DPG method with optimal test functions for a fractional advection diffusion equation *

Vincent J. Ervin[†] Thomas Führer[‡] Norbert Heuer[‡] Michael Karkulik[§]

Abstract

We develop an ultra-weak variational formulation of a fractional advection diffusion problem in one space dimension and prove its well-posedness. Based on this formulation, we define a DPG approximation with optimal test functions and show its quasi-optimal convergence. Numerical experiments confirm expected convergence properties, for uniform and adaptively refined meshes.

Key words: fractional diffusion, Riemann-Liouville fractional integral, DPG method with optimal test functions, ultra-weak formulation
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1 Introduction

In this paper we develop a discontinuous Petrov-Galerkin (DPG) method with optimal test functions for a one-dimensional fractional advection diffusion problem of the form

$$\begin{aligned} -DD^{\alpha-2}Du + bDu + cu &= f \qquad \text{on } I := (0,1), \\ u(0) = u(1) &= 0. \end{aligned} \tag{1}$$

Here, D denotes a single spatial derivative, and $D^{\alpha-2}$, for $\alpha \in (1,2)$, represents a fractional integral operator of order $\alpha-2$, cf. Section 2.3 below. Throughout, we assume that $c \in L^\infty([0,1])$, $b \in C^1([0,1])$, and $c - Db/2 \geq 0$.
Fractional advection diffusion equations have been receiving increased attention over the past decade as modeling equations for physical phenomena in such areas as contaminant transport in

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†Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29634-0975. email: `vjervin@clemson.edu`

‡Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Macul, Santiago, Chile, email: `{tofuhrer,nheuer}@mat.puc.cl`

§Department of Mathematics and Statistics, Portland State University, PO Box 751, Portland, OR 97207-0751. email: `mkarkulik@pdx.edu`

ground water flow [3], viscoelasticity [34], turbulent flow [34, 38], and chaotic dynamics [47]. As most models involving fractional order differential equations do not have closed form solutions particular attention has been paid to the development of numerical approximation schemes for these equations. Two phenomena of fractional order differential equations which impact their numerical discretization and approximation are: (i) the fractional differential operator is nonlocal (leading to a dense coefficient matrix), and (ii) the (typical) low regularity of the solution (leading to slow convergence of the numerical solution to the true solution).

The first approximation methods investigated for fractional order differential equations were finite difference schemes proposed by Liu, Ahn and Turner [32], and Meerschaert and Tadjeran [35], (see also [39, 12, 40]). Subsequently, finite element [20, 18, 41, 33, 46, 29, 10] and spectral methods [31, 44, 48, 11] have been developed for the approximation of fractional order differential equations. We note that a finite difference approximation using the Grünwald formula on a uniform mesh leads to a Toeplitz like matrix which significantly reduces the storage required for the coefficient matrix, and whose linear system can be very efficiently solved using a fast Fourier transform [40].

Fractional diffusion problems are inherently difficult to analyze and with our method we open a way to deal with singularly perturbed cases (not considered here). In fact, principal objective of the DPG method is to provide robust discretizations of singularly perturbed problems like convection diffusion [16, 7, 9, 5] and wave problems [49]. The DPG method with optimal test functions has been developed by Demkowicz, Gopalakrishnan and co-workers. In its most common form it combines several ideas. These are ultra-weak variational formulations (cf. [17, 8]) with additional trace and flux unknowns (cf. [4]), and the utilization of specific test functions which are designed for stability (cf. the SUPG method in [28] and test functions in [2]). Demkowicz and Gopalakrishnan combine these ideas in a discontinuous setting and by employing problem-tailored norms. Appropriately combined, the resulting DPG method with optimal test functions delivers robust error control and also gives access to localized a posteriori error estimation (or rather calculation). For details we refer to [14, 15]. In this paper we follow precisely these steps to deal with equations involving fractional diffusion. By writing (1) as a first-order system, cf. (13), we develop an ultra-weak variational formulation in Section 2.4 below. While a weak formulation of (1) leads to a non-symmetric, coercive bilinear form, for the DPG method with optimal test functions the resulting variational formulation is always symmetric, positive definite, implying existence of a unique solution. This is the central result of the DPG method with optimal test functions, stated below in Theorem 1. Necessary conditions for its application are the well-known Babuška-Brezzi conditions (2), which we check in Section 3 for our ultra-weak formulation. A central step will be to extend Riemann-Liouville fractional integral operators to negative order Sobolev spaces and prove their ellipticity. To that end, we extend recent results from [29]. In our main result, Theorem 7, we show well-posedness of the underlying ultra-weak variational formulation and quasi-optimal convergence of the discrete scheme. In particular, we will gain access to error control and adaptivity. In Section 4, we report on several numerical experiments that illustrate convergence orders of variants with uniform meshes and with adaptively refined meshes.

Galerkin formulations can lack ellipticity for certain variable diffusion coefficients, cf. [41]. This led to the use of Petrov-Galerkin formulations in a series of recent papers. First, [41] considered a pure diffusion problem with variable diffusion coefficient and homogeneous boundary conditions in the case $3/2 < \alpha < 2$. Inhomogeneous boundary conditions where then analyzed in [42]. Recently in [30], the authors propose a variational formulation of Petrov-Galerkin type for fractional advection diffusion equations for $3/2 < \alpha < 2$. The works [41, 42, 30] use a-priori chosen test spaces and globally continuous function spaces and discretizations. We note that in [43] the authors propose a simplified Petrov-Galerkin method with optimal test functions for fractional diffusion problems. They still use continuous spaces, which means that optimal test functions are calculated globally. In contrast, we allow for $1 < \alpha < 2$ and develop the fully discontinuous variant that allows for local calculations of test functions. This is particularly important for fractional-order problems where inner products are defined by double integrals so that global calculations are prohibitively costly. Let us also mention that there is DPG-technology available for hypersingular integral equations [27, 26]. Hypersingular operators are of order one with energy spaces of order 1/2. For closed curves/surfaces, DPG theory can be established with integer-order Sobolev spaces and is then simpler in a certain way. For open curves/surfaces however, one has to return to non integer-order spaces. The case of hypersingular operators can be seen as a limit of fractional diffusion operators with orders between one and two, as considered in this paper.

2 Mathematical setting and main results

We use the widespread notation $A \lesssim B$ to denote the fact that $A \le C \cdot B$ where the constant $C > 0$ does not depend on any quantities of interest. By $A \simeq B$ we mean that both $A \lesssim B$ and $B \lesssim A$ hold. Throughout, suprema are taken over the indicated sets *except* 0.

2.1 DPG method with optimal test functions

We briefly recall the premises and results of the DPG method with optimal test functions, cf. [14, 15, 49]. Given a Banach space U, a Hilbert space V, and a bilinear form $b : U \times V \to \mathbb{R}$, we consider the following three conditions:

$$b(\boldsymbol{u}, \boldsymbol{v}) = 0 \text{ for all } \boldsymbol{v} \in V \implies \boldsymbol{u} = 0; \tag{2a}$$

there is a positive constant C_{infsup} such that

$$C_{\mathrm{infsup}} \|\boldsymbol{v}\|_V \le \sup_{\boldsymbol{u} \in U} \frac{b(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{u}\|_U} \qquad \text{for all } \boldsymbol{v} \in V; \tag{2b}$$

there is a positive constant C_{b} such that

$$b(\boldsymbol{u}, \boldsymbol{v}) \le C_{\mathrm{b}} \|\boldsymbol{u}\|_U \|\boldsymbol{v}\|_V \qquad \text{for all } \boldsymbol{u} \in U, \boldsymbol{v} \in V. \tag{2c}$$

Define the so-called trial-to-test operator $\Theta : U \to V$ by

$$\langle \Theta \boldsymbol{u}, \boldsymbol{v} \rangle_V = b(\boldsymbol{u}, \boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in V. \tag{3}$$

The following result is central to the DPG method and is, in the end, consequence of the Babuška-Brezzi theory [1, 6, 45], cf. [14] and related references given in the introduction.

