Generalized Newtonian Fluid Flow through a Porous Medium

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Abstract

We present a model for generalized Newtonian fluid flow through a porous medium. In the model the dependence of the fluid viscosity on the velocity is replaced by a dependence on a smoothed (locally averaged) velocity. With appropriate assumptions on the smoothed velocity, existence of a solution to the model is shown. Two examples of smoothing operators are presented in the appendix. A numerical approximation scheme is presented and an a priori error estimate derived. A numerical example is given illustrating the approximation scheme and the a priori error estimate.

Key words. Darcy equation, Generalized Newtonian fluid

AMS Mathematics subject classifications. 65N30, 75D03, 76A05, 76M10

1 Introduction

Of interest in this article is the modeling and approximation of generalized Newtonian fluid flow through a porous medium. Darcy's modeling equations for a steady-state fluid flow through a porous medium, Ω, are

$$\nu_{eff}\, K^{-1}\, \mathbf{u} \;+\; \nabla p \;=\; 0\,, \text{ in } \Omega\,, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} \;=\; 0\,, \text{ in } \Omega\,. \tag{1.2}$$

where $\mathbf{u}$ and p denote the velocity and pressure of the fluid, respectively. $K(\mathbf{x})$ in (1.1) represents the permeability of the medium at $\mathbf{x} \in \Omega$, which is assumed to be a symmetric, positive definite tensor. As our investigations are not concerned with K, we assume that K is of the form $k(\mathbf{x})\mathbf{I}$ where $k(\mathbf{x})$ is a Lipschitz continuous, positive, bounded and bounded away from zero, scalar function. ν_{eff} in (1.1) represents the effective viscosity of the fluid.

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In the case of a Newtonian fluid we have that ν_{eff} is a positive constant. For a generalized Newtonian fluid ν_{eff} is a function of $|\mathbf{u}|$. Two such examples are

$$\text{Power Law Model: } \nu_{eff}(|\mathbf{u}|) \;=\; c_{\nu}\, |\mathbf{u}|^{r-2} \,, \;\; \text{Cross Model: } \nu_{eff}(|\mathbf{u}|) \;=\; \nu_{\infty} + \frac{\nu_0 \,-\, \nu_{\infty}}{1 \,+\, c_{\nu}|\mathbf{u}|^{2-r}} \,, \tag{1.3}$$

where c_{ν}, ν_0, ν_{∞} and r are fluid dependent constants. For shear thinning fluids $1 < r < 2$. (In modeling the viscosity of shear thinning fluids the Power Law model suffers the criticism that as $|\mathbf{u}| \to 0$ $\nu_{eff} \to \infty$.)

For the case of a Newtonian fluid (1.1), (1.2) are well studied. The two standard approaches in analyzing (1.1), (1.2) are: (i) study (1.1), (1.2) as a mixed formulation problem for $\mathbf{u}$ and p (either $(\mathbf{u}, p) \in H_{div}(\Omega) \times L^2(\Omega)$, or $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$), or (ii) use (1.2) to eliminate $\mathbf{u}$ in (1.1) to obtain a generalized Laplace's equation for p.

For generalized Newtonian fluids, with $\nu_{eff} = \nu_{eff}(|\mathbf{u}|)$, assumptions are required on ν_{eff} in order to establish existence and uniqueness of solutions. Typical assumptions are uniform continuity of $\nu_{eff}(|\mathbf{u}|)\mathbf{u}$ and strong monotonicity of $\nu_{eff}(|\mathbf{u}|)$ [7, 8, 10], i.e., there exists $C > 0$ such that

$$|\nu_{eff}(|\mathbf{u}|)\mathbf{u} \;-\; \nu_{eff}(|\mathbf{v}|)\mathbf{v}| \;\leq\; C\,|\mathbf{u} \;-\; \mathbf{v}|\,, \;\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d\,, \tag{1.4}$$

$$(\nu_{eff}(|\mathbf{u}|)\mathbf{u} \;-\; \nu_{eff}(|\mathbf{v}|)\mathbf{v}) \cdot (\mathbf{u} \;-\; \mathbf{v}) \;\geq\; C\,(\mathbf{u} \;-\; \mathbf{v}) \cdot (\mathbf{u} \;-\; \mathbf{v})\,, \;\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d\,. \tag{1.5}$$

A more general setting where the fluid rheology is defined implicitly has been analyzed in [5, 6]. The case where the fluid viscosity depends on the shear rate and pressure has been studied in [13, 12]. For both of these cases additional structure beyond (1.4) and (1.5) is required in order to establish existence and uniqueness of a solution.

A nonlinear Darcy fluid flow problem, with a permeability dependent upon the pressure was investigated by Azaïez, Ben Belgacem, Bernardi, and Chorfi [2], and Girault, Murat, and Salgado [11]. For a Lipschitz continuous permeability function, bounded above and bounded away from zero, existence of a solution $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$ was established. Important in handling the nonlinear permeability function, in establishing existence of a solution, was the property that $p \in H^1(\Omega)$. In [2] the authors also investigated a spectral numerical approximation scheme for the nonlinear Darcy problem, assuming an axisymmetric domain Ω. A convergence analysis for the finite element discretization of that problem was given in [11].

Our interest in this paper is in relaxing the assumptions (1.4) and (1.5). Specifically, our interest is assuming that $\nu_{eff}(\cdot)$ is only Lipschitz continuous and both bounded above and bounded away from zero. However, relaxing the conditions (1.4) and (1.5) requires us to make an additional assumption regarding the argument of $\nu_{eff}(\cdot)$. In order to obtain a modeling system of equations for which a solution can be shown to exist, we replace $\mathbf{u}$ in $\nu_{eff}(|\mathbf{u}|)$ by a *smoothed* velocity, $\mathbf{u}^s$. The approach of regularizing the model with the introduction of $\mathbf{u}^s$ is, in part, motivated by the fact that the Darcy fluid flow equations can be derived by *averaging*, e.g. volume averaging [16], homogenization [1], or mixture theory [14].

Presented in the Appendix are two smoothing operators for $\mathbf{u}$. One is a local averaging operator, whereby $\mathbf{u}^s(\mathbf{x})$ is obtained by averaging $\mathbf{u}$ in a neighborhood of $\mathbf{x}$. The second smoothing operator, which is nonlocal, computes $\mathbf{u}^s(\mathbf{x})$ using a differential filter applied to $\mathbf{u}$. That is, $\mathbf{u}^s$ is given by the solution to an elliptic differential equation whose right hand side is $\mathbf{u}$. For establishing the existence of a solution to (1.1)-(1.2), the key property of the smoothing operators is that they transform a weakly convergent sequence in $L^2(\Omega)$ into a sequence which converges strongly in $L^\infty(\Omega)$.

For the mathematical analysis of this problem it is convenient to have homogeneous boundary conditions. This is achieved by introducing a suitable change of variables. For example, assuming $\partial\Omega = \Gamma_{in} \cup \Gamma \cup \Gamma_{out}$, in the case the specified boundary conditions are

$$\mathbf{u} \cdot (-\mathbf{n}) = g_{in} \text{ on } \Gamma_{in}, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \quad p = p_{out} \text{ on } \Gamma_{out} ,$$

we introduce functions $\mathbf{b}(\mathbf{x})$ and $p_b(\mathbf{x})$ defined on Ω satisfying

$$\begin{aligned} \nabla \cdot \mathbf{b} &= 0 , \text{ in } \Omega , \\ \mathbf{b} \cdot \mathbf{n} &= -g_{in} , \text{ on } \Gamma_{in} , \\ \mathbf{b} \cdot \mathbf{t}_i &= 0 , \text{ on } \Gamma_{in} , \\ \mathbf{b} &= \mathbf{0} , \text{ on } \partial\Omega \backslash \Gamma_{in} , \end{aligned}$$

where $t_i, \ i = 1, \ldots, (d-1)$ denotes an orthogonal set of tangent vectors on Γ_{in}.

$$\begin{aligned} \nabla \cdot \nabla p_b &= 0 , \text{ in } \Omega , \\ p_b &= p_{out} , \text{ on } \Gamma_{out} , \\ \frac{\partial p_b}{\partial \mathbf{n}} &= 0 , \text{ on } \partial\Omega \backslash \Gamma_{out} . \end{aligned}$$

(In case the pressure is specified on the inflow boundary Γ_{in}, then $\mathbf{b} = \mathbf{0}$, and the definition of p_b is appropriately modified.)

With the change of variables: $\mathbf{u} = \mathbf{u}_0 + \mathbf{b}$ and $p = p_0 + p_b$, and subsequent relabeling $\mathbf{u}_0 = \mathbf{u}$, $p_0 = p$ and $\mathbf{f} = -\nabla p_b$ we obtain the following system of modeling equations:

$$\begin{aligned} \beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u} + \beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b} + \nabla p &= \mathbf{f} , \text{ in } \Omega , && (1.6) \\ \nabla \cdot \mathbf{u} &= 0 , \text{ in } \Omega , && (1.7) \\ \mathbf{u} \cdot \mathbf{n} &= 0 , \text{ on } \Gamma_{in} \cup \Gamma , && (1.8) \\ p &= 0 , \text{ on } \Gamma_{out} , && (1.9) \end{aligned}$$

where $\beta(|\mathbf{u}^s + \mathbf{b}|) = \nu_{eff}(|\mathbf{u}^s + \mathbf{b}|) \, k^{-1}$.

