Therefore, by the Lax-Milgram theorem, there exists a unique $(\boldsymbol{\sigma}^h, \mathbf{u}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{V}^h$ satisfying (3.8)–(3.9). Finally, existence of a unique solution $(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h) \in \boldsymbol{\Sigma}^h \times \mathbf{X}^h \times S^h$ of (3.5)–(3.7) follows from the inf-sup condition (3.1). ■

Theorem 4.2. *Assume the solution of* (2.10)–(2.12) *satisfies* $(\mathbf{u}, \boldsymbol{\sigma}, p) \in \mathbf{H}^3(\Omega) \times \mathbf{H}^2(\Omega) \times H^2(\Omega)$, *and let* $(\mathbf{u}^h, \boldsymbol{\sigma}^h)$ *denote the solution of* (3.8)–(3.9). *Then, for M satisfying* $1 - 2\lambda Md > 0$,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|_0 + \|\mathbf{u} - \mathbf{u}^h\|_1 \le Ch. \tag{4.6}$$

Proof. Subtracting (3.8)–(3.9) from (2.10)–(2.11), we have

$$A((\boldsymbol{\sigma} - \boldsymbol{\sigma}^h, \mathbf{u} - \mathbf{u}^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \lambda B^h(\mathbf{b}, \boldsymbol{\sigma} - \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) - (p, \nabla \cdot \mathbf{v}^h) = 0 \qquad \forall(\boldsymbol{\tau}^h, \mathbf{v}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{V}^h.$$

Adding and subtracting $\tilde{\boldsymbol{\sigma}}^h$, $\tilde{\mathbf{u}}^h$, and using the orthogonality of $\nabla \cdot \mathbf{v}^h$ to S^h, implies

$$A((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \tilde{\mathbf{u}}^h - \mathbf{u}^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) \tag{4.7}$$

$$= A((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\mathbf{u}}^h - \mathbf{u}), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \boldsymbol{\tau}^h) - (\tilde{p}^h - p, \nabla \cdot \mathbf{v}^h). \tag{4.8}$$

Now, if we choose $\boldsymbol{\tau}^h = \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h$, $\mathbf{v}^h = \tilde{\mathbf{u}}^h - \mathbf{u}^h$, using (4.5), we have

$$\begin{aligned} &A((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \tilde{\mathbf{u}}^h - \mathbf{u}^h), (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \tilde{\mathbf{u}}^h - \mathbf{u}^h)) + \lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) \\ &\quad \ge (1 - 2\lambda Md)\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2 + 4\alpha(1-\alpha)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0^2 + \frac{\lambda}{2}\langle\langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+ - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^-\rangle\rangle_{h,\mathbf{b}}^2. \end{aligned} \tag{4.9}$$

Correspondingly, we bound the RHS of (4.8) as follows:

$$\begin{aligned} &A((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\mathbf{u}}^h - \mathbf{u}), (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \tilde{\mathbf{u}}^h - \mathbf{u}^h)) + \lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) - (\tilde{p}^h - p, \nabla \cdot (\tilde{\mathbf{u}}^h - \mathbf{u}^h)) \\ &\quad \le \|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0 + \lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) + \lambda(g_a(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \nabla\mathbf{b}), \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) \\ &\quad\quad + 2\alpha\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0 + 2\alpha\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0 \\ &\quad\quad + 4\alpha(1-\alpha)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0 + \|p - \tilde{p}^h\|_0\|\nabla \cdot (\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0. \end{aligned} \tag{4.10}$$

Note that, using (3.4) and (3.16),

$$\begin{aligned} &\lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) = \\ &\quad -\lambda((\mathbf{b}\cdot\nabla)(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h), \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma})_h + \lambda\langle\langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma})^-, (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^- - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+\rangle\rangle_{h,\mathbf{b}} \\ &\quad \le \lambda d\|\mathbf{b}\|_\infty\|\nabla(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)\|_{0,h}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0 + \lambda\langle\langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^- \\ &\quad - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+\rangle\rangle_{h,\mathbf{b}}\langle\langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma})^-\rangle\rangle_{h,\mathbf{b}} \le \lambda dMh^{-1}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0 \\ &\quad + \lambda\|\mathbf{b}\|_\infty^{1/2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_{\Gamma^h,0}\langle\langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^- - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+\rangle\rangle_{h,\mathbf{b}} \le \frac{\lambda^2M^2d^2}{4\epsilon_1}h^{-2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \epsilon_1\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2 \\ &\quad + \frac{\lambda^2M}{4\epsilon_2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_{\Gamma,0}^2 + \epsilon_2\langle\langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^- - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+\rangle\rangle_{h,\mathbf{b}}^2 \end{aligned} \tag{4.11}$$

and

$$\lambda(g_a(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \nabla \mathbf{b}), \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) \leq 2\lambda Md\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0$$
$$\leq \frac{\lambda^2M^2d^2}{\epsilon_3}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \epsilon_3\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2. \quad (4.12)$$

Therefore, by (4.10), (4.11), (4.12), and Korn's inequality [17] with constant C_K,

$$\begin{aligned} &A((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\mathbf{u}}^h - \mathbf{u}), (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h, \tilde{\mathbf{u}}^h - \mathbf{u}^h)) + \lambda B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h) - (\tilde{p}^h - p, \nabla\cdot(\tilde{\mathbf{u}}^h - \mathbf{u}^h)) \\ &\leq \frac{1}{4\epsilon_4}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \epsilon_4\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0 + \frac{\lambda^2M^2d^2}{4\epsilon_1}h^{-2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \epsilon_1\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2 + \frac{\lambda^2M}{4\epsilon_2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_{\Gamma^h,0}^2 \\ &+ \epsilon_2\langle\, \langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^- - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+\rangle\, \rangle_{h,\mathbf{b}}^2 + \frac{\lambda^2M^2d^2}{\epsilon_3}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \epsilon_3\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2 + \frac{\alpha^2}{\epsilon_5}\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0^2 \\ &+ \epsilon_5\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2 + \frac{\alpha^2}{\epsilon_6}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \epsilon_6\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0^2 + 2\alpha(1-\alpha)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0^2 \\ &+ 2\alpha(1-\alpha)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0^2 + \epsilon_7\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0^2 + \frac{dC_K^2}{4\epsilon_7}\|p - \tilde{p}^h\|_0^2. \quad (4.13) \end{aligned}$$

Combining (4.13) with (4.9) and using (3.12)–(3.15), we have

$$\begin{aligned} &(1 - 2\lambda Md - \epsilon_1 - \epsilon_3 - \epsilon_4 - \epsilon_5)\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h\|_0^2 + (2\alpha(1-\alpha) - \epsilon_6 - \epsilon_7)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0^2 \\ &+ \left(\frac{\lambda}{2} - \epsilon_2\right)\langle\, \langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^+ - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h)^-\rangle\, \rangle_{h,\mathbf{b}}^2 \leq \left(\frac{\lambda^2M^2d^2}{4\epsilon_1}h^{-2} + \frac{\lambda^2M^2d^2}{\epsilon_3} + \frac{1}{4\epsilon_4} + \frac{\alpha^2}{\epsilon_6}\right)\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 \\ &+ \left(\frac{\alpha^2}{\epsilon_5} + 2\alpha(1-\alpha)\right)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0^2 + \frac{\lambda^2M}{4\epsilon_2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_{\Gamma,0}^2 + \frac{dC_K^2}{4\epsilon_7}\|p - \tilde{p}^h\|_0^2. \quad (4.14) \end{aligned}$$

If ϵ_i, $i = 1, 2, \ldots, 7$ are chosen appropriately, then (4.6) follows from (3.12)–(3.15), (4.14), and the triangle inequality. ∎

Remark. The assumption $1 - 2\lambda Md > 0$ is a sufficient condition to guarantee invertibility of $A((\cdot,\cdot),(\cdot,\cdot)) + \lambda B^h(\cdot,\cdot)$ on $\boldsymbol{\Sigma}^h \times \mathbf{V}^h$.