Theorem 1. *Suppose that* (2a)–(2c) *hold for a Banach space U, a Hilbert space V, and a bilinear form $b : U \times V \to \mathbb{R}$. Then, an equivalent norm on U is given by*

$$\|\boldsymbol{u}\|_E := \sup_{\boldsymbol{v}\in V} \frac{b(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_V}, \qquad \text{and} \qquad C_{\text{infsup}}\|\boldsymbol{u}\|_U \le \|\boldsymbol{u}\|_E.$$

Furthermore, for any $\ell \in V'$, the problem

$$\text{find } \boldsymbol{u} \in U \text{ such that} \quad b(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V \tag{4}$$

has a unique solution, and

$$\|\boldsymbol{u}\|_E \le \|\ell\|_{V'}. \tag{5}$$

In addition, if $U_{\text{hp}} \subset U$ is a finite-dimensional subspace, then the problem

$$\text{find } \boldsymbol{u}_{\text{hp}} \in U_{\text{hp}} \text{ such that} \quad b(\boldsymbol{u}_{\text{hp}},\boldsymbol{v}_{\text{hp}}) = \ell(\boldsymbol{v}_{\text{hp}}) \quad \text{for all } \boldsymbol{v}_{\text{hp}} \in \Theta(U_{\text{hp}}) \tag{6}$$

has a unique solution, and

$$\|\boldsymbol{u} - \boldsymbol{u}_{\text{hp}}\|_E = \inf_{\boldsymbol{u}'_{\text{hp}}\in U_{\text{hp}}} \|\boldsymbol{u} - \boldsymbol{u}'_{\text{hp}}\|_E. \tag{7}$$

2.2 Sobolev spaces

For $s \in \mathbb{R}$ with $s \ge 0$ and an open interval $M = (a,b) \subseteq \mathbb{R}$, the Sobolev spaces $H^s(M)$ are defined via distributional derivatives and the Sobolev-Slobodeckij seminorm $|\cdot|_{H^s(M)}$ and norm $\|\cdot\|_{H^s(M)}$. The space $\widetilde{H}^s(M)$ is defined as the space of functions whose extension by zero is in $H^s(\mathbb{R})$, equipped with the norm $\|\widetilde{u}\|_{H^s(\mathbb{R})}$ and seminorm $|\widetilde{u}|_{H^s(\mathbb{R})}$, where $\widetilde{u}$ is the extension of u by zero. The space $H^{-s}(M)$ denotes the topological dual space of $\widetilde{H}^s(M)$, while $\widetilde{H}^{-s}(M)$ denotes the dual of $H^s(M)$. For a finite partition $\mathcal{T}$ of $I = (0,1)$ into open, disjoint, and connected sets, we define $H^s(\mathcal{T}) := \prod_{T\in\mathcal{T}} H^s(T)$, or, likewise, $\widetilde{H}^s(\mathcal{T}) := \prod_{T\in\mathcal{T}} \widetilde{H}^s(T)$, with product norms $\|v\|^2_{H^s(\mathcal{T})} := \sum_{T\in\mathcal{T}} \|v|_T\|^2_{H^s(T)}$ and $\|v\|^2_{\widetilde{H}^s(\mathcal{T})} := \sum_{T\in\mathcal{T}} \|v|_T\|^2_{\widetilde{H}^s(T)}$ (seminorms on these spaces are defined analogously). We also write $\widetilde{H}^{-s}(\mathcal{T})$ or $H^{-s}(\mathcal{T})$ for the duals of product spaces. By $N := \#\mathcal{T}$ we denote the number of elements in the partition and for $v \in H^s(\mathcal{T})$, $1/2 < s$, we define the jump $[v] \in \mathbb{R}^{N+1}$ as the vector of the differences of the traces of v on the elements to the right and to the left of all nodes $x = \overline{T_-} \cap \overline{T_+}$. For the boundary nodes (i.e., 0 and 1), we just take traces. Jumps are measured in discrete ℓ_2 norms $|[v]|$. For $v \in H^s(\mathcal{T})$, $1/2 < s$, we also define the average $\{v\}$ as the vector of mean values of the traces of v on the elements to the right and to the left of all nodes. We will need certain results for this kind of spaces. From now on, we assume that partitions are quasi-uniform, i.e., for all $T \in \mathcal{T}$ holds $|T| \simeq N^{-1}$ for $N := \#\mathcal{T}$ being the number of elements in the partition $\mathcal{T}$, and the constant involved is independent of $\mathcal{T}$. We denote by $D_{\mathcal{T}}$ the $\mathcal{T}$-piecewise distributional derivative.

Lemma 2. *The following statements hold with constants which only depend on s:*

- *Let $s \in (0, 1/2)$. There holds*

$$\|v\|_{H^s(I)} \lesssim N^s \|v\|_{H^s(\mathcal{T})} \qquad \text{for all } v \in H^s(I). \tag{8}$$

- *Let $s \in (1/2, 1)$. There holds*

$$\|D_{\mathcal{T}} v\|_{\widetilde{H}^{s-1}(I)} \lesssim N^{1-s} |v|_{H^s(\mathcal{T})} \qquad \text{for all } v \in H^s(\mathcal{T}). \tag{9}$$

- *Let $s \in (1/2, 1]$. There holds*

$$|[v]| \lesssim N^{1/2} \|v\|_{H^s(\mathcal{T})} \qquad \text{for all } v \in H^s(\mathcal{T}). \tag{10}$$

Proof. Estimate (8) is seen as follows: First, for $\widehat{T}$ a reference interval with fixed diameter, there is a constant $C_s > 0$ such that $\|\widehat{v}\|_{\widetilde{H}^s(\widehat{T})} \le C_s \|\widehat{v}\|_{H^s(\widehat{T})}$, cf. [22] and [23, Proof of Lemma 5]. Second, scaling arguments show that $\|v\|_{\widetilde{H}^s(T)} \lesssim N^s \|v\|_{H^s(T)}$ for all $T \in \mathcal{T}$. Now,

$$\|v\|^2_{H^s(I)} \lesssim \|v\|^2_{\widetilde{H}^s(I)} \lesssim \sum_{T \in \mathcal{T}} \|v\|^2_{\widetilde{H}^s(T)}, \tag{11}$$

where the second estimate follows from [19, Lemma 20]. To show estimate (9), we proceed as before and use an affine transformation on every element $T \in \mathcal{T}$,

$$\|Dv\|^2_{\widetilde{H}^{s-1}(T)} \lesssim N \|\widehat{D}\widehat{v}\|^2_{\widetilde{H}^{s-1}(\widehat{T})} \lesssim N \|\widehat{D}\widehat{v}\|^2_{H^{s-1}(\widehat{T})}.$$

Here the second estimate follows as the norms involved are dual to norms on which we can use [22] and [23, Proof of Lemma 5]. A quotient space argument on the reference element $\widehat{T}$, cf. [24], shows

$$\|\widehat{D}\widehat{v}\|_{H^{s-1}(\widehat{T})} \lesssim \inf_{c \in \mathbb{R}} \|\widehat{v} + c\|_{H^s(\widehat{T})} \simeq |\widehat{v}|_{H^s(\widehat{T})}.$$

Estimate (9) then follows by application of the scaling argument $|\widehat{v}|^2_{H^s(\widehat{T})} \lesssim N^{1-2s} |v|^2_{H^s(T)}$. Estimate (10) follows easily from, e.g.,

$$|v(x)| \le \|v\|_{L^\infty(\widehat{T})} \lesssim \|\widehat{v}\|_{H^s(\widehat{T})} \lesssim N^{1/2} \|v\|_{H^s(T_+)}.$$

Here, for example, $x = \overline{T_-} \cap \overline{T_+}$, the second estimate follows by the Sobolev Embedding theorem, and the third one again by a scaling argument. $\square$

Lemma 3. *There holds*

$$\|\tau\|_{L_2(I)} \lesssim \|D_{\mathcal{T}} \tau\|_{L_2(I)} + N^{1/2} |[\tau]| \qquad \text{for all } \tau \in H^1(\mathcal{T}),$$

and the hidden constant is independent of $\mathcal{T}$.