In the next section we show that, under suitable assumptions on $\beta(\cdot)$ and $\mathbf{u}^s$, there exists a unique solution to (1.6)-(1.9). An approximation scheme is presented in Section 3, and an a priori error estimate derived. A numerical example illustrating the approximation scheme and the a priori error estimate is presented in Section 4.

2 Existence and Uniqueness

In this section we investigate the existence and uniqueness of solutions to the nonlinear system equations (1.6)-(1.9). We assume that $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a convex polyhedral domain and for vectors in $\mathbb{R}^d$ $|\cdot|$ denotes the Euclidean norm.

Throughout, we use C to denote a generic nonnegative constant, independent of the mesh parameter h, whose actual value may change from line to line in the analysis.

We make the following assumptions on $\beta(\cdot)$ and $\mathbf{u}^s$.

Assumptions on $\beta(\cdot)$

$\mathbf{A}\beta\mathbf{1}$: $\beta(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$,
$\mathbf{A}\beta\mathbf{2}$: $0 < \beta_{min} \le \beta(s) \le \beta_{max}, \ \forall s \in \mathbb{R}^+$,
$\mathbf{A}\beta\mathbf{3}$: β is Lipschitz continuous, $|\beta(s_1) - \beta(s_2)| \le C_\beta |s_1 - s_2|$.

Assumptions on $\mathbf{u}^s$

$\mathbf{Au^s1}$: For $\mathbf{u} \in L^2(\Omega)$, $\|\mathbf{u}^s\|_{L^\infty(\Omega)} \le C_s \|\mathbf{u}\|_{L^2(\Omega)}$,
$\mathbf{Au^s2}$: For $\{\mathbf{u}_n\}_{n=1}^\infty \subset L^2(\Omega)$, with $\mathbf{u}_n$ converging weakly to $\mathbf{u} \in L^2(\Omega)$, then $\{\mathbf{u}_n^s\}_{n=1}^\infty$ converges to $\mathbf{u}^s$ in $L^\infty(\Omega)$,
$\mathbf{Au^s3}$: The mapping $\mathbf{u} \mapsto \mathbf{u}^s$ is linear.

Weak formulation of (1.6)-(1.9)

Let $X = \{\mathbf{v} \in H_{div}(\Omega) : \mathbf{v}\cdot\mathbf{n} = 0\,, \text{ on } \Gamma_{in} \cup \Gamma\}$. We use

$$(f\,,\,g) := \int_\Omega f\cdot g\, d\Omega\,, \quad \text{and} \quad \|f\| := (f\,,\,f)^{1/2}$$

to denote the L^2 inner product and the L^2 norm over Ω, respectively, for both scalar and vector valued functions. Additionally, we introduce the norm

$$\|\mathbf{v}\|_X = \left(\int_\Omega (\nabla\cdot\mathbf{v}\,\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v})\, d\Omega\right)^{1/2} .$$

Remark: For $\mathbf{v} \in H_{div}(\Omega)$ it follows that $\mathbf{v}\cdot\mathbf{n} \in H^{-1/2}(\partial\Omega)$. For the interpretation of the condition $\mathbf{v}\cdot\mathbf{n} = 0$ on $\Gamma_{in} \cup \Gamma$ see [9, 15].

We restate (1.6)-(1.9) as: *Given* $\mathbf{b}$, $\mathbf{f} \in L^2(\Omega)$*, find* $(\mathbf{u}, p) \in X \times L^2(\Omega)$*, such that for all* $\mathbf{v} \in X$ *and* $q \in L^2(\Omega)$

$$(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}\,,\,\mathbf{v}) - (p\,,\,\nabla\cdot\mathbf{v}) + (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) = (\mathbf{f}\,,\,\mathbf{v})\,, \tag{2.1}$$
$$(q\,,\,\nabla\cdot\mathbf{u}) = 0\,. \tag{2.2}$$

For the spaces X and $L^2(\Omega)$ we have the following inf-sup condition

$$\inf_{q\in L^2(\Omega)} \sup_{\mathbf{v}\in X} \frac{(q\,,\,\nabla\cdot\mathbf{v})}{\|q\|\,\|\mathbf{v}\|_X} \ge c_0 > 0\,. \tag{2.3}$$

We begin by establishing boundedness of any solution to (2.1)-(2.2).

Lemma 2.1 *Any solution* $(\mathbf{u}, p) \in X \times L^2(\Omega)$ *to (2.1)-(2.2) satisfies*

$$\|\mathbf{u}\|_X + \|p\| \le C\,(\|\mathbf{b}\| + \|\mathbf{f}\|)\,. \tag{2.4}$$

Proof: From (2.2) and that $\nabla\cdot X \subset L^2(\Omega)$ we have that any solution $\mathbf{u}$ to (2.1)-(2.2) satisfies

$$\|\nabla\cdot\mathbf{u}\| = 0\,. \tag{2.5}$$

With the choice $\mathbf{v} = \mathbf{u}$, $q = p$, subtracting (2.2) from (2.1), and using assumption $\mathbf{A}\beta\mathbf{2}$ yields

$$\begin{aligned} (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}\,,\,\mathbf{u}) &= -\,(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{u}) \,+\, (\mathbf{f}\,,\,\mathbf{u})\,, \\ \beta_{min}\,\|\mathbf{u}\|^2 &\leq \beta_{max}\,\|\mathbf{b}\|\,\|\mathbf{u}\| \,+\, \|\mathbf{f}\|\,\|\mathbf{u}\|\,. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) we obtain the stated bound for $\mathbf{u}$. The estimate for p is obtained using the inf-sup condition (2.3).

$$\begin{aligned} \|p\| &\leq \frac{1}{c_0}\sup_{\mathbf{v}\in X}\frac{(p\,,\,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X} \;=\; \frac{1}{c_0}\sup_{\mathbf{v}\in X}\frac{(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}\,,\,\mathbf{v}) \,+\, (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) \,-\, (\mathbf{f}\,,\,\mathbf{v})}{\|\mathbf{v}\|_X} \\ &\leq \frac{1}{c_0}\left(\|\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}\| \,+\, \|\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\| \,+\, \|\mathbf{f}\|\right) \\ &\leq \frac{1}{c_0}\left(\beta_{max}\,(\|\mathbf{u}\| \,+\, \|\mathbf{b}\|) \,+\, \|\mathbf{f}\|\right)\,, \end{aligned}$$

from which the stated bound follows.

■

$$\text{Define}\quad Z \;=\; \left\{\mathbf{v}\in X \,:\, (q\,,\,\nabla\cdot\mathbf{v}) \,=\, 0\,, \forall q\in L^2(\Omega)\right\}\,.$$

Because of the inf-sup condition (2.3), the weak formulation (2.1)-(2.2) can be equivalently stated as: *Given* $\mathbf{b}$, $\mathbf{f}\in L^2(\Omega))$*, find* $\mathbf{u}\in Z$*, such that for all* $\mathbf{v}\in Z$

$$(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}\,,\,\mathbf{v}) \;+\; (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) \;=\; (\mathbf{f}\,,\,\mathbf{v})\,. \tag{2.7}$$

Remark: For $\mathbf{v}\in Z$, $\|\mathbf{v}\|_X \,=\, \|\mathbf{v}\|$, as $\|\nabla\cdot\mathbf{v}\| \,=\, 0$.

To establish the existence of a solution to (2.7) we use the Leray-Schauder fixed point theorem. To do this we show that a solution to (2.7) is a fixed point of a compact mapping Φ.

Theorem 2.1 *For* $\beta(\cdot)$ *and* $\mathbf{u}^s$ *satisfying assumptions* $\mathbf{A}\beta\mathbf{1} - \mathbf{A}\beta\mathbf{3}$ *and* $\mathbf{Au^s1} - \mathbf{Au^s2}$*, respectively, there exists a solution* $\mathbf{u}$ *to (2.7).*

Proof: Let $\Phi\,:\,Z \longrightarrow Z$ be defined by $\Phi(\mathbf{u}) \,=\, \mathbf{w}$, where $\mathbf{w}$ satisfies

$$(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{w}\,,\,\mathbf{v}) \;+\; (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) \;=\; (\mathbf{f}\,,\,\mathbf{v})\,. \tag{2.8}$$

That Φ is well defined follows from $\mathbf{A}\beta\mathbf{2}$ and the Lax-Milgram theorem.