5. DEFECT-CORRECTION METHOD

The defect-correction method for (3.5)–(3.7) may be described as follows.

Choose $\bar{\lambda}$ and $\tilde{\lambda}$ such that $\bar{\lambda}, \tilde{\lambda} \leq \lambda$.

Let $(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h, p_0^h) \in \mathbf{X}^h \times \boldsymbol{\Sigma}^h \times S^h$ denote the solution of

$$(\boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) + \bar{\lambda}B^h(\mathbf{b}, \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_0^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}_0^h), \boldsymbol{\tau}^h) = 0 \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h,$$

$$(\boldsymbol{\sigma}_0^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_0^h), \mathbf{D}(\mathbf{v}^h)) - (p_0^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h,$$

$$(q^h, \nabla\cdot\mathbf{u}_0^h) = 0 \qquad \forall\, q^h \in S^h.$$

The associated defects $R_1(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h)$, $R_2(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h, p_0^h)$, and $R_3(\mathbf{u}_0^h)$ for equations (3.5), (3.6), and (3.7), respectively, are defined by

$$(R_1(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h), \boldsymbol{\tau}^h) := -(\boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) - \lambda B^h(\mathbf{b}, \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) - \lambda(g_a(\boldsymbol{\sigma}_0^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) + 2\alpha(\mathbf{D}(\mathbf{u}_0^h), \boldsymbol{\tau}^h), \tag{5.1}$$

$$(R_2(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h, p_0^h), \mathbf{v}^h) := (\mathbf{f}, \mathbf{v}^h) - (\boldsymbol{\sigma}_0^h, \mathbf{D}(\mathbf{v}^h)) - 2(1-\alpha)(\mathbf{D}(\mathbf{u}_0^h), \mathbf{D}(\mathbf{v}_0^h)) + (p_0^h, \nabla\cdot\mathbf{v}^h), \tag{5.2}$$

$$(R_3(\mathbf{u}_0^h, q^h) := -(q^h, \nabla\cdot\mathbf{u}_0^h). \tag{5.3}$$

The error or correction $(\boldsymbol{\epsilon}_0^h, \boldsymbol{\xi}_0^h, \rho_0^h) \in \mathbf{X}^h \times \boldsymbol{\Sigma}^h \times S^h$ then satisfy

$$(\boldsymbol{\xi}_0^h, \boldsymbol{\tau}^h) + \bar{\lambda}B^h(\mathbf{b}, \boldsymbol{\xi}_0^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\xi}_0^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\boldsymbol{\epsilon}_0^h), \boldsymbol{\tau}^h)) = (R_1(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h), \boldsymbol{\tau}^h) \qquad \forall\boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h, \tag{5.4}$$

$$(\boldsymbol{\xi}_0^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\boldsymbol{\epsilon}_0^h), \mathbf{D}(\mathbf{v}^h)) - (\rho_0^h, \nabla\cdot\mathbf{v}^h) = (R_2(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h, p_0^h), \mathbf{v}^h) \qquad \forall\mathbf{v}^h \in \mathbf{X}^h, \tag{5.5}$$

$$(q^h, \nabla\cdot\boldsymbol{\epsilon}_0^h) = (R_3(\mathbf{u}_0^h), q^h) \qquad \forall q^h \in S^h. \tag{5.6}$$

We expect $(\mathbf{u}_1^h, \boldsymbol{\sigma}_1^h, p_1^h) := (\mathbf{u}_0^h + \boldsymbol{\epsilon}_0^h, \boldsymbol{\sigma}_0^h + \boldsymbol{\xi}_0^h, p_0^h + \rho_0^h)$ to be a better approximation to $(\mathbf{u}^h, \boldsymbol{\sigma}^h, p^h)$ than $(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h, p_0^h)$. Note that combining (5.1)–(5.3) with (5.4)–(5.6), $(\mathbf{u}_1^h, \boldsymbol{\sigma}_1^h, p_1^h)$ satisfies

$$\begin{aligned}(\boldsymbol{\sigma}_1^h, \boldsymbol{\tau}^h) + \bar{\lambda}B^h(\mathbf{b}, \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_1^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}_1^h), \boldsymbol{\tau}^h)) &= -(\lambda-\bar{\lambda})B^h(\mathbf{b}, \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) \\ &\quad - (\lambda-\tilde{\lambda})(g_a(\boldsymbol{\sigma}_0^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) \qquad \forall\boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h,\end{aligned}$$

$$(\boldsymbol{\sigma}_1^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_1^h), \mathbf{D}(\mathbf{v}^h)) - (p_1^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h,$$

$$(q^h, \nabla\cdot\mathbf{u}_1^h) = 0 \qquad \forall\, q^h \in S^h.$$

The algorithm is summarized as follows.

Algorithm 5.1 (Defect-correction method)

Step 1. Solve the defected problem: Find $\mathbf{u}_0^h \in \mathbf{X}^h$, $p_0^h \in S^h$, $\boldsymbol{\sigma}_0^h \in \boldsymbol{\Sigma}^h$ such that

$$(\boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) + \bar{\lambda}B^h(\mathbf{b}, \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_0^h, \nabla\mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}_0^h), \boldsymbol{\tau}^h) = 0 \qquad \forall\boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h,$$

$$(\boldsymbol{\sigma}_0^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_0^h), \mathbf{D}(\mathbf{v}^h)) - (p_0^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h,$$

$$(q^h, \nabla\cdot\mathbf{u}_0^h) = 0 \qquad \forall\, q^h \in S^h,$$

where $\bar{\lambda}$ and $\tilde{\lambda}$ are chosen to be less than or equal to λ.