Proof. Let $\phi \in \widetilde{H}^1(I)$ be the weak solution of $-D^2\phi = \tau$. Then $D\phi \in H^1(I)$ with distributional derivative $D^2\phi = -\tau$, and integration by parts yields

$$(\tau\,,\tau) = -(\tau\,, D^2\phi) = (D_{\mathcal{T}}\tau\,, D\phi) + \langle [\tau]\,, D\phi\rangle.$$

Cauchy-Schwarz and Lemma 2, eq. (10) imply

$$\|\tau\|_{L_2(I)}^2 \lesssim \|D_{\mathcal{T}}\tau\|_{L_2(I)}\|D\phi\|_{L_2(I)} + N^{1/2}|[\tau]|\|D\phi\|_{H^1(I)}.$$

By construction, $\|D\phi\|_{H^1(I)} \lesssim \|\tau\|_{L_2(I)}$, which concludes the proof. $\square$

We will need the following result on fractional seminorms.

Lemma 4. *Let $s \in (0,1)$ be fixed and $J \subset \mathbb{R}$ be an interval. There holds*

$$|u|_{H^s(J)} \lesssim \|Du\|_{H^{s-1}(J)} \qquad \text{for all } u \in H^s(J).$$

where Du is the distributional derivative of u. The hidden constant does not depend on I.

Proof. In [13, Proposition 1] it is shown that the derivative operator is an isomorphism from $\left\{u \in H^s(J) \mid \int_J u(s)\,ds = 0\right\}$ to $H^{s-1}(J)$. In a first step, this shows the statement of the Lemma with a constant depending on J. Scaling arguments then prove that the constant does not depend on J, cf. [25]. $\square$

2.3 Fractional integral operators

The fractional integral operators that we will use are of so-called Riemann-Liouville type. For $\beta > 0$ we denote by ${}_0D^{-\beta}$ and $D_1^{-\beta}$ the left and right-sided versions of these operators, defined on $I = (0,1)$ by

$${}_0D^{-\beta}u(x) := \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}u(s)\,ds \quad \text{and} \quad D_1^{-\beta}u(x) := \frac{1}{\Gamma(\beta)}\int_x^1 (s-x)^{\beta-1}u(s)\,ds.$$

In the following we will use $-1 < \beta - 1 < 0$, such that the integrals above contain singular kernels. We also abbreviate $D^{-\beta} := {}_0D^{-\beta}$. A standard textbook on this kind of operators is [37]. Recently, classical results regarding boundedness and ellipticity of these operators were extended in [18, 29]. In order to obtain a variational formulation suited for DPG analysis, we need to extend these operators to negative order Sobolev spaces and show their ellipticity. To this end, let $\mathcal{F}$ denote the Fourier transforms on the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions, cf. [36, Chapter 7], defined by $\mathcal{F}T(\varphi) := T(\mathcal{F}\varphi)$, where

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-1/2}\int_{\mathbb{R}} \varphi(x)e^{-ix\xi}\,dx$$

is the Fourier transform on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$. Choosing a space of test functions which is invariant under the action of $D^{-\beta}$ and $D^{-\beta}$, these operators can be extended to the associated spaces of distributions, cf. [37, §8]. In the present setting, a different argument can be used.

Lemma 5. *For every $s \in \mathbb{R}$ with $-\beta \le s$ and $\beta > 0$, the operator $D^{-\beta}$ can be extended to a bounded linear operator $D^{-\beta} : \widetilde{H}^s(I) \to H^{s+\beta}(I)$.*

Proof. For $0 \le s$, the statement was shown for ${}_0D^{-\beta}$ and $D_1^{-\beta}$ in Theorem 3.1 of [29]. It therefore remains to consider $-\beta \le s < 0$. We will show the statement for $s = -\beta$, the remaining cases follow by interpolation. We already know that $D_1^{-\beta} : L_2(I) \to H^\beta(I)$ is a linear and bounded operator. According to [37, Corollary of Thm. 3.5], it holds that

$$(D^{-\beta}u\,, v) = (u\,, D_1^{-\beta}v) \quad \text{for all } u, v \in L_2(I). \tag{12}$$

Hence, the right-hand side of (12) extends $D^{-\beta}$ to a linear, bounded operator $D^{-\beta} : \widetilde{H}^{-\beta} \to L_2(I)$. $\square$

Lemma 6. *The operator $D^{-\beta}$ is elliptic on $H^{-\beta/2}(\mathcal{T})$ for $0 < \beta < 1$.*

Proof. For a test function $\varphi \in \mathcal{D}(I)$ holds $\mathcal{F}(D^{-\beta}\varphi)(\xi) = (i\xi)^{-\beta}\mathcal{F}(\varphi)(\xi)$, cf. [37, Thm. 7.1]. Then, a short computation (cf. [18, Proof of Lemma 2.4]) shows

$$\begin{aligned}
(D^{-\beta}\varphi\,, \varphi) &= ((i\xi)^{-\beta}\mathcal{F}(\varphi)\,, \overline{\mathcal{F}(\varphi)}) = ((i\xi)^{-\beta/2}\mathcal{F}(\varphi)\,, \overline{(-i\xi)^{\beta/2}\mathcal{F}(\varphi)}) \\
&= \cos(-\pi\beta/2)((i\xi)^{-\beta/2}\mathcal{F}(\varphi)\,, \overline{(i\xi)^{\beta/2}\mathcal{F}(\varphi)}) \\
&\quad + i\sin(-\pi\beta/2)\Big(\int_0^\infty (i\xi)^{-\beta/2}\mathcal{F}(\varphi)\overline{(i\xi)^{-\beta/2}\mathcal{F}(\varphi)}d\xi \\
&\qquad - \int_{-\infty}^0 (i\xi)^{-\beta/2}\mathcal{F}(\varphi)\overline{(i\xi)^{-\beta/2}\mathcal{F}(\varphi)}d\xi\Big)
\end{aligned}$$

As the left-hand side of this identity is real, the imaginary part on the right-hand side vanishes. Furthermore, $\cos(-\pi\beta/2) > 0$ for $0 < \beta < 1$. We obtain

$$(D^{-\beta}\varphi\,, \varphi) \gtrsim \|(\xi^2)^{-\beta/4}\mathcal{F}(\varphi)\|^2_{L_2(\mathbb{R})} \gtrsim \|(1+\xi^2)^{-\beta/4}\mathcal{F}(\varphi)\|^2_{L_2(\mathbb{R})}.$$

The right-hand side is equivalent to the norm $\|\varphi\|_{H^{-\beta/2}(\mathbb{R})}$. A density argument shows the ellipticity on $H^{-\beta/2}(\mathbb{R})$. Since on $H^{-\beta/2}(\mathcal{T})$ it holds $\|\cdot\|_{H^{-\beta/2}(\mathbb{R})} \gtrsim \|\cdot\|_{H^{-\beta/2}(I)} \gtrsim \|\cdot\|_{H^{-\beta/2}(\mathcal{T})}$, cf. (11), the proof is finished. $\square$

2.4 Ultra-weak formulation and main result

We write (1) as first-order system

$$\begin{aligned}
\sigma - Du &= 0, \\
-DD^{\alpha-2}\sigma + bDu + cu &= f.
\end{aligned} \tag{13}$$

Then, we multiply these equations with τ respectively v, integrate by parts piecewise on a partition $\mathcal{T}$ and rename the appearing boundary terms of $D^{\alpha-2}\sigma$ and u by $\widehat{\sigma}$ and $\widehat{u}$ to obtain

$$(\sigma\,, \tau) + (u\,, D_{\mathcal{T}}\tau) - \langle \widehat{u}\,, [\tau]\rangle = 0 \tag{14a}$$

$$(D^{\alpha-2}\sigma\,, D_{\mathcal{T}}v) + (b\sigma\,, v) + (cu\,, v) - \langle \widehat{\sigma}\,, [v]\rangle = (f\,, v). \tag{14b}$$

The left and right-hand sides of the preceding equations define our bilinear form and linear form via

$$
\begin{aligned}
b(\boldsymbol{u},\boldsymbol{v}) := b(\sigma,u,\widehat{\sigma},\widehat{u};\tau,v) &:= (\sigma\,,\tau + D^{(\alpha-2)\star} D_{\mathcal{T}} v + bv) + (u\,, D_{\mathcal{T}}\tau + cv) - \langle \widehat{u}\,, [\tau]\rangle - \langle \widehat{\sigma}\,, [v]\rangle,\\
\ell(\boldsymbol{v}) := \ell(\tau,v) &:= (f\,,v).
\end{aligned}
$$