To show that Φ is a compact operator, let $\{\mathbf{u}_n\}_{n=1}^{\infty}$ denote a bounded sequence in Z. From $\{\mathbf{u}_n\}_{n=1}^{\infty}$ we can extract a subsequence, which we again denote as $\{\mathbf{u}_n\}_{n=1}^{\infty}$, such that $\{\mathbf{u}_n\}_{n=1}^{\infty}$ converges weakly to $\mathbf{u}\in Z$. For $\mathbf{w}_n \,=\, \Phi(\mathbf{u}_n)$, using (2.8)

$$\begin{aligned} (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{w}\,,\,\mathbf{v}) \;-\; (\beta(|\mathbf{u}_n^s + \mathbf{b}|)\mathbf{w}_n\,,\,\mathbf{v}) &= -\,(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) \;+\; (\beta(|\mathbf{u}_n^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) \\ \Longleftrightarrow\;\; (\beta(|\mathbf{u}_n^s + \mathbf{b}|)\,(\mathbf{w} \,-\, \mathbf{w}_n)\,,\,\mathbf{v}) &= -\,((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_n^s + \mathbf{b}|))\,\mathbf{w}\,,\,\mathbf{v}) \\ &\quad -\; ((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_n^s + \mathbf{b}|))\,\mathbf{b}\,,\,\mathbf{v})\,. \end{aligned}$$

With $\mathbf{v} = \mathbf{w} - \mathbf{w}_n$, and using $\mathbf{A}\beta\mathbf{2}$ and $\mathbf{A}\beta\mathbf{3}$

$$\begin{aligned} \beta_{min}\|\mathbf{w} - \mathbf{w}_n\|^2 &\leq \|C_\beta \,|\,(|\mathbf{u}^s + \mathbf{b}| - |\mathbf{u}_n^s + \mathbf{b}|)\,|\,\mathbf{w}\|\,\|\mathbf{w} - \mathbf{w}_n\| \\ &\quad + \|C_\beta \,|\,(|\mathbf{u}^s + \mathbf{b}| - |\mathbf{u}_n^s + \mathbf{b}|)\,|\,\mathbf{b}\|\,\|\mathbf{w} - \mathbf{w}_n\| \\ &\leq \|C_\beta\,|\mathbf{u}^s - \mathbf{u}_n^s|\,\mathbf{w}\|\,\|\mathbf{w} - \mathbf{w}_n\| + \|C_\beta\,|\mathbf{u}^s - \mathbf{u}_n^s|\,\mathbf{b}\|\,\|\mathbf{w} - \mathbf{w}_n\| \\ &\leq C_\beta\sqrt{d}\,\|\mathbf{u}^s - \mathbf{u}_n^s\|_{L^\infty(\Omega)}\,\|\mathbf{w}\|\,\|\mathbf{w} - \mathbf{w}_n\| \\ &\quad + C_\beta\sqrt{d}\,\|\mathbf{u}^s - \mathbf{u}_n^s\|_{L^\infty(\Omega)}\,\|\mathbf{b}\|\,\|\mathbf{w} - \mathbf{w}_n\| \\ \Rightarrow \quad \|\mathbf{w} - \mathbf{w}_n\|_X &= \|\mathbf{w} - \mathbf{w}_n\| \leq \frac{C_\beta\sqrt{d}}{\beta_{min}}\,\|\mathbf{u}^s - \mathbf{u}_n^s\|_{L^\infty(\Omega)}\,(\|\mathbf{w}\| + \|\mathbf{b}\|)\,, \end{aligned}$$

from which, with $\mathbf{Au^s2}$, we can conclude that Φ is a compact operator.

For $r = \frac{\beta_{max}}{\beta_{min}}(\|\mathbf{b}\| + \|\mathbf{f}\|)$, from Lemma 2.1 we have that $\|\Phi(\mathbf{u})\| \leq r\,,\ \forall \mathbf{u} \in Z$. Then, applying the Leray-Schauder fixed point theorem [17] we obtain that there exists a $\mathbf{u} \in Z$ such that $\mathbf{u} = \Phi(\mathbf{u})$.

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Under small data conditions we have the following theorem guaranteeing uniqueness of solutions to (2.7).

Theorem 2.2 *With the stated assumptions* $\mathbf{A}\beta\mathbf{1} - \mathbf{A}\beta\mathbf{3}$ *and* $\mathbf{Au^s1} - \mathbf{Au^s2}$*, and the condition that* $\|\mathbf{b}\| \leq \max\left\{\beta_{min}/\beta_{max}\,,\,\beta_{min}/(C_\beta\sqrt{d}\,C_s)\right\}$*, if a solution* $\mathbf{u}$ *to (2.7) exists satisfying*

$$\|\mathbf{u}\| < \max\left\{\frac{\beta_{min}}{\beta_{max}}\,,\,\frac{\beta_{min}}{C_\beta\sqrt{d}\,C_s}\right\} - \|\mathbf{b}\|\,, \tag{2.9}$$

then there is no other solution to (2.7).

Proof: Suppose that both $\mathbf{u}$ and $\mathbf{w} \in Z$ satisfy (2.7), i.e., together with (2.7) we have that

$$(\beta(|\mathbf{w}^s + \mathbf{b}|)\mathbf{w}\,,\,\mathbf{v}) + (\beta(|\mathbf{w}^s + \mathbf{b}|)\mathbf{b}\,,\,\mathbf{v}) - (\mathbf{f}\,,\,\mathbf{v}) = 0\,,\quad \forall \mathbf{v} \in Z\,. \tag{2.10}$$

With $\mathbf{v} = \mathbf{u} - \mathbf{w}$, subtracting (2.10) from (2.7) and using the bounds for $\beta(\cdot)$ we obtain

$$(\beta(|\mathbf{w}^s + \mathbf{b}|)(\mathbf{u} - \mathbf{w})\,,\,(\mathbf{u} - \mathbf{w})) + ((\beta(|\mathbf{u}^s + \mathbf{b}|) - \beta(|\mathbf{w}^s + \mathbf{b}|))\mathbf{u}\,,\,(\mathbf{u} - \mathbf{w}))$$
$$+ ((\beta(|\mathbf{u}^s + \mathbf{b}|) - \beta(|\mathbf{w}^s + \mathbf{b}|))\mathbf{b}\,,\,(\mathbf{u} - \mathbf{w})) = 0 \tag{2.11}$$
$$\Rightarrow \quad \beta_{min}\,\|\mathbf{u} - \mathbf{w}\|^2 \leq \beta_{max}\|\mathbf{u}\|\,\|\mathbf{u} - \mathbf{w}\| + \beta_{max}\|\mathbf{b}\|\,\|\mathbf{u} - \mathbf{w}\|$$
$$\Rightarrow \quad (\beta_{min} - \beta_{max}(\|\mathbf{u}\| + \|\mathbf{b}\|))\|\mathbf{u} - \mathbf{w}\| \leq 0 \tag{2.12}$$
$$\Rightarrow \quad \mathbf{w} = \mathbf{u}\,,\ \text{provided} \qquad \|\mathbf{u}\| < \frac{\beta_{min}}{\beta_{max}} - \|\mathbf{b}\|\,. \tag{2.13}$$

Alternatively, from (2.11), using $\mathbf{A}\beta\mathbf{3}$ and $\mathbf{Au^s1}$,

$$\begin{aligned} \beta_{min}\,\|\mathbf{u} - \mathbf{w}\|^2 &\leq C_\beta\sqrt{d}\,\|\mathbf{u}^s - \mathbf{w}^s\|_{L^\infty(\Omega)}\|\mathbf{u}\|\,\|\mathbf{u} - \mathbf{w}\| \\ &\quad + C_\beta\sqrt{d}\,\|\mathbf{u}^s - \mathbf{w}^s\|_{L^\infty(\Omega)}\|\mathbf{b}\|\,\|\mathbf{u} - \mathbf{w}\| \\ &\leq C_\beta\sqrt{d}\,C_s\,(\|\mathbf{u}\| + \|\mathbf{b}\|)\,\|\mathbf{u} - \mathbf{w}\|^2 \\ \Rightarrow \quad (\beta_{min} - C_\beta\sqrt{d}\,C_s\,(\|\mathbf{u}\| &+ \|\mathbf{b}\|))\|\mathbf{u} - \mathbf{w}\|^2 \leq 0 \\ \Rightarrow \quad \mathbf{w} = \mathbf{u}\,,\ \text{provided} \qquad \|\mathbf{u}\| &< \frac{\beta_{min}}{C_\beta\sqrt{d}\,C_s} - \|\mathbf{b}\|\,. \end{aligned}$$

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3 Finite Element Approximation

In this section we investigate the finite element approximation to ($\mathbf{u}$, p) satisfying (1.6)-(1.9).