Step 2. For $i = 0, 1, 2, \ldots$, solve the following problem for the correction: Find $\mathbf{u}_{i+1}^h \in \mathbf{X}^h$, $p_{i+1}^h \in S^h$, $\boldsymbol{\sigma}_{i+1}^h \in \boldsymbol{\Sigma}^h$ such that

$$(\boldsymbol{\sigma}_{i+1}^h, \boldsymbol{\tau}^h) + \bar{\lambda} B^h(\mathbf{b}, \boldsymbol{\sigma}_{i+1}^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_{i+1}^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}_{i+1}^h), \boldsymbol{\tau}^h)) = -(\lambda - \bar{\lambda}) B^h(\mathbf{b}, \boldsymbol{\sigma}_i^h, \boldsymbol{\tau}^h) - (\lambda - \tilde{\lambda})(g_a(\boldsymbol{\sigma}_i^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h,$$

$$(\boldsymbol{\sigma}_{i+1}^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_{i+1}^h), \mathbf{D}(\mathbf{v}^h)) - (p_{i+1}^h, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h,$$

$$(q^h, \nabla \cdot \mathbf{u}_{i+1}^h) = 0 \qquad \forall\, q^h \in S^h.$$

We now derive an error estimate for the corrected solution $(\mathbf{u}_1^h, \boldsymbol{\sigma}_1^h, p_1^h)$ obtained after one correction step.

Theorem 5.2. *Let $(\mathbf{u}_1^h, \boldsymbol{\sigma}_1^h, p_1^h)$ be a solution to Step 2 for $i = 0$. Then, for $1 - 2\lambda M d > 0$, there exists a constant C, independent of h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_1^h\|_0 + \|\mathbf{D}(\mathbf{u} - \mathbf{u}_1^h)\|_0 \le C[h + (\lambda - \bar{\lambda})h^{-1}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_0 + (\lambda - \tilde{\lambda})\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_0]. \tag{5.7}$$

Proof. Using the discrete div free space $\mathbf{V}^h$, the problem in Step 2 may be written in the equivalent form: Find $\mathbf{u}_1^h \in \mathbf{V}^h$, $p_1^h \in S^h$, $\boldsymbol{\sigma}_1^h \in \boldsymbol{\Sigma}^h$ such that

$$(\boldsymbol{\sigma}_1^h, \boldsymbol{\tau}^h) + \bar{\lambda} B^h(\mathbf{b}, \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_1^h, \mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}_1^h), \boldsymbol{\tau}^h) = -(\lambda - \bar{\lambda}) B^h(\mathbf{b}, \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) - (\lambda - \tilde{\lambda})(g_a(\boldsymbol{\sigma}_0^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h, \tag{5.8}$$

$$(\boldsymbol{\sigma}_1^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_1^h), \mathbf{D}(\mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathbf{V}^h. \tag{5.9}$$

To simplify expressions, define the bilinear form $\tilde{A}$ defined on $\boldsymbol{\Sigma} \times \mathbf{X}$ by

$$\tilde{A}((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) := (\boldsymbol{\sigma}, \boldsymbol{\tau}) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}, \nabla \mathbf{b}), \boldsymbol{\tau}) - 2\alpha(\mathbf{D}(\mathbf{u}), \boldsymbol{\tau}) + 2\alpha(\boldsymbol{\sigma}, \mathbf{D}(\mathbf{v})) + 4\alpha(1-\alpha)(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})). \tag{5.10}$$

Subtracting (5.8) and (5.9) from (2.10) and (2.11), respectively, and combining them, we have

$$\tilde{A}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_1^h, \mathbf{u} - \mathbf{u}_1^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \bar{\lambda} B^h(\mathbf{b}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}^h) = -(\lambda - \bar{\lambda}) B^h(\mathbf{b}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) - (\lambda - \tilde{\lambda})(g_a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) + (p, \nabla \cdot \mathbf{v}^h) \qquad \forall (\boldsymbol{\tau}^h, \mathbf{v}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{X}^h. \tag{5.11}$$

Adding and subtracting $\tilde{\mathbf{u}}^h$, $\tilde{\boldsymbol{\sigma}}^h$ in (5.11) and using the orthogonality of $\nabla \cdot \mathbf{v}^h$ to S^h, we have

$$\tilde{A}((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h, \tilde{\mathbf{u}}^h - \mathbf{u}_1^h), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \bar{\lambda} B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}^h) \tag{5.12}$$

$$= \tilde{A}((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\mathbf{u}}^h - \mathbf{u}), (\boldsymbol{\tau}^h, \mathbf{v}^h)) + \bar{\lambda} B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \boldsymbol{\tau}^h) + (p - \tilde{p}^h, \nabla \cdot \mathbf{v}^h) \tag{5.13}$$

$$-(\lambda - \bar{\lambda}) B^h(\mathbf{b}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) - (\lambda - \tilde{\lambda})(g_a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h, \nabla \mathbf{b}), \boldsymbol{\tau}^h) \qquad \forall (\boldsymbol{\tau}^h, \mathbf{v}^h) \in \boldsymbol{\Sigma}^h \times \mathbf{X}^h. \tag{5.14}$$

With $\boldsymbol{\tau}^h = \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h$ and $\mathbf{v}^h = \tilde{\mathbf{u}}^h - \mathbf{u}_1^h$, a lower bound for (5.12) is found in the similar manner to (4.9):

$$\tilde{A}((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h, \tilde{\mathbf{u}}^h - \mathbf{u}_1^h), (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h, \tilde{\mathbf{u}}^h - \mathbf{u}_1^h)) + \bar{\lambda} B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h) \geq (1 - 2\tilde{\lambda} M d)$$
$$\times \|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h\|_0^2 + 4\alpha(1-\alpha)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}_1^h)\|_0^2 + \frac{\bar{\lambda}}{2}\langle\, \langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^+ - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^-\rangle\,\rangle_{h,\mathbf{b}}^2. \quad (5.15)$$

Also, a bound for (5.13) can be found in the similar manner to (4.13): for some ϵ_1, ϵ_2, $\epsilon_7 > 0$:

$$\tilde{A}((\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\mathbf{u}}^h - \mathbf{u}), (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h, \tilde{\mathbf{u}}^h - \mathbf{u}_1^h)) + \bar{\lambda} B^h(\mathbf{b}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h) + (p - \tilde{p}^h, \nabla \cdot (\tilde{\mathbf{u}}^h - \mathbf{u}_1^h))$$
$$\leq \frac{C_1}{\epsilon_1}(\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2 + \|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0^2) + \epsilon_1(\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h\|_0^2 + \|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}_1^h)\|_0^2) + \frac{\bar{\lambda}^2 M^2 d^2}{4\epsilon_2} h^{-2}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2$$
$$+ \epsilon_2\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1\|_0^2 + \frac{\bar{\lambda}^2 M}{\epsilon_3}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_{\Gamma^h,0}^2 + \epsilon_3\langle\, \langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^- - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^+\rangle\,\rangle_{h,\mathbf{b}}^2 + 2\alpha(1-\alpha)$$
$$\times \|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0^2 + 2\alpha(1-\alpha)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}_1^h)\|_0^2 + \epsilon_7\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}_1^h)\|_0^2 + \frac{dC_k^2}{4\epsilon_7}\|p - \tilde{p}^h\|_0^2. \quad (5.16)$$