Here and from now on, $D^{(\alpha-2)\star} : \widetilde{H}^{\alpha/2-1}(I) \to H^{1-\alpha/2}(I)$ denotes the conjugate of $D^{\alpha-2}$. Define $U_\alpha := \widetilde{H}^{\alpha/2-1}(I) \times L_2(I) \times \mathbb{R}^{N+1} \times \mathbb{R}^{N-1}$ and $V_\alpha := H^1(\mathcal{T}) \times H^{\alpha/2}(\mathcal{T})$, where N is the number of elements of $\mathcal{T}$, with product norms

$$
\begin{aligned}
\|\boldsymbol{u}\|_{U_\alpha}^2 &:= \|\sigma\|_{\widetilde{H}^{\alpha/2-1}(I)}^2 + \|u\|_{L_2(I)}^2 + N^{-3}(|\widehat{\sigma}|^2 + |\widehat{u}|^2), \text{ and}\\
\|\boldsymbol{v}\|_{V_\alpha}^2 &:= \|\tau\|_{H^1(\mathcal{T})}^2 + \|v\|_{H^{\alpha/2}(\mathcal{T})}^2.
\end{aligned}
$$

By $|\cdot|$, we mean the usual Euclidean norm. Our ultra-weak formulation now reads as follows: given $\ell \in V_\alpha'$, we aim to find $\boldsymbol{u} \in U_\alpha$ such that

$$
b(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V_\alpha. \tag{15}
$$

For a discrete subspace $U_{\mathrm{hp}} \subset U_\alpha$, the DPG method with optimal test functions is to find $\boldsymbol{u}_{\mathrm{hp}} \in U_{\mathrm{hp}}$ such that

$$
b(\boldsymbol{u}_{\mathrm{hp}},\boldsymbol{v}_{\mathrm{hp}}) = \ell(\boldsymbol{v}_{\mathrm{hp}}) \quad \text{for all } \boldsymbol{v}_{\mathrm{hp}} \in \Theta_\alpha(U_{\mathrm{hp}}), \tag{16}
$$

where $\Theta_\alpha : U_\alpha \to V_\alpha$ is the trial-to-test operator associated with b, cf. (3). The following theorem is the main result of this work. It states unique solvability and stability of the continuous and discrete formulations (15) and (16), as well as a best approximation result.

Theorem 7. *For $\alpha \in (1,2)$, $f \in L_2(I)$, and arbitrary partition $\mathcal{T}$, the variational formulation* (15) *has a unique solution $\boldsymbol{u} \in U_\alpha$, and*

$$
\|\boldsymbol{u}\|_{U_\alpha} \lesssim \|f\|_{L_2(I)}.
$$

Furthermore, the discrete problem (16) *has a unique solution $\boldsymbol{u}_{\mathrm{hp}} \in U_{\mathrm{hp}}$, and*

$$
\|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{hp}}\|_{U_\alpha} \lesssim \inf_{(\sigma'_{\mathrm{hp}},u'_{\mathrm{hp}},\widehat{\sigma}'_{\mathrm{hp}},\widehat{u}'_{\mathrm{hp}})\in U_{\mathrm{hp}}} \Big(N^{1-\alpha/2}\|\sigma - \sigma'_{\mathrm{hp}}\|_{\widetilde{H}^{\alpha/2-1}(I)} + \|u - u'_{\mathrm{hp}}\|_{L_2(I)} \Big).
$$

Proof. We are going to apply Theorem 1, hence we check (2a)–(2c). The condition (2a) follows from Lemma 9. The condition (2c) follows from Lemma 8. It remains to check condition (2b). To that end, observe first that

$$
\sup_{\boldsymbol{u}\in U_\alpha} \frac{b(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{u}\|_{U_\alpha}} = \Big(\|\tau + D^{(\alpha-2)\star} D_{\mathcal{T}} v + bv\|_{H^{1-\alpha/2}(I)}^2 + \|D_{\mathcal{T}}\tau + cv\|_{L_2(I)}^2 + N^3(|[\tau]| + |[v]|)^2 \Big)^{1/2} \tag{17}
$$

For given $\boldsymbol{v} = (\tau, v) \in V_\alpha$ we define $\tau_1 \in H^1(I)$ and $v_1 \in \widetilde{H}^{\alpha/2}(I)$ as the solution of Lemma 10 with data $F := D_\mathcal{T}\tau + cv$ and $G = \tau + D^{(\alpha-2)\star}D_\mathcal{T}v + bv$, and write $\tau = \tau_0 + \tau_1$ and $v = v_0 + v_1$. The functions τ_0 and v_0 then fulfill the assumptions of Lemma 11. The triangle inequality and Lemmas 10 and 11 show

$$\|\boldsymbol{v}\|_{V_\alpha} \lesssim \|\tau + D^{(\alpha-2)\star}D_\mathcal{T}v + bv\|_{H^{1-\alpha/2}(I)} + \|D_\mathcal{T}\tau + cv\|_{L_2(I)} + N^{3/2}(|[\tau]| + |[v]|). \tag{18}$$

The equations (17) and (18) show condition (2b). Theorem 1 shows that there are unique solutions $\boldsymbol{u}$ and $\boldsymbol{u}_{\mathrm{hp}}$ of the problems (15) and (16) which fulfill stability (5) and best approximation (7), and that

$$C_{\mathrm{infsup}}\|\boldsymbol{u}\|_{U_\alpha} \le \|\boldsymbol{u}\|_{E_\alpha} := \sup_{\boldsymbol{v}\in V_\alpha} \frac{b(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{V_\alpha}}.$$

Lemma 8 shows that

$$\inf_{\boldsymbol{u}'_{\mathrm{hp}}\in U_{\mathrm{hp}}} \|\boldsymbol{u} - \boldsymbol{u}'_{\mathrm{hp}}\|_{E_\alpha} \lesssim \inf_{(\sigma'_{\mathrm{hp}}, u'_{\mathrm{hp}}, \widehat{\sigma}'_{\mathrm{hp}}, \widehat{u}'_{\mathrm{hp}})\in U_{\mathrm{hp}}} \left(N^{1-\alpha/2}\|\sigma - \sigma'_{\mathrm{hp}}\|_{\widetilde{H}^{\alpha/2-1}(I)} + \|u - u'_{\mathrm{hp}}\|_{L_2(I)}\right).$$

Here, owing to the fact that $\widehat{u}$ and $\widehat{\sigma}$ are just finite-dimensional vectors, the functions $\widehat{\sigma}'_{\mathrm{hp}}$ and $\widehat{u}'_{\mathrm{hp}}$ can be omitted on the right-hand side. $\square$

3 Technical results

The first lemma states boundedness of the bilinear form b.

Lemma 8. *For $\alpha \in (1,2)$,*

$$|b(\boldsymbol{u},\boldsymbol{v})| \lesssim \left(N^{2-\alpha}\|\sigma\|^2_{\widetilde{H}^{\alpha/2-1}(I)} + \|u\|^2_{L_2(I)} + N|\hat{u}|^2 + N|\hat{\sigma}|^2\right)^{1/2} \|\boldsymbol{v}\|_{V_\alpha},$$

with a constant independent of $\mathcal{T}$. In particular, $|b(\boldsymbol{u},\boldsymbol{v})| \le C_{\mathrm{b}}\|\boldsymbol{u}\|_{U_\alpha}\|\boldsymbol{v}\|_{V_\alpha}$, where the constant C_{b} depends on N.