Let T_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrons (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\overline{\Omega} \ = \ \cup_{K \in T_h} \overline{K} .$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \le h_K \le c_2 \rho_K \, ,$$

where h_K is the diameter of triangle (tetrahedron) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K \in T_h} h_K$. For $k \in \mathbb{N}$, let $P_k(A)$ denote the space of polynomials on A of degree no greater than k, and $RT_k(T_h)$ the (Piola) affine transformation of the Raviart-Thomas elements of order k on the unit triangle. We define the finite element spaces X_h, X_h^s and Q_h as follows.

$$X_h \ := \ \left\{ RT_k(T_h) \cap X \right\} \, , \tag{3.1}$$

$$X_h^s \ := \ \left\{ \mathbf{v} \in X \cap C^0(\overline{\Omega}) \ : \ \mathbf{v}|_K \in P_l(K), \forall K \in T_h \right\} \, , \tag{3.2}$$

$$Q_h \ := \ \left\{ q \in L^2(\Omega) \ : \ q|_K \in P_k(K), \forall K \in T_h \right\} . \tag{3.3}$$

$$\text{Additionally, let } \ Z_h \ := \ \{ \mathbf{v} \in X_h \, : \, (q \, , \, \mathbf{v}) \ = \ 0 \, , \ \forall q \in Q_h \} \ . \tag{3.4}$$

Note that as $\nabla \cdot X_h \ \subset \ Q_h$, for $\mathbf{v} \in Z_h$ we have that $\|\nabla \cdot \mathbf{v}\| \ = \ 0$, thus $\|\mathbf{v}\|_X \ = \ \|\mathbf{v}\|$.

For X_h and Q_h defined in (3.1) and (3.3), the following discrete inf-sup condition is satisfied

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q \, , \, \nabla \cdot \mathbf{v})}{\|q\|_Q \, \|\mathbf{v}\|_X} \ \ge \ c_0 \ > \ 0 \, . \tag{3.5}$$

With X_h, Z_h, Q_h defined above, we have the following approximation properties [4, 3]. For $\mathbf{u} \in Z \cap H^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$

$$\inf_{\mathbf{v} \in Z_h} \|\mathbf{u} \, - \, \mathbf{v}\|_X \ = \ \inf_{\mathbf{v} \in Z_h} \|\mathbf{u} \, - \, \mathbf{v}\| \ \le \ C \inf_{\mathbf{v} \in X_h} \|\mathbf{u} \, - \, \mathbf{v}\| \ = \ C \, h^{k+1} \|\mathbf{u}\|_{H^{k+1}(\Omega)} \, , \tag{3.6}$$

$$\inf_{q \in Q_h} \|p \, - \, q\| \ \le \ C \, h^{k+1} \|p\|_{H^{k+1}(\Omega)} \, . \tag{3.7}$$

The approximation scheme we investigate is: *Given* $\mathbf{b}$, $\mathbf{f} \in L^2(\Omega)$, *determine* $(\mathbf{u}_h, \, p_h) \in X_h \times Q_h$, *satisfying*

$$(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h \, , \, \mathbf{v}) \ - \ (p_h \, , \, \nabla \cdot \mathbf{v}) \ + \ (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b} \, , \, \mathbf{v}) \ = \ (\mathbf{f} \, , \, \mathbf{v}) \ , \ \forall \mathbf{v} \in X_h \tag{3.8}$$

$$(q \, , \, \nabla \cdot \mathbf{u}_h) \ = \ 0 \, , \ \forall q \in Q_h . \tag{3.9}$$

Regarding $\mathbf{u}_h^s$, note that applying a smoother to a function $\mathbf{v} \in X_h$ (typically) does not result in $\mathbf{v}^s \in X_h^s$. Therefore, we let $\tilde{\mathbf{u}}_h^s \in H^{l+1}(\Omega) \cap C^0(\Omega)$ denote the result of the smoother applied to $\mathbf{u}_h$, and define

$$\mathbf{u}_h^s(x) \ = \ I_h \tilde{\mathbf{u}}_h^s(x) \, , \tag{3.10}$$

where $I_h : C^0(\Omega) \longrightarrow X_h^s$ denotes an interpolation operator.

We assume that the smoothed velocity $\tilde{\mathbf{u}}_h^s$ is sufficiently regular such that there exists a constant dependent on $\tilde{\mathbf{u}}_h^s$, $C_{\tilde{\mathbf{u}}_h^s}$ such that

$$\|\tilde{\mathbf{u}}_h^s \, - \, I_h \tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)} \ \leq \ C_{\tilde{\mathbf{u}}_h^s} h^{l+1} \, . \tag{3.11}$$

The precise dependence of $C_{\tilde{\mathbf{u}}_h^s}$ on $\tilde{\mathbf{u}}_h^s$ will depend on the particular smoother used.

The existence, uniqueness, and boundedness of the solutions $(\mathbf{u}_h^n, \, p_h^n)$ to (3.8)-(3.9) are established in a completely analogous manner as for the continuous problem.

Corollary 3.1 *(See Lemma 2.1.) Any solution $(\mathbf{u}, \, p) \in X_h \times Q_h$ to (3.8)-(3.9) satisfies*

$$\|\mathbf{u}_h\|_X \ + \ \|p_h\| \ \leq \ C \, (\|\mathbf{b}\| \, + \, \|\mathbf{f}\|) \, . \tag{3.12}$$

Corollary 3.2 *(See Theorem 2.1.) For $\beta(\cdot)$ and $\mathbf{u}_h^s$ satisfying assumptions $\mathbf{A}\beta\mathbf{1}-\mathbf{A}\beta\mathbf{3}$ and $\mathbf{Au^s1}-\mathbf{Au^s2}$, respectively, there exists a solution $(\mathbf{u}_h, p_h)$ to (3.8)-(3.9).*

Proof: The existence of $\mathbf{u}_h$ is established as that for $\mathbf{u}$ in Theorem 2.1. The existence of p_h then follows from the discrete inf-sup condition (3.5).

■

In the next lemma we present the a priori error estimate for the approximation given by (3.8)-(3.9).

Lemma 3.1 *For $(\mathbf{u}, p) \in H^{k+1}(\Omega) \cap X \times H^{k+1}(\Omega)$ satisfying (2.1)-(2.2), $(\mathbf{u}_h, p_h)$ satisfying (3.8)-(3.9), and $\mathbf{u}$ satisfying the small data condition*

$$C_\beta \sqrt{d} \, C_s \, (\|\mathbf{u}\| \, + \, \|\mathbf{b}\|) \ < \ \beta_{min} \, , \tag{3.13}$$

and assuming that $C_{\tilde{\mathbf{u}}_h^s}$ given in (3.11) is bounded by a constant $C_{\mathbf{u}}$, we have that there exists $C > 0$ such that

$$\|\mathbf{u} \, - \, \mathbf{u}_h\|_X \ + \ \|p \, - \, p_h\| \ \leq \ C \, \left(h^{k+1} \, \|\mathbf{u}\|_{H^{k+1}(\Omega)} \ + \ h^{k+1} \, \|p\|_{H^{k+1}(\Omega)} \, + \, C_{\mathbf{u}} h^{l+1} \right) \, . \tag{3.14}$$

Remark: The condition (3.13) guarantees uniqueness of the solution to (3.8)-(3.9), see Theorem 2.2.

Proof: We have that the solutions $\mathbf{u}_h$ and $\mathbf{u}$ to (3.8)-(3.9) and (2.1)-(2.2), respectively, satisfy the following equations for all $\mathbf{v} \in Z_h$:

$$(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h \, , \, \mathbf{v}) \ + \ (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b} \, , \, \mathbf{v}) \ = \ (\mathbf{f} \, , \, \mathbf{v}) \ , \tag{3.15}$$

and

$$\begin{aligned}(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u} \, , \, \mathbf{v}) \ + \ (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b} \, , \, \mathbf{v}) \ &= \ (\mathbf{f} \, , \, \mathbf{v}) \ - \ ((\beta(|\mathbf{u}^s + \mathbf{b}|) \, - \, \beta(|\mathbf{u}_h^s + \mathbf{b}|)) \, \mathbf{u} \, , \, \mathbf{v}) \\ &\quad - \ ((\beta(|\mathbf{u}^s + \mathbf{b}|) \, - \, \beta(|\mathbf{u}_h^s + \mathbf{b}|)) \, \mathbf{b} \, , \, \mathbf{v}) \ . \end{aligned} \tag{3.16}$$

With $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$, subtracting equations (3.15) and (3.16) we obtain

$$\begin{aligned}(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{e}\,,\,\mathbf{v}) \;&=\; -\,((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{u}\,,\,\mathbf{v}) \\ &\quad -\,((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{b}\,,\,\mathbf{v})\;,\;\forall \mathbf{v} \in Z_h\,. \end{aligned} \tag{3.17}$$

For $\mathbf{U} \in Z_h$, let $\mathbf{e} = (\mathbf{u} - \mathbf{U}) + (\mathbf{U} - \mathbf{u}_h) := \mathbf{\Lambda} + \mathbf{E}$. Then, for $\mathbf{v} = \mathbf{E}$, (3.17) becomes

$$\begin{aligned}(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{E}\,,\,\mathbf{E}) \;&=\; -\,(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{\Lambda}\,,\,\mathbf{E}) \,-\, ((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{u}\,,\,\mathbf{E}) \\ &\quad -\,((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{b}\,,\,\mathbf{E})\;. \end{aligned} \tag{3.18}$$

Next we bound each of the terms in (3.18).