With $\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}$ replaced by $\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h$ and $\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}^h$ by $\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h$ in (4.11) and (4.12), we get

$$-(\lambda - \bar{\lambda})B^h(\mathbf{b}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h, \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h) - (\lambda - \tilde{\lambda})(g_a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h, \nabla\mathbf{b}), \tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h) \leq C_2\Bigg((\lambda - \bar{\lambda})^2 h^{-2}\frac{1}{\epsilon_4}$$
$$+ (\lambda - \tilde{\lambda})^2\frac{1}{\epsilon_5}\Bigg)\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_0^2 + (\epsilon_4 + \epsilon_5)\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h\|_0^2 + (\lambda - \bar{\lambda})^2\frac{M}{4\epsilon_6}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_{\Gamma^h,0}^2$$
$$+ \epsilon_6\langle\, \langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^+ - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^-\rangle\,\rangle_{h,\mathbf{b}}. \quad (5.17)$$

Using (5.15), (5.16), and (5.17), we have

$$(1 - 2\tilde{\lambda} M d - \epsilon_1 - \epsilon_2 - \epsilon_4 - \epsilon_5)\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h\|_0^2 + (2\alpha(1-\alpha) - \epsilon_1 - \epsilon_7)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u}_1^h)\|_0^2$$
$$+ \left(\frac{\bar{\lambda}}{2} - \epsilon_3 - \epsilon_6\right)\langle\, \langle(\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^+ - (\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}_1^h)^-\rangle\,\rangle_{h,\mathbf{b}} \leq C\Bigg[\left(\frac{1}{\epsilon_1} + \frac{\bar{\lambda}^2 M^2 d^2}{4\epsilon_2} h^{-2}\right)\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_0^2$$
$$+ \left(\frac{1}{\epsilon_1} + 2\alpha(1-\alpha)\right)\|\mathbf{D}(\tilde{\mathbf{u}}^h - \mathbf{u})\|_0^2 + \frac{\bar{\lambda}^2 M}{\epsilon_3}\|\tilde{\boldsymbol{\sigma}}^h - \boldsymbol{\sigma}\|_{\Gamma^h,0}^2 + \frac{1}{\epsilon_7}\|p - \tilde{p}^h\|_0^2$$
$$+ (\lambda - \bar{\lambda})^2\frac{M}{\epsilon_6}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_{\Gamma^h,0}^2 + \left((\lambda - \bar{\lambda})^2 h^{-2}\frac{1}{\epsilon_4} + (\lambda - \tilde{\lambda})^2\frac{1}{\epsilon_5}\right)\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_0^2\Bigg]. \quad (5.18)$$

If ϵ_i, $i = 1, \ldots, 7$ are chosen appropriately, then (5.7) follows from (3.12)–(3.15), (5.18), and the triangle inequality. ■

6. NUMERICAL RESULTS

In this section we investigate Algorithm 5.1, specifically, choices for $\bar{\lambda}$ and $\tilde{\lambda}$. We consider two examples. The first is a nonphysical problem with domain $\Omega = [0, 1] \times [0, 1]$ and a specified

TABLE I. Example 1. Errors using the exact solution of (3.5)–(3.7).

λ	h	$\|\mathbf{u} - \mathbf{u}^h\|_1$	Cvge. rate for $\mathbf{u}^h$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|_0$	Cvge. rate for $\boldsymbol{\sigma}^h$
2	1/5	$2.908 \cdot 10^{-2}$		$2.070 \cdot 10^{-2}$	
	1/10	$6.998 \cdot 10^{-3}$	2.1	$5.024 \cdot 10^{-3}$	2.0
	1/15	$3.119 \cdot 10^{-3}$	2.0	$2.241 \cdot 10^{-3}$	2.0
5	1/5	$3.232 \cdot 10^{-2}$		$3.404 \cdot 10^{-2}$	
	1/10	$7.408 \cdot 10^{-3}$	2.1	$7.689 \cdot 10^{-3}$	2.1
	1/15	$3.269 \cdot 10^{-3}$	2.0	$3.306 \cdot 10^{-3}$	2.1
8	1/5	$3.775 \cdot 10^{-2}$		$4.898 \cdot 10^{-2}$	
	1/10	$8.002 \cdot 10^{-3}$	2.2	$1.089 \cdot 10^{-2}$	2.2
	1/15	$3.457 \cdot 10^{-3}$	2.1	$4.614 \cdot 10^{-3}$	2.1

solution. The second example is a 4-to-1 contraction channel flow problem, with a corner singularity.

For the numerical tests of the defect-correction method we considered three cases for several values of λ:

Case 1: $\bar{\lambda}, \tilde{\lambda} < \lambda$ (both the B^h and g_a were defected)
Case 2: $\bar{\lambda} < \lambda$, $\tilde{\lambda} = \lambda$ (only the B^h term was defected)
Case 3: $\bar{\lambda} = \lambda$, $\tilde{\lambda} < \lambda$ (only the g_a term was defected)

Example 1. The function $\mathbf{b}$ was chosen to be the exact solution $\mathbf{u}$. The right-hand side functions in (2.1)–(2.3) were appropriately given so that the exact solution was

$$\begin{cases} \mathbf{u} = \begin{bmatrix} \sin(\pi x)y(y-1) \\ \sin(x)(x-1)y\cos(\pi y/2) \end{bmatrix} \\ p = \cos(2\pi x)y(y-1) \\ \boldsymbol{\sigma} = 2\alpha \mathbf{D}(\mathbf{u}). \end{cases}$$

The parameters α and a in the equations were chosen as 0.5 and 0, respectively.

For comparison of accuracy purposes, the solution to (3.5)–(3.7), for $\lambda = 2$, 5, and 8, was computed. Table I shows errors for $\mathbf{u}$ and $\boldsymbol{\sigma}$ for different mesh sizes. The computed experimental convergence rates are optimal with respect to the approximating elements used (i.e., continuous piecewise quadratics for $\mathbf{u}$, discontinuous piecewise linears for $\boldsymbol{\sigma}$), which are better than the theoretically predicted result in (4.6).

Tables II and III present errors for the defect-correction method using various pairs of $(\bar{\lambda}, \tilde{\lambda})$ for $\lambda = 5$ and 8, respectively. Note that no defect-correction method was used if $\bar{\lambda} = \tilde{\lambda} = \lambda$. (Errors for this case are included for comparison with the defect-correction method.) These

TABLE II. Example 1. Errors using defect-correction for $\lambda = 5$ and $h = 1/8$.