Proof. By Lemma 2, eq. (10), we have

$$|\langle \widehat{u}, [\tau]\rangle| \lesssim N^{1/2}|\widehat{u}|\|\tau\|_{H^1(\mathcal{T})} \quad \text{and} \quad |\langle \widehat{\sigma}, [v]\rangle| \lesssim N^{1/2}|\widehat{\sigma}|\|v\|_{H^{\alpha/2}(\mathcal{T})}.$$

The triangle inequality, Lemmas 2 and 5, the definition of $D^{(\alpha-2)\star}$, $c \in L^\infty([0,1])$, $b \in C^1([0,1])$, and $1 \le \alpha$ show

$$\begin{aligned}\|\tau + D^{(\alpha-2)\star}D_\mathcal{T}v + bv\|_{H^{1-\alpha/2}(I)} &\lesssim \|\tau\|_{H^{1-\alpha/2}(I)} + \|D_\mathcal{T}v\|_{\widetilde{H}^{\alpha/2-1}(I)} + \|bv\|_{H^{1-\alpha/2}(I)} \\ &\lesssim N^{1-\alpha/2}\left(\|\tau\|_{H^1(\mathcal{T})} + \|v\|_{H^{\alpha/2}(\mathcal{T})}\right)\end{aligned}$$

and

$$\|D_{\mathcal{T}}\tau + cv\|_{L_2(I)} \lesssim \|\tau\|_{H^1(\mathcal{T})} + \|v\|_{H^{\alpha/2}(\mathcal{T})}.$$

We finish the proof with the triangle and Cauchy-Schwarz inequalities. $\square$

Lemma 9. *It holds that*

$$b(\boldsymbol{u}, \boldsymbol{v}) = 0 \textit{ for all } \boldsymbol{v} \in \mathbf{V} \iff \boldsymbol{u} = 0.$$

Proof. The direction $\Leftarrow$ is clear, and we proceed with the implication $\Rightarrow$. Using $\tau \in C_0^\infty(I)$ in (14a) shows that the distributional derivative of u fulfills $Du = \sigma$. As $\sigma \in \widetilde{H}^{\alpha/2-1}(I)$, we conclude that $u \in H^{\alpha/2}(I)$. In a second step, using functions $\tau \in C^\infty(T)$ for all $T \in \mathcal{T}$ in (14a) and integrating by parts shows that $\widehat{u} = u$ at inner nodes as well as $u(a) = u(b) = 0$. Hence $u \in \widetilde{H}^{\alpha/2}(I)$. We plug in $\sigma = Du$ in (14b) and obtain the variational formulation

$$(D^{\alpha-2}Du\,, Dv) + (bDu\,, v) + (cu\,, v) = 0 \quad \text{ for all } v \in \widetilde{H}^{\alpha/2}(I).$$

According to [18, Section 3], the bilinear form on the left-hand side of this formulation is elliptic on $\widetilde{H}^{\alpha/2}(I)$. We conclude that $u = 0$ and hence $\sigma = 0$. Then, $\widehat{u} = 0$ and $\widehat{\sigma} = 0$ follow immediately. $\square$

3.1 Analysis of the adjoint problem

Lemma 10. *For $F \in L_2(I)$ and $G \in H^{1-\alpha/2}(I)$, there exists a solution $\tau \in H^1(I)$, $v \in \widetilde{H}^{\alpha/2}(I)$ of*

$$\begin{aligned} D\tau + cv &= F \\ \tau + D^{(\alpha-2)\star}Dv + bv &= G \end{aligned} \tag{19}$$

such that

$$\|v\|_{H^{\alpha/2}(I)} + \|\tau\|_{H^1(I)} \lesssim \|F\|_{L_2(I)} + \|G\|_{H^{1-\alpha/2}(I)}. \tag{20}$$

Proof. Consider the variational formulation to find $v \in \widetilde{H}^{\alpha/2}(I)$ such that

$$(Dv\,, D^{\alpha-2}D\phi) + (bv\,, D\phi) + (cv\,, \phi) = (F\,, \phi) - (DG\,, \phi) \quad \text{ for all } \phi \in \widetilde{H}^{\alpha/2}(I).$$

According to [18, Section. 3], the bilinear form of this formulation is elliptic on $\widetilde{H}^{\alpha/2}(I)$. The linear functional on the right-hand side is bounded in $\widetilde{H}^{\alpha/2}(I)$ with constant $\|F\|_{L_2(I)} + \|G\|_{H^{1-\alpha/2}(I)}$. Hence, there exists a unique solution $v \in \widetilde{H}^{\alpha/2}(I)$ which satisfies

$$\|v\|_{\widetilde{H}^{\alpha/2}(I)} \lesssim \|F\|_{L_2(I)} + \|G\|_{H^{1-\alpha/2}(I)}$$

Now define $\tau := -D^{(\alpha-2)\star}Dv - bv + G$. A priori, $\tau \in H^{1-\alpha/2}(I)$, but the definition of v shows

$$(\tau\,, D\phi) = (cv - F\,, \phi) \quad \text{ for all } \phi \in C_0^\infty(I).$$

Hence, $\tau \in H^1(I)$ and $D\tau = F - cv$. The bounds on τ follow immediately. $\square$

Lemma 11. *Suppose that $\tau \in H^1(\mathcal{T})$ and $v \in H^{\alpha/2}(\mathcal{T})$ fulfill*

$$D_{\mathcal{T}}\tau + cv = 0 \tag{21a}$$

$$\tau + D^{(\alpha-2)\star} D_{\mathcal{T}} v + bv = 0, \tag{21b}$$

on I. Then

$$\|\tau\|_{H^1(\mathcal{T})} + \|v\|_{H^{\alpha/2}(\mathcal{T})} \lesssim N^{3/2}\left(|[v]| + |[\tau]|\right).$$

Proof. We proceed in three steps.

Step 1: Let $\psi \in \widetilde{H}^{\alpha/2}(I)$ be the unique variational solution of $-DD^{\alpha-2}D\psi + bD\psi + c\psi = -v$, cf. [18, Section 3], i.e.,

$$(D^{\alpha-2}D\psi\,, D\phi) + (bD\psi\,, \phi) + (c\psi\,, \phi) = -(v\,, \phi) \quad \text{for all } \phi \in \widetilde{H}^{\alpha/2}(I),$$

so that $\|\psi\|_{\widetilde{H}^{\alpha/2}(I)} \lesssim \|v\|_{L_2(I)}$. Due to Lemma 5 and $\alpha - 2 \leq \alpha/2 - 1$, it holds

$$\|D^{\alpha-2}D\psi\|_{L_2(I)} \lesssim \|D\psi\|_{\widetilde{H}^{\alpha-2}(I)} \lesssim \|D\psi\|_{\widetilde{H}^{\alpha/2-1}} \lesssim \|\psi\|_{\widetilde{H}^{\alpha/2}(I)} \lesssim \|v\|_{L_2(I)}. \tag{22}$$

The equation solved by ψ implies that the distributional derivative of $D^{\alpha-2}D\psi$ is given by $DD^{\alpha-2}D\psi = bD\psi + c\psi + v \in H^{\alpha/2-1}(I)$, such that $D^{\alpha-2}D\psi \in H^{\alpha/2}(I)$. Using Lemma 4, we see

$$\begin{aligned} |D^{\alpha-2}D\psi|_{H^{\alpha/2}(I)} &\lesssim \|DD^{\alpha-2}D\psi\|_{H^{\alpha/2-1}(I)} = \|bD\psi + c\psi + v\|_{H^{\alpha/2-1}(I)} \\ &\lesssim \|\psi\|_{\widetilde{H}^{\alpha/2}(I)} + \|v\|_{L_2(I)} \lesssim \|v\|_{L_2(I)}. \end{aligned} \tag{23}$$

We may also integrate by parts and use (21) to obtain

$$\begin{aligned} (v\,, v) &= -(D^{\alpha-2}D\psi\,, D_{\mathcal{T}}v) - (bD\psi\,, v) - (c\psi\,, v) + \langle D^{\alpha-2}D\psi\,, [v]\rangle \\ &= (D\psi\,, -D^{(\alpha-2)\star}D_{\mathcal{T}}v - bv - \tau) + \langle \psi\,, [\tau]\rangle + \langle D^{\alpha-2}D\psi\,, [v]\rangle \\ &= \langle \psi\,, [\tau]\rangle + \langle D^{\alpha-2}D\psi\,, [v]\rangle. \end{aligned}$$

Lemma 2, eq. (10), estimates (22), (23), and stability of ψ yield

$$\|v\|_{L_2(I)}^2 \lesssim N^{1/2}\left(|[\tau]| + |[v]|\right)\|v\|_{H^{\alpha/2}(\mathcal{T})}. \tag{24}$$