$$(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{E}\,,\,\mathbf{E}) \;\ge\; \beta_{min}\|\mathbf{E}\|^2\,. \tag{3.19}$$

$$-\,(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{\Lambda}\,,\,\mathbf{E}) \;\le\; \beta_{max}\,\|\mathbf{\Lambda}\|\,\|\mathbf{E}\| \;\le\; \epsilon_1\|\mathbf{E}\|^2 \,+\, \frac{1}{4\epsilon}\beta_{max}^2\,\|\mathbf{\Lambda}\|^2\,. \tag{3.20}$$

$$\begin{aligned} &-\,((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{u}\,,\,\mathbf{E}) \;\le\; \|\,(\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{u}\|\,\|\mathbf{E}\| \\ &\le\; C_\beta\sqrt{d}\,\|\mathbf{u}^s \,-\, \mathbf{u}_h^s\|_{L^\infty(\Omega)}\,\|\mathbf{u}\|\,\|\mathbf{E}\| \\ &\le\; C_\beta\sqrt{d}\,\left(\|\mathbf{u}^s \,-\, \tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)} \,+\, \|\tilde{\mathbf{u}}_h^s \,-\, \mathbf{u}_h^s\|_{L^\infty(\Omega)}\right)\,\|\mathbf{u}\|\,\|\mathbf{E}\| \\ &\le\; C_\beta\sqrt{d}\,\left(C_s\,\|\mathbf{u} \,-\, \mathbf{u}_h\| \,+\, \|\tilde{\mathbf{u}}_h^s \,-\, I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}\right)\,\|\mathbf{u}\|\,\|\mathbf{E}\| \\ &\le\; C_\beta\sqrt{d}\,\left(C_s\,(\|\mathbf{\Lambda}\| \,+\, \|\mathbf{E}\|) \,+\, \|\tilde{\mathbf{u}}_h^s \,-\, I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}\right)\,\|\mathbf{u}\|\,\|\mathbf{E}\| \\ &\le\; C_\beta\sqrt{d}\,C_s\,\|\mathbf{u}\|\,\|\mathbf{E}\|^2 \,+\, \epsilon_2\|\mathbf{E}\|^2 \,+\, \frac{1}{2\epsilon_2}C_\beta^2\,d\,\|\mathbf{u}\|^2\,\left(C_s^2\,\|\mathbf{\Lambda}\|^2 \,+\, \|\tilde{\mathbf{u}}_h^s \,-\, I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}^2\right) \end{aligned} \tag{3.21}$$

A similar bound to that given in (3.21) holds for the third term on the right hand side of (3.18). Combining the estimates (3.19)-(3.21) with (3.18) we have

$$\begin{aligned} &\left(\beta_{min} \,-\, \epsilon_1 \,-\, C_\beta\sqrt{d}\,C_s\,(\|\mathbf{u}\| \,+\, \|\mathbf{b}\|) \,-\, 2\epsilon_2\right)\|\mathbf{E}\|^2 \;\le \\ &\quad \left(\frac{1}{4\,\epsilon_1}\beta_{max}^2 \,+\, \frac{1}{2\,\epsilon_2}C_\beta^2\,d\,C_s^2\,(\|\mathbf{u}\|^2 \,+\, \|\mathbf{b}\|^2)\right)\,\|\mathbf{\Lambda}\|^2 \\ &\quad +\, \frac{1}{2\,\epsilon_2}C_\beta^2\,d\,\left(\|\mathbf{u}\|^2 \,+\, \|\mathbf{b}\|^2\right))\,\|\tilde{\mathbf{u}}_h^s \,-\, I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}^2\,. \end{aligned} \tag{3.22}$$

Hence, in view of the stated hypothesis (3.13), there exists $C > 0$ such that $\|\mathbf{E}\| \le C\,\left(\|\mathbf{\Lambda}\| \,+\, \|\tilde{\mathbf{u}}_h^s \,-\, I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}\right)$. Finally, from the triangle inequality and (3.6) we have that

$$\|\mathbf{u} - \mathbf{u}_h\|_X \;=\; \|\mathbf{u} - \mathbf{u}_h\| \;\le\; \|\mathbf{\Lambda}\| + \|\mathbf{E}\| \;\le\; C\,\left(h^{k+1}\|\mathbf{u}\|_{H^{k+1}(\Omega)} \,+\, C_{\mathbf{u}}\,h^{l+1}\right)\,. \tag{3.23}$$

To obtain the error estimate for the pressure, let $P \in Q_h$. Then, from (3.5) we have that there exists $\mathbf{v} \in X_h$ such that

$$c_0\,\|P \,-\, p_h\| \;\le\; \frac{(P \,-\, p_h\,,\,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X} \;=\; \frac{(P\,,\,\nabla\cdot\mathbf{v}) \,-\, (p_h\,,\,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X}\,.$$

Using (3.8) and (2.1) we obtain

$$
\begin{aligned}
c_0 \, \|\mathbf{v}\|_X \, \|P - p_h\| \;&\le\; (P \,,\, \nabla \cdot \mathbf{v}) \,+\, (\mathbf{f} \,,\, \mathbf{v}) \,-\, (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h \,,\, \mathbf{v}) \,-\, (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b} \,,\, \mathbf{v}) \\
&=\; (P \,,\, \nabla \cdot \mathbf{v}) \,-\, (p \,,\, \nabla \cdot \mathbf{v}) \,+\, (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u} \,,\, \mathbf{v}) \,+\, (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b} \,,\, \mathbf{v}) \\
&\quad -\, (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h \,,\, \mathbf{v}) \,-\, (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b} \,,\, \mathbf{v}) \\
=\; (P - p \,,\, \nabla \cdot \mathbf{v}) \;&+\; (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\,(\mathbf{u} - \mathbf{u}_h) \,,\, \mathbf{v}) + ((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{u} \,,\, \mathbf{v}) \\
&\quad +\, ((\beta(|\mathbf{u}^s + \mathbf{b}|) \,-\, \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{b} \,,\, \mathbf{v}) \\
\Rightarrow c_0 \, \|P - p_h\| \;&\le\; \|P - p\| \,+\, \beta_{max} \, \|\mathbf{u} - \mathbf{u}_h\| \\
&\quad +\, C_\beta \sqrt{d} \, \left(C_s \, \|\mathbf{u} \,-\, \mathbf{u}_h\| \,+\, \|\tilde{\mathbf{u}}_h^s \,-\, I_h \tilde{\mathbf{u}}_h\|_{L^\infty(\Omega)}\right) (\|\mathbf{u}\| \,+\, \|\mathbf{b}\|) \,.
\end{aligned}
$$

Using the triangle inequality, (2.4), (3.7), (3.11) and (3.23) we obtain the stated estimate for $\|p - p_h\|$. ■

Remark: The $L^\infty(\Omega)$ norm used for the term $(\tilde{\mathbf{u}}_h^s \,-\, I_h \tilde{\mathbf{u}}_h^s)$ and the $L^2(\Omega)$ norm used for $\mathbf{u}$ and $\mathbf{b}$ in (3.22) may be interchanged, assuming that the functions $\mathbf{u}$ and $\mathbf{b}$ are sufficiently regular.

4 Numerical Computations

In this section we present a numerical example to demonstrate the numerical approximation scheme (3.8)-(3.9), and investigate the a priori error estimate (3.14).

Let $\Omega \,=\, (-1 \,,\, 1) \times (0 \,,\, 1)$, $\beta(s) \,=\, v_\infty \,+\, (v_0 - v_\infty)/(1 \,+\, k s^{2-r})$, with parameters $v_\infty = 1$, $v_0 = 5$, $k = 1$, and $r = 1/2$. ($\beta(\cdot)$ represents the Cross model for the effective viscosity for a generalized Newtonian fluid.) The true solution $\mathbf{u}$ and p are taken to be

$$
\mathbf{u}(x, y) \;=\; \begin{bmatrix} \sin(\pi x) \, \cos(\pi y) \\ \cos(\pi x) \, \sin(\pi y) \end{bmatrix} \,, \quad p(x, y) \;=\; xy \,. \tag{4.1}
$$

For this choice of $\mathbf{u}$, $\nabla \cdot \mathbf{u} \neq 0$, hence a right hand side function is added to (3.9). The boundary conditions used are $\mathbf{u} \cdot \mathbf{n}$ along $\{1\} \times (0 \,,\, 1)$, $(-1 \,,\, 1) \times \{1\}$, $\{-1\} \times (0 \,,\, 1)$, with $p = 0$ weakly imposed along $(-1 \,,\, 1) \times \{0\}$. A computation mesh corresponding to mesh parameter $h \,=\, 1/4$ is presented in Figure 4.1. Plots of $\beta(|\mathbf{u}|)$, $\mathbf{u}$ and p are given in Figures 4.2, 4.3 and 4.4, respectively.

Example 1.

For $\mathbf{u}_h^s$, the interpolate of $\tilde{\mathbf{u}}_h^s$ (the *smoothed* function of $\mathbf{u}_h$), we compute a continuous, piecewise quadratic, velocity by taking a simple average of $\mathbf{u}_h$ at the nodal points of $\mathbf{u}_h^s$. Computations were performed using $RT_0 - discP_0$, $RT_1 - discP_1$, and $RT_2 - discP_2$ elements for the velocity and pressure. (By RT_k we are referring to Raviart-Thomas elements of degree k, and $discP_k$ refers to the space of discontinuous scalar functions which are polynomials of degree less that or equal to k on each triangle in the triangulation.) The results, together with the experimental convergence rates are presented in Table 4.1. The experimental convergence rates are consistent with those predicted by (3.14) for $l \,=\, 2$. (Regarding the $O(h^4)$ experimental convergence rate for the pressure using $RT_2 - discP_2$ elements, note that the true solution for the pressure lies in the $discP_2$ approximation space.)