$\bar{\lambda}$	$\tilde{\lambda}$	No. of correction steps: i	$\|\mathbf{u} - \mathbf{u}_i^h\|_0$	$\|\mathbf{u} - \mathbf{u}_i^h\|_1$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_i^h\|_0$
5	5	0	$2.110 \cdot 10^{-4}$	$1.173 \cdot 10^{-2}$	$1.232 \cdot 10^{-2}$
4	4	2	$2.350 \cdot 10^{-4}$	$1.247 \cdot 10^{-2}$	$1.208 \cdot 10^{-2}$
4	5	2	$2.411 \cdot 10^{-4}$	$1.249 \cdot 10^{-2}$	$1.208 \cdot 10^{-2}$
5	4	1	$2.073 \cdot 10^{-4}$	$1.172 \cdot 10^{-2}$	$1.224 \cdot 10^{-2}$
3	3	10	$3.499 \cdot 10^{-4}$	$1.446 \cdot 10^{-2}$	$1.694 \cdot 10^{-2}$
3	5	10	$3.476 \cdot 10^{-4}$	$1.445 \cdot 10^{-2}$	$1.692 \cdot 10^{-2}$
5	3	1	$2.260 \cdot 10^{-4}$	$1.177 \cdot 10^{-2}$	$1.219 \cdot 10^{-2}$

TABLE III. Example 1. Errors using defect-correction for $\lambda = 8$ and $h = 1/8$.

$\bar{\lambda}$	$\tilde{\lambda}$	No. of correction steps: i	$\|\mathbf{u} - \mathbf{u}_i^h\|_0$	$\|\mathbf{u} - \mathbf{u}_i^h\|_1$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_i^h\|_0$
8	8	0	$2.681 \cdot 10^{-4}$	$1.293 \cdot 10^{-2}$	$1.761 \cdot 10^{-2}$
6	6	1	$2.604 \cdot 10^{-4}$	$1.290 \cdot 10^{-2}$	$1.754 \cdot 10^{-2}$
6	8	4	$2.595 \cdot 10^{-4}$	$1.381 \cdot 10^{-2}$	$1.641 \cdot 10^{-2}$
8	6	3	$2.831 \cdot 10^{-4}$	$1.398 \cdot 10^{-2}$	$1.699 \cdot 10^{-2}$
4	4	10	$1.692 \cdot 10^{-3}$	$4.370 \cdot 10^{-2}$	$6.095 \cdot 10^{-2}$
4	8	10	$2.745 \cdot 10^{-3}$	$7.552 \cdot 10^{-2}$	$2.864 \cdot 10^{-2}$
8	4	2	$2.681 \cdot 10^{-4}$	$1.292 \cdot 10^{-2}$	$1.761 \cdot 10^{-2}$

tables demonstrate the behavior of Algorithm 5.1 for choices of the parameters $(\bar{\lambda}, \tilde{\lambda})$. Typically the larger the *defects*, i.e., $(\lambda - \bar{\lambda})$, $(\lambda - \bar{\lambda})$, the more correction iterations required. Additionally, the tables indicate that defecting $\bar{\lambda}$ gave rise to significantly more correction iterations than defecting $\tilde{\lambda}$. Note that, from our analysis, $\bar{\lambda}$ and $\tilde{\lambda}$ have difference influences on the approximation scheme. For example,

- in establishing the positivity of $\tilde{A}((\cdot, \cdot), (\cdot, \cdot)) + \bar{\lambda} B^h(\mathbf{b}, \cdot, \cdot)$ in (5.15), $\bar{\lambda}$ is the coefficient of a positive term, whereas $\tilde{\lambda}$ is the coefficient of an indefinite term,
- in (5.7) we have as the coefficients for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0^h\|_0$: $(\lambda - \bar{\lambda})h^{-1} + (\lambda - \tilde{\lambda})$.

(The specific dependence of the number of correction steps on the *defects* is currently being investigated.)

Based on the errors in the tables, the defect-correction method performed best for $\bar{\lambda} = \lambda$ and $\tilde{\lambda} < \lambda$, i.e., when only λ in the g_a term was reduced.

Example 2. The second example was a 4-to-1 contraction channel flow problem with a corner singularity. The computational domain (assuming symmetry) with a typical velocity field is illustrated in Fig. 1. Along the inflow and outflow boundaries the velocity was specified to

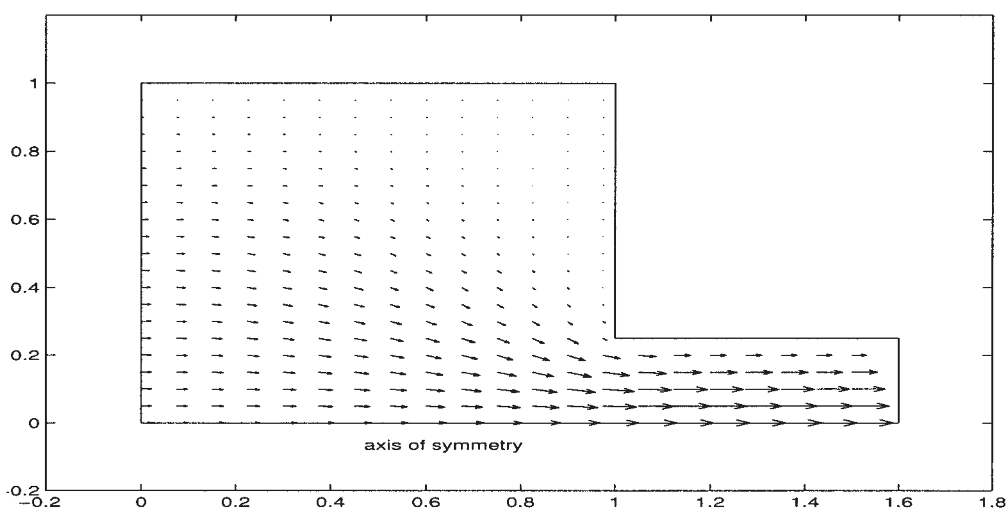

FIG. 1. Example 2. Computational domain with a typical velocity field.

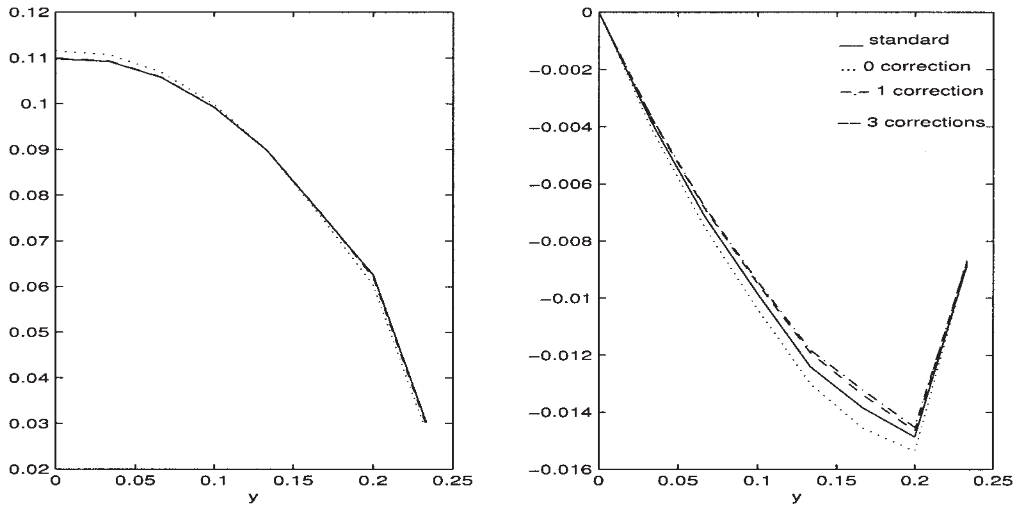

FIG. 2. Example 2. Horizontal and vertical velocity near reentrant corner $\lambda = 1$, Case 1: $\bar{\lambda} = \tilde{\lambda} = 0.7$.

have a parabolic profile. The stress boundary condition at the inflow was specified as $\boldsymbol{\sigma} = 2\alpha\mathbf{D}(\mathbf{u})$.