Step 2: Piecewise integration by parts shows

$$\begin{aligned} (D_{\mathcal{T}}(bv)\,, v) &= (vDb\,, v) + (bD_{\mathcal{T}}v\,, v) \\ &= (vDb\,, v) - (v\,, D_{\mathcal{T}}(bv)) - \langle [v]\,, \{bv\}\rangle - \langle \{v\}\,, [bv]\rangle, \end{aligned}$$

which gives

$$(D_{\mathcal{T}}(bv)\,, v) = (vDb/2\,, v) - 1/2\langle[v]\,, \{bv\}\rangle - 1/2\langle\{v\}\,, [bv]\rangle. \tag{25}$$

Now we multiply (21a) with v and insert (21b) as well as (25). Then, as $D^{(\alpha-2)\star}D_{\mathcal{T}}v \in H^{\alpha/2}(\mathcal{T})$ by (21b), integration by parts gives

$$\begin{aligned}0 &= (D_{\mathcal{T}}v\,, D^{\alpha-2}D_{\mathcal{T}}v) + (v(c - Db/2)\,, v)\\ &\quad + \langle[D^{(\alpha-2)\star}D_{\mathcal{T}}v]\,, \{v\}\rangle + \langle\{D^{(\alpha-2)\star}D_{\mathcal{T}}v\}\,, [v]\rangle + 1/2\langle[v]\,, \{bv\}\rangle + 1/2\langle\{v\}\,, [bv]\rangle\end{aligned}$$

As $c - Db/2 \geq 0$ and $D^{\alpha-2}$ is elliptic in $H^{\alpha/2-1}(\mathcal{T})$ due to Lemma 6, we obtain with the triangle inequality

$$\|D_{\mathcal{T}}v\|^2_{H^{\alpha/2-1}(\mathcal{T})} \lesssim |\langle[\tau]\,, \{v\}\rangle| + |\langle\{\tau\}\,, [v]\rangle| + |\langle[v]\,, \{bv\}\rangle| + |\langle[bv]\,, \{v\}\rangle|.$$

All terms on the right-hand side of this inequality are treated with Lemma 2, eq. (10). For the second term, we additionally use Lemma 3 and (21a) and get

$$\begin{aligned}|\langle\{\tau\}\,, [v]\rangle| &\lesssim N^{1/2}\|\tau\|_{H^1(\mathcal{T})} \cdot |[v]| \lesssim N^{1/2}\|v\|_{L_2(I)}|[v]| + N|[\tau]| \cdot |[v]|\\ &\lesssim N^{1/2}\|v\|_{L_2(I)}|[v]| + N^{3/2}\|v\|_{H^{\alpha/2}(\mathcal{T})} \cdot |[\tau]|.\end{aligned}$$

We conclude that

$$|v|^2_{H^{\alpha/2}(\mathcal{T})} \lesssim \|D_{\mathcal{T}}v\|^2_{H^{\alpha/2-1}(\mathcal{T})} \lesssim N^{3/2}\left(|[\tau]| + |[v]|\right)\|v\|_{H^{\alpha/2}(\mathcal{T})}, \tag{26}$$

where we have used Lemma 4 for the first estimate. Adding (24) and (26) and dividing by $\|v\|_{H^{\alpha/2}(\mathcal{T})}$ gives

$$\|v\|_{H^{\alpha/2}(\mathcal{T})} \lesssim N^{3/2}\left(|[\tau]| + |[v]|\right). \tag{27}$$

Step 3: It remains to show the bound for τ. As $\tau \in L_2(I)$, we can write $\tau = D\psi + t$ with $\psi \in \widetilde{H}^1(I)$ and $t \in \mathbb{R}$ such that $\|\psi\|_{H^1(I)} + |t| \lesssim \|\tau\|_{L_2(I)}$. Integration by parts, identities (21), Lemma 2 eq. (10), and Cauchy-Schwarz show

$$\begin{aligned}(\tau\,, \tau) &= (cv\,, \psi) + \langle[\tau]\,, \psi\rangle - (D^{(\alpha-2)\star}D_{\mathcal{T}}v + bv\,, t)\\ &\lesssim \Big(\|v\|_{L_2(I)} + N^{1/2}|[\tau]| + (D^{(\alpha-2)\star}D_{\mathcal{T}}v\,, 1)\Big)\,\|\tau\|_{L_2(I)}.\end{aligned} \tag{28}$$

For the last term, we use $(D^{\alpha-2}1)(x) = x^{2-\alpha}/\Gamma(2-\alpha+1)$, cf. [37, Section 2.5], and integration by parts to compute

$$\begin{aligned}(D^{(\alpha-2)\star}D_{\mathcal{T}}v\,, 1) &= -(v\,, x^{1-\alpha}) + \frac{\langle[v]\,, x^{2-\alpha}\rangle}{\Gamma(2-\alpha+1)}\\ &\lesssim \|v\|_{H^{\alpha/2}(\mathcal{T})} + N^{1/2}|[v]|.\end{aligned} \tag{29}$$

Here, the last estimate follows by direct computation. Combining the estimates (28), (29), and (27), we obtain

$$\|\tau\|_{L_2(I)} \lesssim N^{3/2}\left(|[\tau]| + |[v]|\right).$$

An estimate for $D_{\mathcal{T}}\tau$ is obtained from (21a) and (27). This concludes the proof. $\square$

4 Numerical Examples

4.1 Discretization and approximate optimal test functions

Let us briefly fix some notation: We consider the discrete subspace

$$U_{\text{hp}}(\mathcal{T}) := U^p(\mathcal{T}) \times U^q(\mathcal{T}) \times \mathbb{R}^{N+1} \times \mathbb{R}^{N-1} \subset U_\alpha,$$

where

$$U^p(\mathcal{T}) := \{v \in L_2(I) \,:\, v|_T \text{ is polynomial of degree at most } p \;\forall T \in \mathcal{T}\}$$

is the space of $\mathcal{T}$-elementwise polynomials of degree $p \in \mathbb{N}_0$. Note that $\dim(U_{\text{hp}}(\mathcal{T})) = (p + q + 4)N$. Given a basis $\{\boldsymbol{u}_j \mid j = 1, \dots, \dim(U_{\text{hp}}(\mathcal{T}))\}$ of $U_{\text{hp}}(\mathcal{T})$, the optimal test functions $\Theta(\boldsymbol{u}_j) \in V_\alpha$ $(j = 1, \dots, \dim(U_{\text{hp}}(\mathcal{T}))$ are computed by solving the problems

$$\langle \Theta(\boldsymbol{u}_j)\,, \boldsymbol{v}\rangle_{V_\alpha} = b(\boldsymbol{u}_j, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V_\alpha = H^1(\mathcal{T}) \times H^{\alpha/2}(\mathcal{T}). \tag{30}$$

For $\boldsymbol{v} = (\tau, v), \boldsymbol{w} = (\rho, w) \in V_\alpha$ the V_α-inner product is given by

$$\langle \boldsymbol{v}\,, \boldsymbol{w}\rangle_{V_\alpha} = (\tau, \rho)_I + (D_\mathcal{T}\tau, D_\mathcal{T}\rho)_I + (v, w)_I + \sum_{T\in\mathcal{T}} \int_T \int_T \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{1+\alpha}}\, dy\, dx,$$

which induces our chosen local norm $\|\boldsymbol{v}\|_{V_\alpha}^2 = \|\tau\|_{H^1(\mathcal{T})}^2 + \|v\|_{H^{\alpha/2}(\mathcal{T})}^2$ on V_α. Since the definition of the optimal test functions (30) involves the infinite-dimensional space V_α, we approximate $\Theta_\alpha(\boldsymbol{u}_j) \in V_\alpha$ by $\Theta_{\alpha,h}(\boldsymbol{u}_j) \in V_{\text{hp}}(\mathcal{T}) := U^m(\mathcal{T}) \times U^n(\mathcal{T})$ with $m, n \in \mathbb{N}_0$, i.e., instead of (30) we solve for $j = 1, \dots, \dim(U_{\text{hp}}(\mathcal{T}))$ the problem

$$\langle \Theta_{\alpha,h}(\boldsymbol{u}_j)\,, \boldsymbol{v}_k\rangle_{V_\alpha} = b(\boldsymbol{u}_j, \boldsymbol{v}_k) \quad k = 1, \dots, \dim(V_{\text{hp}}(\mathcal{T})). \tag{31}$$