Example 2.

In order to investigate the dependence of the approximation on the interpolant of the smoother,

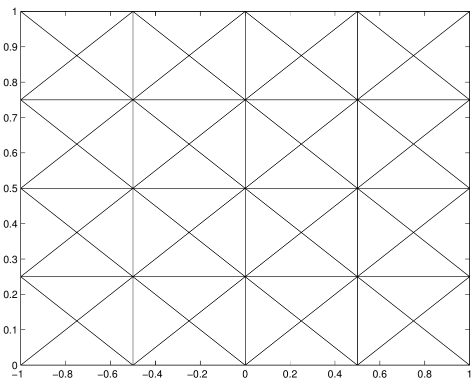

Figure 4.1: Computational mesh for $h = 1/4$.

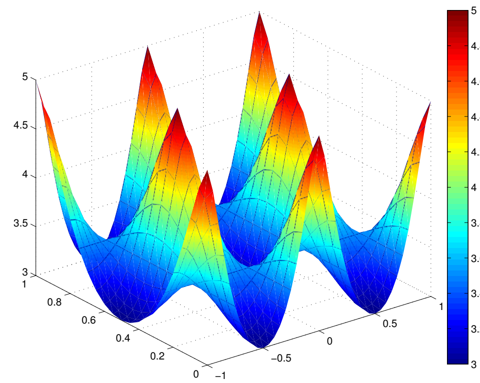

Figure 4.2: Plot of $\beta(|\mathbf{u}|)$.

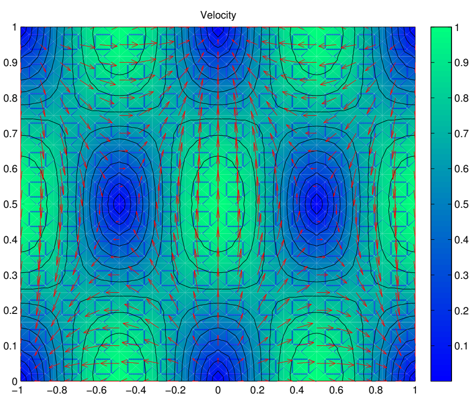

Figure 4.3: Plot of the velocity flow field $\mathbf{u}$.

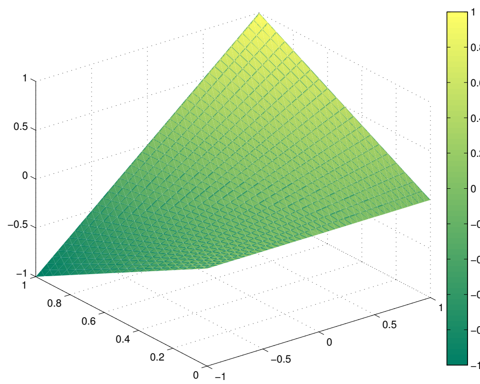

Figure 4.4: Plot of the pressure function p.

in this case we take $\mathbf{u}_h^s$ to be a continuous, piecewise linear function, obtained by taking a simple average of $\tilde{\mathbf{u}}_h^s$ at the vertices of the triangles in the triangulations. The results obtained using $RT1 - discP1$, and $RT2 - discP2$ approximating elements are presented in Table 4.2. In this case $(l = 1)$ we observe optimal convergence for $RT1 - discP1$ (and $RT0 - discP0$, results not included). However, the experimental convergence rates for the $RT2 - discP2$ approximation is limited to 2 for the velocity and pressure, consistent with (3.14).

A Example of a local smoothing function

In this section we give an example of a local smoothing function which satisfies properties $\mathbf{Au^s1}$ and $\mathbf{Au^s2}$ presented in Section 2. The smoothing function is a simple averaging operator. We use the term *domain* to refer to an open connected set in $\mathbb{R}^n$.

For simplicity we present the case for a scalar function $u(\mathbf{x})$. For a vector valued function the smoother is simply applied to each of the coordinate functions.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}$	Cvg. rate	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. rate
			$X_h = RT_0 \quad Q_h = discP_0$			
1/4	3.543E-01	0.98	1.274E+00	0.97	9.212E-2	1.29
1/6	2.376E-01	0.98	8.589E-01	0.99	5.464E-2	1.10
1/8	1.790E-01	1.00	6.468E-01	0.99	3.981E-2	1.08
1/10	1.433E-01	1.00	5.184E-01	0.99	3.131E-2	1.05
1/12	1.195E-01		4.325E-01		2.588E-2	
Pred.		1.0		1.0		1.0
			$X_h = RT_1 \quad Q_h = discP_1$			
1/4	5.645E-02	1.94	2.020E-01	1.97	5.680E-03	2.80
1/6	2.574E-02	1.98	9.089E-02	1.99	1.824E-03	2.44
1/8	1.456E-02	1.99	5.134E-02	1.99	9.049E-04	2.30
1/10	9.344E-03	1.99	3.292E-02	1.99	5.419E-04	2.21
1/12	6.495E-03		2.289E-02		3.619E-04	
Pred.		2.0		2.0		2.0
			$X_h = RT_2 \quad Q_h = discP_2$			
1/4	6.661E-03	3.09	2.268E-02	2.97	9.877E-04	3.98
1/6	1.905E-03	3.06	6.788E-03	2.99	1.966E-04	3.94
1/8	7.905E-04	3.02	2.874E-03	2.99	6.328E-05	4.02
1/10	4.028E-04	3.02	1.474E-03	3.00	2.578E-05	3.98
1/12	2.321E-04		8.537E-04		1.247E-05	
Pred.		3.0		3.0		3.0

Table 4.1: Example 1, $\mathbf{u}_h^s$ a quadratic interpolant of $\tilde{\mathbf{u}}_h^s$.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}$	Cvg. rate	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. rate
			$X_h = RT_1 \quad Q_h = discP_1$			
1/4	6.744E-2	1.88	2.020E-01	1.97	2.420E-2	1.99
1/6	3.150E-2	1.93	9.089E-02	1.99	1.079E-2	2.06
1/8	1.808E-2	1.95	5.134E-02	1.99	5.960E-3	2.01
1/10	1.170E-2	1.97	3.292E-02	1.99	3.802E-3	2.01
1/12	8.169E-3		2.289E-02		2.634E-3	
Pred.		2.0		2.0		2.0
			$X_h = RT_2 \quad Q_h = discP_2$			
1/4	3.635E-2	1.84	2.268E-02	2.97	2.770E-2	1.17
1/6	1.727E-2	1.97	6.788E-03	2.99	1.727E-2	3.15
1/8	9.804E-3	1.97	2.874E-03	2.99	6.984E-3	2.00
1/10	6.310E-3	1.97	1.474E-03	3.00	4.473E-3	2.00
1/12	4.404E-3		8.537E-04		3.107E-3	
Pred.		2.0		2.0		2.0

Table 4.2: Example 2, $\mathbf{u}_h^s$ a linear interpolant of $\tilde{\mathbf{u}}_h^s$.

Let Ω denote a bounded domain in $\mathbb{R}^n$ and $\mathcal{L}(\Omega)$ the Lebesgue measurable sets in Ω. Let $\delta > 0$ denote the (fixed) volume measure over which we average a function to obtain its *smoothed* value.

For $\mathbf{x} \in \Omega$ the typical averaging volume which comes to mind is $B(\mathbf{x}, r_\delta)$, where $B(\mathbf{x}, r_\delta)$ denotes the ball centered at $\mathbf{x}$ of radius r_δ having volume δ. As δ is fixed the difficulty in using $B(\mathbf{x}, r_\delta)$ arises for points whose distance from $\partial\Omega$ is less that r_δ. This requires us to consider averaging volumes other than balls. Namely, for each point $\mathbf{x} \in \Omega$ we associate a domain $V(\mathbf{x})$ having a volume of δ. We require that the association of $\mathbf{x}$ with $V(\mathbf{x})$ be continuous. This continuity is formally described in the next paragraph.

Let ν denote the Lebesgue measure in $\mathbb{R}^n$. For $S_1, S_2 \in \mathcal{L}(\Omega)$, introduce the metric $d(S_1\,,\,S_2)$ defined by

$$d(S_1\,,\,S_2) \;:=\; \nu(S_1 \triangle S_2)\,,\ \text{where } S_1 \triangle S_2 \;:=\; (S_1 \backslash S_2) \cup (S_2 \backslash S_1)\,. \tag{A.1}$$

Now, let $V\,:\,\overline{\Omega} \longrightarrow \mathcal{L}(\Omega)$ satisfy: (i) $V(\mathbf{x})$ is a domain with $\nu(V(\mathbf{x})) \;=\; \delta$ for all $\mathbf{x} \in \Omega$, and (ii) $d(V(\mathbf{x})\,,\,V(\mathbf{y})) \;=\; \nu(V(\mathbf{x}) \triangle V(\mathbf{y})) \;\le\; C_V\,|\mathbf{x} \,-\, \mathbf{y}|$ for all $\mathbf{x}$, $\mathbf{y} \in \Omega$, where C_V a fixed constant. For convenience we denote the domain $V(\mathbf{x})$ as $V_{\mathbf{x}}$.