The parameters α and a in the equations were chosen as 0.89 and 0, respectively. The function $\mathbf{b}$ was chosen as the solution $\mathbf{u}^h$ of (3.5)–(3.7), for $\lambda = 0$, computed over this domain.

Figures 2–4 present the horizontal and vertical velocities near the reentrance corner along the vertical line $x = 1.003$ for $\lambda = 1$ and various pairs of values for $(\bar{\lambda}, \tilde{\lambda})$. The velocity solution $\mathbf{u}^h$ of (3.5)–(3.7), labeled `standard` in the figures, is included for comparison with the defect-correction iterates. From Fig. 3 note that reducing λ only in the B^h term caused little change in the computed velocity profile in the defect step of the algorithm. Figure 4 shows that

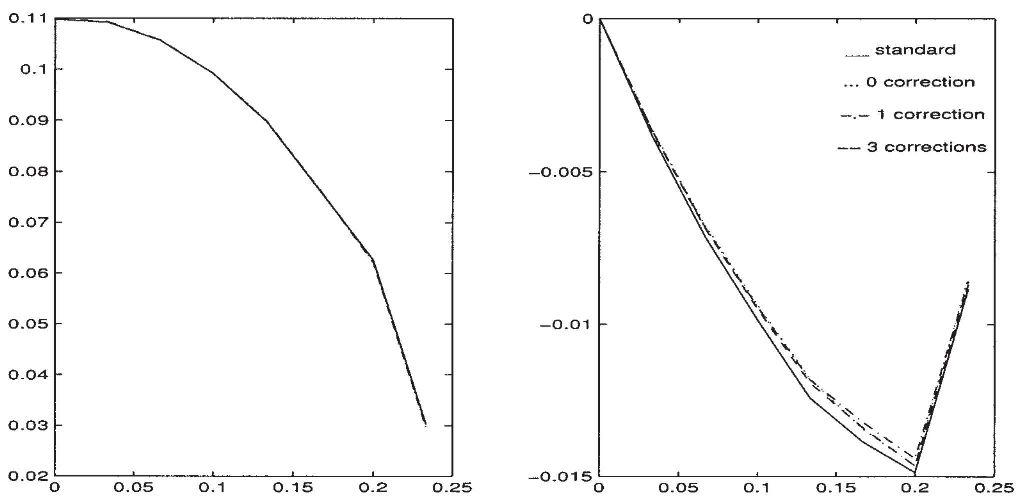

FIG. 3. Example 2. Horizontal and vertical velocity near reentrant corner $\lambda = 1$, Case 2: $\bar{\lambda} = 0.7$, $\tilde{\lambda} = 1$.

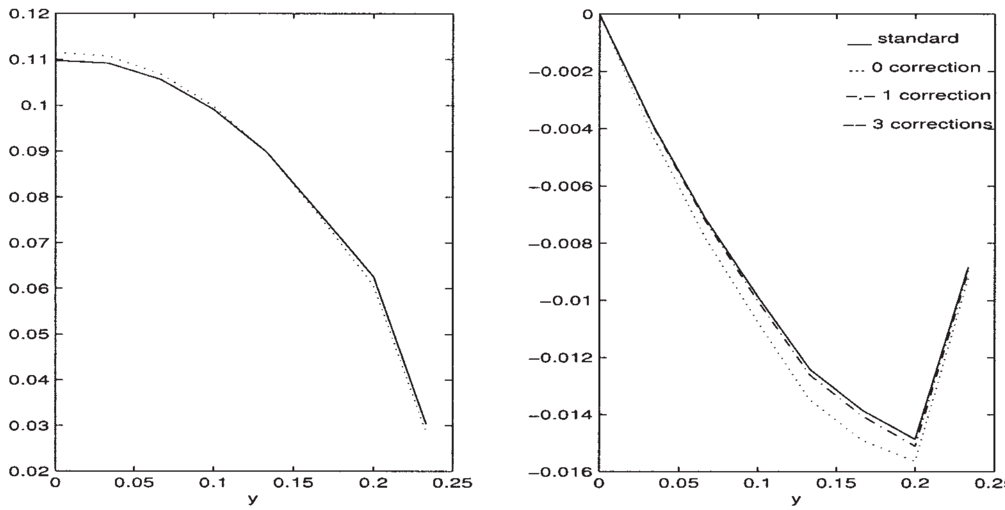

FIG. 4. Example 2. Horizontal and vertical velocity near reentrant corner $\lambda = 1$, Case 3: $\bar{\lambda} = 1$, $\tilde{\lambda} = 0.7$.

the defect-correction method performed effectively when only the g_a terms were defected, which is consistent with that observed in Example 1.

Remark. The data for Examples 1 and 2 do not satisfy the sufficiency condition of Theorems 4.1, 4.2, $1 - 2\lambda Md > 0$.

7. EXTENSION TO JOHNSON-SEGALMAN PROBLEM

In this section the defect-correction method is applied to the nonlinear Johnson-Segalman problem (2.1)–(2.3) (Fig. 5). A detailed analysis of the defect-correction method for this problem is still under investigation.

Algorithm 7.1 (Defect-correction method for the Johnson-Segalman model)

Step 1. Solve the nonlinear defected problem: Find $\mathbf{u}_0^h \in \mathbf{X}^h$, $p_0^h \in S^h$, $\boldsymbol{\sigma}_0^h \in \boldsymbol{\Sigma}^h$ such that

$$(\boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) + \bar{\lambda}B^h(\mathbf{u}_0^h, \boldsymbol{\sigma}_0^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_0^h, \nabla\mathbf{u}_0^h), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u}_0^h), \boldsymbol{\tau}^h) = 0 \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h,$$

$$(\boldsymbol{\sigma}_0^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_0^h), \mathbf{D}(\mathbf{v}^h)) - (p_0^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h,$$

$$(q^h, \nabla\cdot\mathbf{u}_0^h) = 0 \qquad \forall\, q^h \in S^h,$$

where $\bar{\lambda}$ and $\tilde{\lambda}$ are chosen to be less than or equal to λ.