The inner product $\langle \boldsymbol{v}\,, \boldsymbol{w}\rangle_{V_\alpha}$ is computed analytically for functions $\boldsymbol{v}, \boldsymbol{w} \in V_{\text{hp}}(\mathcal{T})$. It is seen immediately that choosing m and n too small in comparison with p and q leads to a system which is not well posed. This question is investigated in [21]. The authors show that in the case of the Poisson equation in $\mathbb{R}^d$ and $p = q$, using polynomial degrees $n = m$ which are higher than $p + d$ is sufficient in order to obtain well-posedness and best approximation results. We point out that the operators constructed in [21] are not applicable in our case. Altogether we have to assemble the matrices $\mathbf{B} := (\mathbf{B}_{kj})$ and $\boldsymbol{\Theta} := (\boldsymbol{\Theta}_{k\ell})$ with

$$\mathbf{B}_{kj} := b(\boldsymbol{u}_j, \boldsymbol{v}_k) \quad \text{and} \quad \boldsymbol{\Theta}_{k\ell} := \langle \boldsymbol{v}_\ell\,, \boldsymbol{v}_k\rangle_{V_\alpha},$$

where $\boldsymbol{u}_j$ and $\boldsymbol{v}_k$, $j = 1, \dots, \dim(U_{\text{hp}}(\mathcal{T}))$, $k = 1, \dots, \dim(V_{\text{hp}}(\mathcal{T}))$, are the basis functions described above. Note that $\boldsymbol{\Theta}$ has a sparse structure, whereas $\mathbf{B}$ contains a dense block corresponding to the discretization of the fractional integral operator. With the definition of the right-hand side vector

$$\mathbf{f}_j := \ell(\boldsymbol{v}_j) \quad \text{for all } j = 1, \dots, \dim(V_{\text{hp}}(\mathcal{T}))$$

the computation of the DPG solution (6) consists in solving the linear system

$$\mathbf{B}^T \boldsymbol{\Theta}^{-1} \mathbf{B}\mathbf{x} = \mathbf{B}^T \boldsymbol{\Theta}^{-1} \mathbf{f}. \tag{32}$$

An advantage of the DPG method is that, by design, we can evaluate the error in the energy norm. We define the local contributions of the error in the energy norm on an element $T \in \mathcal{T}$, $\mathrm{est}(T)$, as

$$\mathrm{est}(T)^2 := \sum_{\{j\,:\, \boldsymbol{v}_j|_{T'}=0 \text{ for } T' \neq T\}} (\mathbf{f} - \mathbf{B}\mathbf{x})_j \left(\boldsymbol{\Theta}^{-1}(\mathbf{f} - \mathbf{B}\mathbf{x})\right)_j \tag{33}$$

Then, with $r_h \in V_{\mathrm{hp}}$ denoting the element corresponding to the vector $\boldsymbol{\Theta}^{-1}(\mathbf{f} - \mathbf{B}\mathbf{x})$ it holds

$$\mathrm{est}^2 := \sum_{T \in \mathcal{T}} \mathrm{est}(T)^2 = (\mathbf{f} - \mathbf{B}\mathbf{x})^T \left(\boldsymbol{\Theta}^{-1}(\mathbf{f} - \mathbf{B}\mathbf{x})\right) = \|r_h\|_{V_\alpha}^2. \tag{34}$$

Let us discuss the convergence rates we can expect. Due to standard approximation results of the L_2-orthogonal projection $\pi_p : L_2(I) \to U^p(\mathcal{T})$ we have

$$\begin{aligned}\inf_{\sigma'_{\mathrm{hp}} \in U^p(\mathcal{T})} N^{1-\alpha/2} \|\sigma - \sigma'_{\mathrm{hp}}\|_{H^{\alpha/2-1}(I)} &\leq N^{1-\alpha/2} \|\sigma - \pi_p \sigma\|_{H^{\alpha/2-1}(I)} \\ &\lesssim \|\sigma - \pi_p \sigma\|_{L_2(I)} \lesssim N^{-\min(p+1,s)} \|\sigma\|_{H^s(I)}\end{aligned}$$

and

$$\inf_{u'_{\mathrm{hp}} \in U^q(\mathcal{T})} \|u - u'_{\mathrm{hp}}\|_{L_2(I)} \lesssim N^{-\min(q+1,r)} \|u\|_{H^r(I)}.$$

According to Theorem 7, this yields

$$\|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{hp}}\|_{U_\alpha} \lesssim \mathrm{est} \lesssim N^{-\min(q+1,p+1,r,s)} \left(\|\sigma\|_{H^s(I)} + \|u\|_{H^r(I)}\right). \tag{35}$$

Here, the fact that est can be included in this estimate in this way follows from Theorem 1. For the numerical examples where the exact solution $\boldsymbol{u} = (\sigma, u, \widehat{\sigma}, \widehat{u})$ is known, we can compute the exact error $\|\boldsymbol{u}\|_{U_\alpha}$. For this we define the quantities

$$\begin{aligned}\mathrm{err}(u_h) &:= \|u - u_h\|_{L_2(I)}, \\ \mathrm{err}(\sigma_h) &:= N^{1-\alpha/2} \|h^{\alpha/2-1}(\sigma - \sigma_h)\|_{L_2(I)}, \\ \mathrm{err}(\widehat{u}_h) &:= N^{-1/2} |\widehat{u} - \widehat{u}_h|, \\ \mathrm{err}(\widehat{\sigma}_h) &:= N^{-1/2} |\widehat{\sigma} - \widehat{\sigma}_h|.\end{aligned}$$

Here, $\widehat{u}$ are the evaluations of the function u at the interior nodes (i.e. without the endpoints of the interval $I = (0,1)$) of the mesh $\mathcal{T}$ and $\widehat{\sigma}$ are the evaluations of $D^{2-\alpha}Du$ at the nodes of $\mathcal{T}$. We emphasize that the norms $\mathrm{err}(\sigma_h)$, $\mathrm{err}(\widehat{u}_h)$, and $\widehat{(\sigma_h)}$ to measure the error of approximations to σ, $\widehat{u}$, and $\widehat{\sigma}$ are *stronger* than those contained in the norm $\|\boldsymbol{u}\|_{U_\alpha}$ on the left-hand side of (35). However, the experiments show that we have optimal convergence rates also in these stronger norms. We emphasize that we even have the rigorous error bound

$$\|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{hp}}\|_{U_\alpha}^2 \lesssim \mathrm{est}^2 \lesssim \mathrm{err}(\sigma_h)^2 + \mathrm{err}(u_h)^2.$$