Definition: Local Smoothing Operator
For $u \in L^2(\Omega)$, define u^s as

$$u^s(\mathbf{x}) \;=\; \frac{1}{\delta}\int_{V_{\mathbf{x}}} u(\mathbf{z})\,d\Omega\,. \tag{A.2}$$

We have the following properties for $u^s(\mathbf{x})$.

Lemma A.1 *For $u \in L^2(\Omega)$, u^s defined by (A.2) satisfies the following properties.*
(i) $\|u^s\|_{L^\infty(\Omega)} \;\le\; \delta^{-1/2}\,\|u\|_{L^2(\Omega)}$.
(ii) $u^s\,:\,\overline{\Omega} \longrightarrow \mathbb{R}$ is uniformly continuous.
(iii) Suppose that $\{u_n\}_{n=1}^\infty \subset L^2(\Omega)$ and that u_n converges weakly to $u \in L^2(\Omega)$. Then $\{u_n^s\}_{n=1}^\infty$ converges to u^s in $L^\infty(\Omega)$.

Proof: Let $1_S \in L^2(\Omega)$ denote the characteristic function of the domain S. From (A.2), for $\mathbf{x} \in \Omega$

$$\begin{aligned}
u^s(\mathbf{x}) \;&=\; \frac{1}{\delta}\int_{V_{\mathbf{x}}} u(\mathbf{z})\,d\Omega \;=\; \frac{1}{\delta}\int_{\Omega} 1_{V_{\mathbf{x}}}\,u(\mathbf{z})\,d\Omega \\
&\le\; \frac{1}{\delta}\left(\int_{\Omega} (1_{V_{\mathbf{x}}})^2\,d\Omega\right)^{1/2}\left(\int_{\Omega} u(\mathbf{z})^2\,d\Omega\right)^{1/2} \\
&=\; \delta^{-1/2}\,\|u\|_{L^2(\Omega)}\,,
\end{aligned}$$

which establishes (i).

For $\mathbf{x}$, $\mathbf{y} \in \Omega$,

$$\begin{aligned}
|u^s(\mathbf{x}) \;-\; u^s(\mathbf{y})| \;&\le\; \frac{1}{\delta}\int_{\Omega} \left|1_{V_{\mathbf{x}}} - 1_{V_{\mathbf{y}}}\right|\,|u(\mathbf{z})|\,d\Omega \\
&=\; \frac{1}{\delta}\left(\int_{\Omega} \left(1_{V_{\mathbf{x}}} - 1_{V_{\mathbf{y}}}\right)^2\,d\Omega\right)^{1/2}\left(\int_{\Omega} u(\mathbf{z})^2\,d\Omega\right)^{1/2} \\
&=\; \frac{1}{\delta}\,\|u\|_{L^2(\Omega)}\,d(V(\mathbf{x})\,,\,V(\mathbf{y}))^{1/2} \\
&=\; \frac{C_V^{1/2}}{\delta}\,\|u\|_{L^2(\Omega)}\,|\mathbf{x} \,-\, \mathbf{y}|^{1/2}\,,
\end{aligned}$$

which establishes the uniform continuity of u^s. As u^s is bounded on Ω then u^s can be continuously extended to $\partial\Omega$.

To establish (iii), as $\{u_n\}$ converges weakly, let $\sup_n \|u_n\| = M < \infty$. In addition, for $\epsilon > 0$, $\sigma = \left(\epsilon/(6\, M\, C_V^{1/2})\right)^2$, let $\{\mathbf{z}_i\}_{i=1}^N$ denote a σ-net of $\overline{\Omega}$, i.e., for all $\mathbf{x} \in \Omega$ there exists an $i_{\mathbf{x}} \in \{1, 2, \ldots, N\}$ such that $|\mathbf{x} - \mathbf{z}_{i_{\mathbf{x}}}| < \sigma$.

Now,

$$\begin{aligned}
|u_n^s(\mathbf{x}) - u^s(\mathbf{x})| &= \left| \int_{V_{\mathbf{x}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| \\
&= \left| \int_{V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega + \int_{V_{\mathbf{x}} \setminus V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| \\
&\leq \left| \int_{V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| + \int_{V_{\mathbf{x}} \triangle V_{\mathbf{z}_{i_{\mathbf{x}}}}} |u_n(\mathbf{y}) - u(\mathbf{y})|\, d\Omega\,. \qquad \text{(A.3)}
\end{aligned}$$

Since $\{u_n\}$ converges weakly to u in $L^2(\Omega)$, for all $w \in L^2(\Omega)$ there exists N_w such that for $n > N_w$

$$\left| \int_{\Omega} (u_n - u)\, w\, d\Omega \right| < \frac{\epsilon}{3}\,. \qquad \text{(A.4)}$$

Let $N_\star = \max_{i=1,2,\ldots,N} \left\{ N_{1_{V_{\mathbf{z}_i}}} \right\}$. Then, for $n > N_\star$

$$\left| \int_{V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| = \left| \int_{\Omega} (u_n(\mathbf{y}) - u(\mathbf{y}))\, 1_{V_{\mathbf{z}_i}}\, d\Omega \right| < \frac{\epsilon}{3}\,.$$

For the second term on the right hand side of (A.3) we have

$$\begin{aligned}
\int_{V_{\mathbf{x}} \triangle V_{\mathbf{z}_{i_{\mathbf{x}}}}} |u_n(\mathbf{y}) - u(\mathbf{y})|\, d\Omega &\leq \left(\int_{V_{\mathbf{x}} \triangle V_{\mathbf{z}_{i_{\mathbf{x}}}}} |u_n(\mathbf{y}) - u(\mathbf{y})|^2\, d\Omega \right)^{1/2} \left(\int_{V_{\mathbf{x}} \triangle V_{\mathbf{z}_{i_{\mathbf{x}}}}} 1\, d\Omega \right)^{1/2} \\
&\leq 2M\, \nu(V_{\mathbf{x}} \triangle V_{\mathbf{z}_{i_{\mathbf{x}}}})^{1/2} \\
&\leq 2M\, C_V^{1/2}\, |\mathbf{x} - \mathbf{z}_{i_{\mathbf{x}}}|^{1/2} \leq 2M\, C_V^{1/2}\, \sigma^{1/2} \\
&= \frac{\epsilon}{3}\,. \qquad \text{(A.5)}
\end{aligned}$$

Thus, from (A.3)-(A.5) it follows that for all $\mathbf{x} \in \Omega$, for $n > N_\star$

$$|u_n^s(\mathbf{x}) - u^s(\mathbf{x})| < \frac{2}{3}\epsilon, \quad \text{i.e., } \|u_n^s - u^s\|_{L^\infty(\Omega)} < \frac{2}{3}\epsilon < \epsilon\,.$$

■

A.1 Regularity of u^s (for $u \in L^\infty(\Omega)$)

If, in place of $u \in L^2(\Omega)$, we have $u \in L^\infty(\Omega)$ then u^s defined by (A.2) is a $H^1(\Omega)$ function. To establish this regularity result we begin by citing a characterization of the $W^{1,p}(\mathbb{R}^n)$ function space.

Theorem A.1 ([18], Theorem 2.1.6) *Let $1 < p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and*

$$\left(\int_{\mathbb{R}^n} \left| \frac{u(\mathbf{x}+\mathbf{h}) \,-\, u(\mathbf{x})}{|\mathbf{h}|} \right|^p d\mathbf{x} \right)^{1/p} \;=\; |\mathbf{h}|^{-1} \, \| u(\mathbf{x}+\mathbf{h}) \,-\, u(\mathbf{x}) \|_{L^p(\mathbb{R}^n)} \tag{A.6}$$

remains bounded for all $\mathbf{h} \in \mathbb{R}^n$.

■

Theorem A.2 *If $u \in L^\infty(\Omega)$ then, for u^s defined by (A.2), $u^s \in H^1(\Omega)$.*

Proof: In order to apply Theorem A.1 we need to define an extension of u to $\mathbb{R}^n$. Let

$$\tilde{u}(\mathbf{x}) \;=\; \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega \\ 0, & \mathbf{x} \notin \Omega \end{cases} \;, \quad \text{and } \widetilde{V} \,:\, \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n)$$

denote an extension of V satisfying properties (i) and (ii) (with Ω replaced by $\mathbb{R}^n$), and additionally that there exists constants $C_1 > 0$ and $C_2 \geq 0$ such that (iii) diameter$(\widetilde{V}(\mathbf{z})) \,\leq\, C_1$ for all $\mathbf{z} \in \mathbb{R}^n$, and (iv) $\sup_{\mathbf{z}\in\mathbb{R}^n} \inf_{\mathbf{y}\in\widetilde{V}(\mathbf{z})} |\mathbf{z} - \mathbf{y}| \,\leq\, C_2$.