Step 2. For $i = 0, 1, 2, \ldots$, solve the following problem for the correction: Find $\mathbf{u}_{i+1}^h \in \mathbf{X}^h$, $p_{i+1}^h \in S^h$, $\boldsymbol{\sigma}_{i+1}^h \in \boldsymbol{\Sigma}^h$ such that

$$\begin{aligned}(\boldsymbol{\sigma}_{i+1}^h, \boldsymbol{\tau}^h) &+ \bar{\lambda}B^h(\mathbf{u}_{i+1}^h, \boldsymbol{\sigma}_i^h, \boldsymbol{\tau}^h) + \bar{\lambda}B^h(\mathbf{u}_i^h, \boldsymbol{\sigma}_{i+1}^h, \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_{i+1}^h, \nabla\mathbf{u}_i^h), \boldsymbol{\tau}^h) + \tilde{\lambda}(g_a(\boldsymbol{\sigma}_i^h, \nabla\mathbf{u}_{i+1}^h), \boldsymbol{\tau}^h)\\ &- 2\alpha(\mathbf{D}(\mathbf{u}_{i+1}^h), \boldsymbol{\tau}^h)) = -(\lambda - 2\bar{\lambda})B^h(\mathbf{u}_i^h, \boldsymbol{\sigma}_i^h, \boldsymbol{\tau}^h) - (\lambda - 2\tilde{\lambda})(g_a(\boldsymbol{\sigma}_i^h, \nabla\mathbf{u}_i^h), \boldsymbol{\tau}^h) \qquad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h,\end{aligned}$$

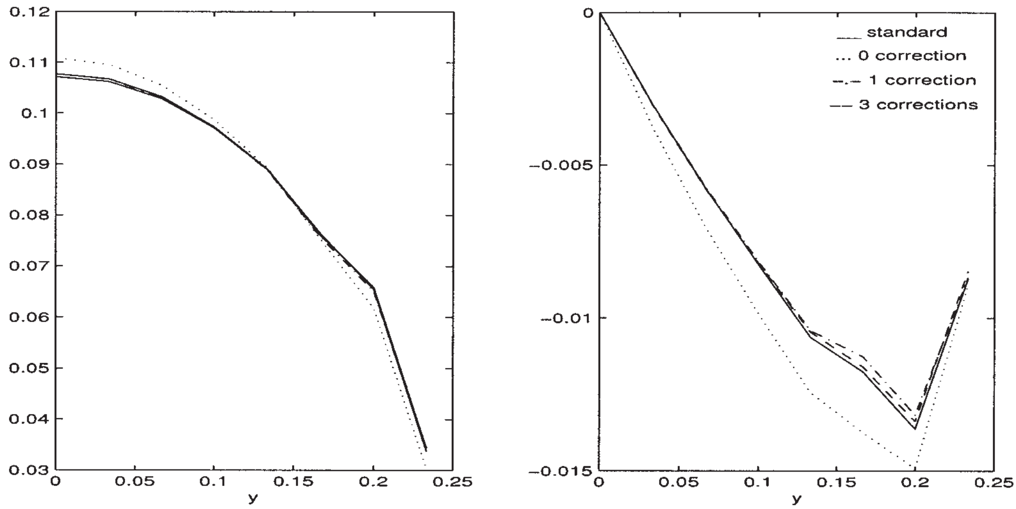

FIG. 5. Johnson-Segalman. Horizontal and vertical velocity near reentrant corner $\lambda = 1$, Case 1: $\bar{\lambda} = 0.7$, $\tilde{\lambda} = 0.7$.

$$(\boldsymbol{\sigma}_{i+1}^h, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_{i+1}^h), \mathbf{D}(\mathbf{v}^h)) - (p_{i+1}^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h,$$

$$(q^h, \nabla\cdot\mathbf{u}_{i+1}^h) = 0 \qquad \forall\, q^h \in S^h.$$

Algorithm 7.1 was applied to the 4-to-1 contraction problem, Example 2 discussed in Section 6. The `standard` Newton method was used to solve the nonlinear problem in the defect step. It was noted from our pre-calculations that the `standard` Newton iteration converges for λ values up to 1.1. Therefore, for a successful approximation in the defect step of the algorithm,

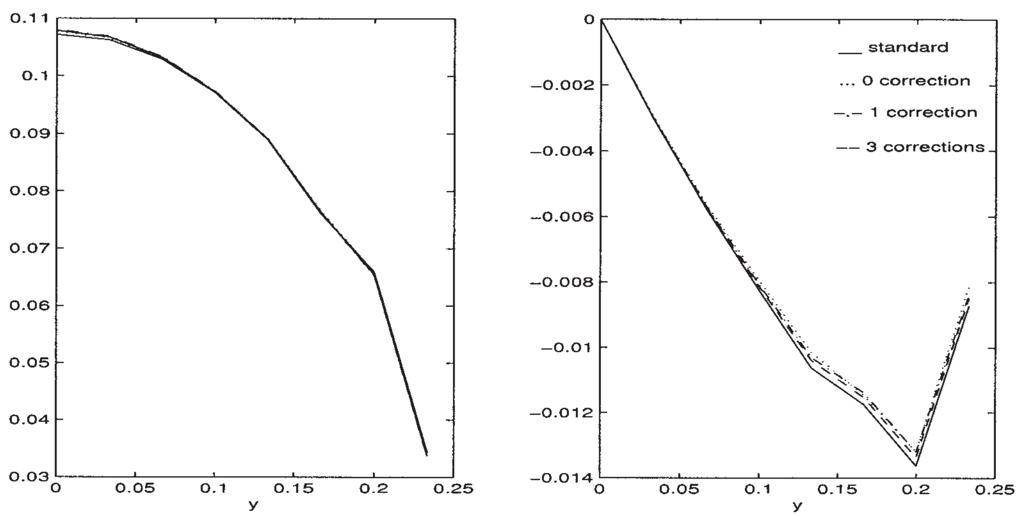

FIG. 6. Johnson-Segalman. Horizontal and vertical velocity near reentrant corner $\lambda = 1$, Case 2: $\bar{\lambda} = 0.7$, $\tilde{\lambda} = 1$.

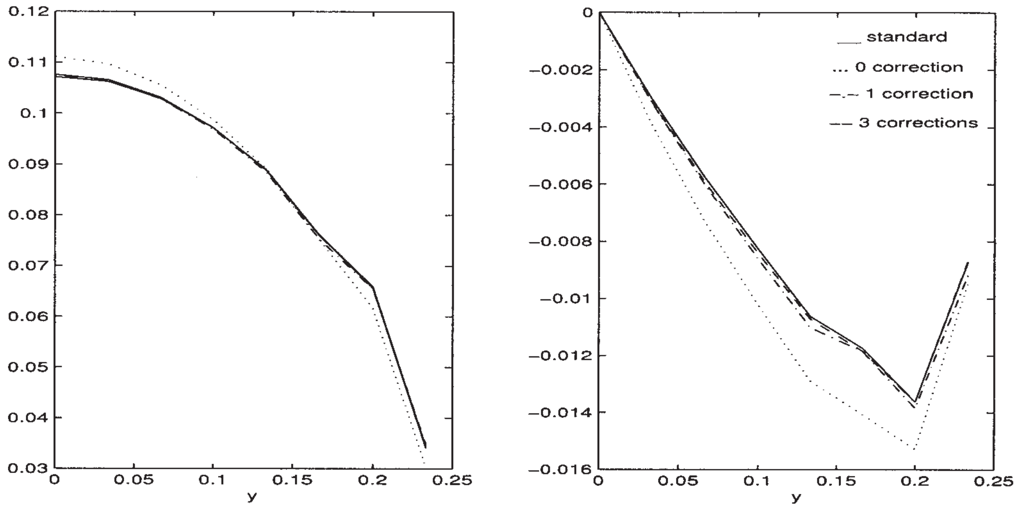

FIG. 7. Johnson-Segalman. Horizontal and vertical velocity near reentrant corner $\lambda = 1$, Case 3: $\bar{\lambda} = 1$, $\tilde{\lambda} = 0.7$.

the reduced Weissenberg number had to be less than or equal to 1.1. Results of the numerical experiments are summarized in Figures 5–9.

As was observed for the Oseen-viscoelastic examples in Section 6, Fig. 6 shows that Case 2 produces little change in the defect approximation from the Newton approximation, portending that if the Newton method has difficulty converging so will the defect step for Case 2. Cases 1 and 3 gave similar results to one another after the same number of correction steps. Figures 8 and 9 show results for $\lambda = 1.5$ and 2, respectively, values for which the standard Newton method failed to converge. For these values of λ Case 2 did not improve the convergence

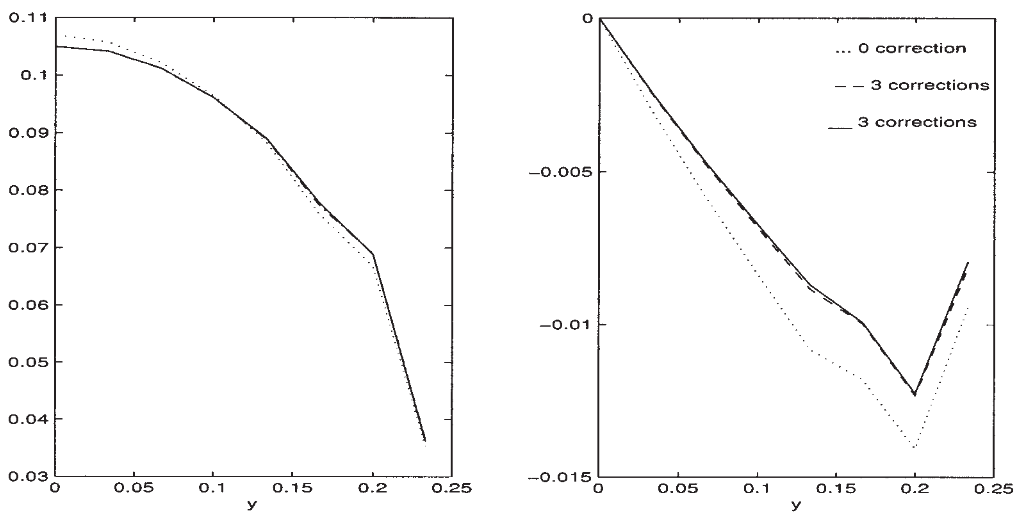

FIG. 8. Johnson-Segalman. Horizontal and vertical velocity near reentrant corner $\lambda = 1.5$, Case 1, 3: $(\bar{\lambda} = 1.1, \tilde{\lambda} = 1.1)$, $(\bar{\lambda} = 1.5, \tilde{\lambda} = 1.1)$.

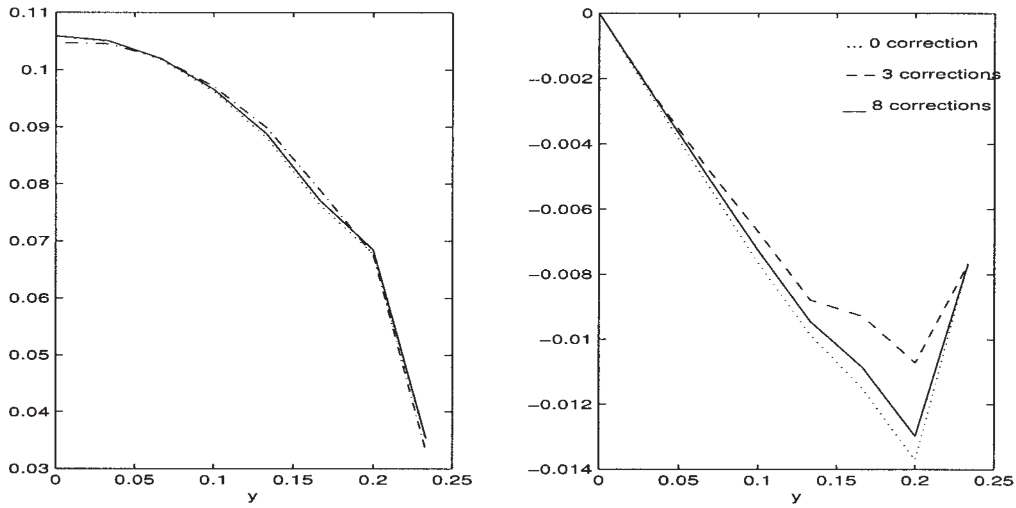

FIG. 9. Johnson-Segalman. Horizontal and vertical velocity near reentrant corner $\lambda = 2$, Case 3: $\bar{\lambda} = 2$, $\tilde{\lambda} = 1.1$.

behavior of the iteration scheme in the defect step. This result was anticipated from Figs. 3 and 7, since reducing the Weissenberg number in the B^h term produced little change in the defect approximation. If the Weissenberg number was moderately reduced, Cases 1 and 3 give similar results as shown for $\lambda = 1$. In fact, when $\bar{\lambda} = \tilde{\lambda} = 1.1$ for $\lambda = 1.5$, velocity profiles are almost identical to Fig. 8. However, when $\bar{\lambda}$ and $\tilde{\lambda}$ were considerably reduced, Case 1 required more correction steps than Case 3. For $\lambda - \tilde{\lambda}$ large, more correction steps are expected to be required to obtain a converged solution in the correction step, as demonstrated in Fig. 9.

8. CONCLUDING REMARKS

The purpose of this article was to investigate the defect-correction method as a stable approximation algorithm for viscoelastic fluid flow at high Weissenberg number. Because of the complexity of analyzing the defect-correction method for the nonlinear Johnson-Segalman equations, a *nearby* problem, the Oseen-viscoelastic problem, was introduced and studied. Numerical tests of the defect-correction method performed for both the Oseen-viscoelastic problem and the Johnson-Segalman problem demonstrated similar behavior. The computations indicate that the numerical solution algorithms are more sensitive to the g_a term in the constitutive equation than the convective derivative.

The defect-correction method, Algorithm 7.1, could be more efficiently implemented when combined with a multilevel method [19]. A multilevel method could be effectively used in both the defect and correction steps. Additionally, a local defect-correction method could be used whereby the Weissenberg number is locally defected to compensate for singularities in the solution, e.g., near boundaries or reentrance corner of a domain.

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