Let Ω_B denote the bounded set, $\Omega_B \,:=\, \{\mathbf{x} \in \mathbb{R}^n \,:\, \inf_{\mathbf{y}\in\Omega} |\mathbf{x} - \mathbf{y}| \,<\, 1 + C_1 + C_2\} \supset \text{support}(\tilde{u}^s)$. Note that for $\mathbf{x} \in \mathbb{R}^n \backslash \Omega_B$ and $|\mathbf{h}| < 1$, $\tilde{u}^s(\mathbf{x}+\mathbf{h}) \,=\, 0$.

Now, for $|\mathbf{h}| \geq 1$,

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \frac{\tilde{u}^s(\mathbf{x}+\mathbf{h}) \,-\, \tilde{u}^s(\mathbf{x})}{|\mathbf{h}|} \right|^2 d\mathbf{x} \;&\leq\; \frac{2}{|\mathbf{h}|^2} \left(\int_{\mathbb{R}^n} (\tilde{u}^s(\mathbf{x}+\mathbf{h}))^2 \, d\mathbf{x} \;+\; \int_{\mathbb{R}^n} (\tilde{u}^s(\mathbf{x}))^2 \, d\mathbf{x} \right) \\ &\leq\; \frac{4}{|\mathbf{h}|^2} \int_{\mathbb{R}^n} (\tilde{u}^s(\mathbf{x}))^2 \, d\mathbf{x} \;\leq\; \frac{4}{|\mathbf{h}|^2} \| \tilde{u}^s \|^2_{L^\infty(\Omega_B)} \, \nu(\Omega_B) \\ &\leq\; 4\,\nu(\Omega_B) \, \|\tilde{u}\|^2_{L^\infty(\mathbb{R}^n)} \;=\; 4\,\nu(\Omega_B) \, \|u\|^2_{L^\infty(\Omega)} \,. \end{aligned} \tag{A.7}$$

For $|\mathbf{h}| < 1$,

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \frac{\tilde{u}^s(\mathbf{x}+\mathbf{h}) \,-\, \tilde{u}^s(\mathbf{x})}{|\mathbf{h}|} \right|^2 d\mathbf{x} \;&=\; \frac{1}{|\mathbf{h}|^2} \int_{\Omega_B} |\tilde{u}^s(\mathbf{x}+\mathbf{h}) \,-\, \tilde{u}^s(\mathbf{x})|^2 \, d\mathbf{x} \\ &=\; \frac{1}{|\mathbf{h}|^2} \int_{\Omega_B} \left| \frac{1}{\delta} \int_{\Omega_B} \tilde{u}(\mathbf{z}) \, \left(1_{\widetilde{V}_{\mathbf{x}+\mathbf{h}}}(\mathbf{z}) \,-\, 1_{\widetilde{V}_{\mathbf{x}}}(\mathbf{z}) \right) \, d\mathbf{z} \right|^2 d\mathbf{x} \\ &\leq\; \frac{1}{|\mathbf{h}|^2} \, \frac{1}{\delta^2} \|\tilde{u}\|^2_{L^\infty(\Omega_B)} \int_{\Omega_B} \left(\int_{\Omega_B} \left| 1_{\widetilde{V}_{\mathbf{x}+\mathbf{h}}}(\mathbf{z}) \,-\, 1_{\widetilde{V}_{\mathbf{x}}}(\mathbf{z}) \right| \, d\mathbf{z} \right)^2 d\mathbf{x} \\ &=\; \frac{1}{|\mathbf{h}|^2} \, \frac{1}{\delta^2} \|\tilde{u}\|^2_{L^\infty(\Omega_B)} \int_{\Omega_B} d(\widetilde{V}_{\mathbf{x}+\mathbf{h}} \,, \widetilde{V}_{\mathbf{x}})^2 \, d\mathbf{x} \\ &\leq\; \frac{1}{|\mathbf{h}|^2} \, \frac{1}{\delta^2} \|\tilde{u}\|^2_{L^\infty(\Omega_B)} \, C_V^2 \, |\mathbf{h}|^2 \, \nu(\Omega_B) \\ &=\; \frac{1}{\delta^2} \, C_V^2 \, \nu(\Omega_B) \, \|\tilde{u}\|^2_{L^\infty(\Omega_B)} \;=\; \frac{1}{\delta^2} \, C_V^2 \, \nu(\Omega_B) \, \|u\|^2_{L^\infty(\Omega)} \,. \end{aligned} \tag{A.8}$$

From (A.7) and (A.8), together with Theorem A.1, we obtain that $\tilde{u}^s \in H^1(\mathbb{R}^n)$. As $u^s \,=\, \tilde{u}^s|_\Omega$, it then follows that $u^s \in H^1(\Omega)$.

■

B Example of a differential smoothing function

As an alternative to the local averaging filter discussed in Section A, in this section we present a differential smoothing filter.

$$\text{Let} \quad X^s \;=\; H_0^1(\Omega) \;=\; \{\mathbf{v} \in H^1(\Omega) \,:\, \mathbf{v} \,=\, \mathbf{0} \ \text{ on } \partial\Omega\} \subset X\,. \tag{B.9}$$

Definition: Differential Smoothing Operator
For $\mathbf{u} \in L^2(\Omega)$, define $\mathbf{u}^s \in X^s$ as

$$(\nabla \mathbf{u}^s\,,\, \nabla \mathbf{v}) \;=\; (\mathbf{u}^s\,,\, \mathbf{v})\,,\ \forall \mathbf{v} \in X^s\,. \tag{B.10}$$

The well posedness of $\mathbf{u}^s$ follows from an application of the Lax-Milgram theorem. Next we show that this smoothing operation satisfies properties $\mathbf{Au^s1}$ and $\mathbf{Au^s2}$ presented in Section 2.

Lemma B.2 *For $\mathbf{u} \in L^2(\Omega)$, $\mathbf{u}^s$ defined by (B.10) satisfies the following properties.*
(i) $\|\mathbf{u}^s\|_{L^\infty(\Omega)} \ \le\ C\,\|\mathbf{u}\|_{L^2(\Omega)}$.
(ii) Suppose that $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset L^2(\Omega)$, and that $\mathbf{u}_n$ converges weakly to $\mathbf{u} \in L^2(\Omega)$. The $\{\mathbf{u}_n^s\}$ converges to $\mathbf{u}^s$ in $L^\infty(\Omega)$.

Proof: From (B.10) we have that $\mathbf{u}^s \in X^s$, and as $\mathbf{u} \in L^2(\Omega)$, from the *shift theorem* (together with a sufficiently smooth $\partial\Omega$), it follows that

$$\mathbf{u}^s \in H^2(\Omega) \cap X^s, \quad \text{with} \ \ \|\mathbf{u}^s\|_{H^2(\Omega)} \ \le\ C\|\mathbf{u}\|\,. \tag{B.11}$$

Using the embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$ we establish (i).

Let $\mathcal{W} \,:\, L^2(\Omega) \longrightarrow H^2(\Omega) \cap X^s$, $\mathcal{W}(\mathbf{u}) := \mathbf{u}^s$, denote the filter mapping. Then from (B.11) $\mathcal{W}$ is a bounded (linear) transformation from $L^2(\Omega) \longrightarrow H^2(\Omega) \cap X^s$.

Let $\mathcal{W}^* \,:\, \left(H^2(\Omega) \cap X^s\right)^* \longrightarrow L^2(\Omega)$ denote the adjoint operator of $\mathcal{W}$. (The existence of $\mathcal{W}^*$ follows immediately from the Riesz Representation Theorem.)

Now, for $\boldsymbol{\eta} \in \left(H^2(\Omega) \cap X^s\right)^*$

$$\begin{aligned}
\langle \mathbf{u}_n^s \,-\, \mathbf{u}^s \,,\, \boldsymbol{\eta}\rangle_{H^2,(H^2)^*} \;&=\; \langle \mathcal{W}(\mathbf{u}_n) \,-\, \mathcal{W}(\mathbf{u}) \,,\, \boldsymbol{\eta}\rangle_{H^2,(H^2)^*} \;=\; \langle \mathcal{W}(\mathbf{u}_n \,-\, \mathbf{u}) \,,\, \boldsymbol{\eta}\rangle_{H^2,(H^2)^*} \\
&=\; (\mathbf{u}_n \,-\, \mathbf{u} \,,\, \mathcal{W}^*(\boldsymbol{\eta})) \\
&\longrightarrow 0 \ \text{ as } n \to \infty\,,
\end{aligned}$$

as $\mathbf{u}_n$ converges weakly in $L^2(\Omega)$ to $\mathbf{u}$. Hence as $H^2(\Omega) \cap X^s$ is compactly embedded in $L^\infty(\Omega) \cap X^s$, then $\mathbf{u}_n^s$ converges to $\mathbf{u}^s$ strongly in $L^\infty(\Omega)$.

■

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