4.2 Example 1

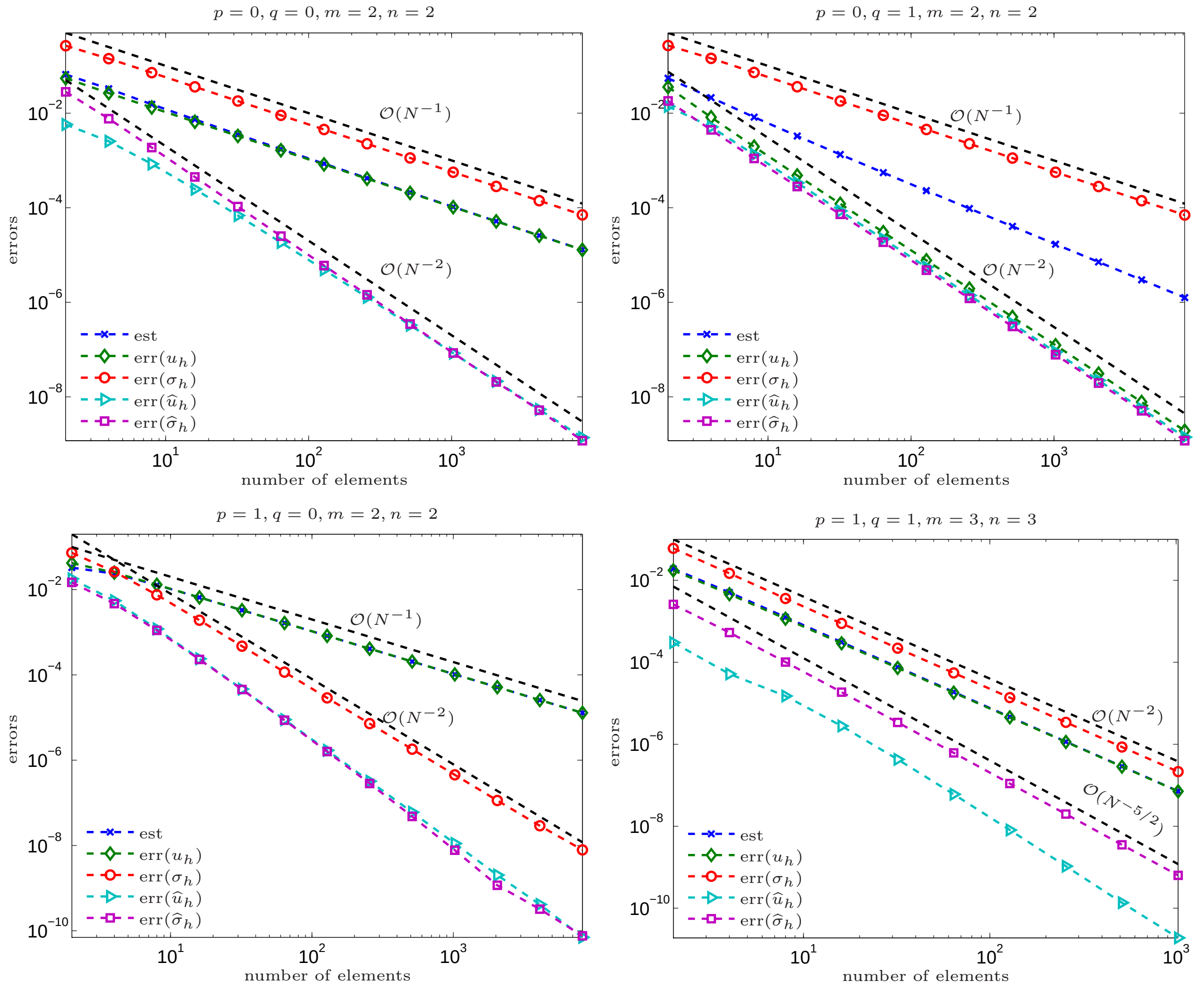

Figure 1: Experimental convergence rates for Example 1 from Section 4.2. Uniform mesh refinement is used througout.

We consider the following example, see also [18, Section 5, Example 2]: Let $I = (0, 1)$, $\alpha = 3/2$, $b(x) := 1/2$, $c(x) := 1/2$ for $x \in I$ and prescribe the exact solution $u(x) = x^2 - x^3$. Then, the right-hand side is given by

$$f(x) = -2\frac{\Gamma(2)}{\Gamma(3-\alpha)}x^{2-\alpha} + 3\frac{\Gamma(3)}{\Gamma(4-\alpha)}x^{3-\alpha} - \frac{1}{2}x^3 - x^2 + x.$$

Furthermore, straightforward calculations show

$$\sigma(x) = Du = 2x - 3x^2,$$
$$D^{2-\alpha}\sigma(x) = D^{2-\alpha}Du = 2\frac{\Gamma(2)}{\Gamma(4-\alpha)}x^{3-\alpha} - 3\frac{\Gamma(3)}{\Gamma(5-\alpha)}x^{4-\alpha}.$$

We consider uniform meshes on I with mesh-size $h = 1/N$ and $N = \#\mathcal{T}$. Figure 1 shows results for different values of p, q, m, n. As u and σ are both smooth, we expect from (35) that $\mathrm{est} = \mathcal{O}(N^{-\min(p+1,q+1)})$, and the numerical experiments reflect this expectation. We even see in the experiments the simultaneous approximation orders $\mathrm{err}(u_h) = \mathcal{O}(N^{-(q+1)})$ and $\mathrm{err}(\sigma_h) = \mathcal{O}(N^{-(p+1)})$. The trace errors $\mathrm{err}(\widehat{u}_h)$ and $\mathrm{err}(\widehat{\sigma}_h)$ show higher convergence rates in all cases. In the case $p = 0, q = 1, m = 2, n = 2$ (upper right plot), est converges slightly faster than $\mathrm{err}(\sigma_h)$ but slower than $\mathrm{err}(u_h)$.

4.3 Example 2

For the next example we prescribe the exact solution $u(x) = x^\lambda - x$ with $1/2 < \lambda < 3/2$ on $I = (0,1)$, see also [18, Section 5, Example 3]. The right-hand side as well as σ are given by

$$f(x) = -\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}x^{\lambda-\alpha} + \frac{1}{\Gamma(2-\alpha)}x^{1-\alpha},$$
$$\sigma(x) = Du = \lambda x^{\lambda-1} - 1,$$
$$D^{\alpha-2}Du(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)}x^{\lambda+1-\alpha} - \frac{1}{\Gamma(3-\alpha)}x^{2-\alpha}.$$

We have $u \in H^{\lambda+1/2-\varepsilon}(I)$ and $\sigma \in H^{\lambda-1/2-\varepsilon}(I)$ for all $\varepsilon > 0$, and hence, due to $1/2 < \lambda < 3/2$, with a view to (35), we expect a convergence rate of $\mathrm{est} = \mathcal{O}(N^{1/2-\lambda})$. However, with a view to the norm $\|\cdot\|_{U_\alpha}$, the expected rate, dictated by σ in this case, would be $\mathcal{O}(N^{1/2-\lambda+\alpha/2-1})$. This is what we will see for uniform refinement. In order to regain the optimal convergence orders $\mathcal{O}(N^{-\min(p+1,q+1)})$, we utilize an adaptive strategy where we use $\mathrm{est}(T)$ as local refinement indicators and mark elements $\mathcal{M} \subseteq \mathcal{T}$ according to Dörfler's marking criterion

$$\theta\,\mathrm{est}^2 \le \sum_{T\in\mathcal{M}} \mathrm{est}(T)^2, \tag{36}$$

where we use $\theta = 0.4$ and $\mathcal{M}$ is a set of minimal cardinality. Note that $\theta = 1$ means uniform refinement, i.e., $\mathcal{M} = \mathcal{T}$. Each marked element $T \in \mathcal{M}$ is bisected such that local quasi-uniformity

$$\max_{\substack{T,T'\in\mathcal{T}\\ \overline{T}\cap\overline{T'}\neq\emptyset}} \frac{\mathrm{diam}(T)}{\mathrm{diam}(T')} \le 2$$

is preserved. Figure 2 shows est and the error quantities for the parameters

$$\lambda = 0.6, \quad \alpha = 1.2.$$

In the upper left plot the results for uniform refinement and $p = q = 0$, $n = m = 2$ are given. We observe the convergence rate est $= \mathcal{O}(N^{1/2-\lambda+\alpha/2-1}) = \mathcal{O}(N^{-\lambda+1/10})$. As expected, also for the separated error contributions, we observe reduced convergence rates. Adaptive refinement recovers the optimal rate $\mathcal{O}(N^{-\min(p+1,q+1)})$, as is seen in the three remaining plots. As in Example 4.2, we see that the traces even have better convergence rates.

4.4 Example 3

In the last experiment we set $f(x) := \log(x)$ for $x \in I = (0,1)$ and note that $f \in L_2(I)$. For this right-hand side we do not know the explicit form of the solution u. Therefore, we only plot the error in the energy norm est for different values of p, q, m, n and α, respectively. Throughout, we set $p = q$ as well as $m = n = p + 2$. Figure 3 shows the error in the energy norm est for $\alpha = 1.6$ (left) and $\alpha = 1.8$ (right). We compare uniform refinement ($\theta = 1$) and adaptive refinement with $\theta = 0.4$ for $p = q = 0$. Moreover, we plot the results in the adaptive case with $p = q = 1$ resp. $p = q = 2$. We observe that for adaptive refinement we obtain convergence rates $p+1$, i.e., est $= \mathcal{O}(N^{-(p+1)})$, whereas for uniform refinement we get only the suboptimal rate $\alpha/2 - 1/2$.

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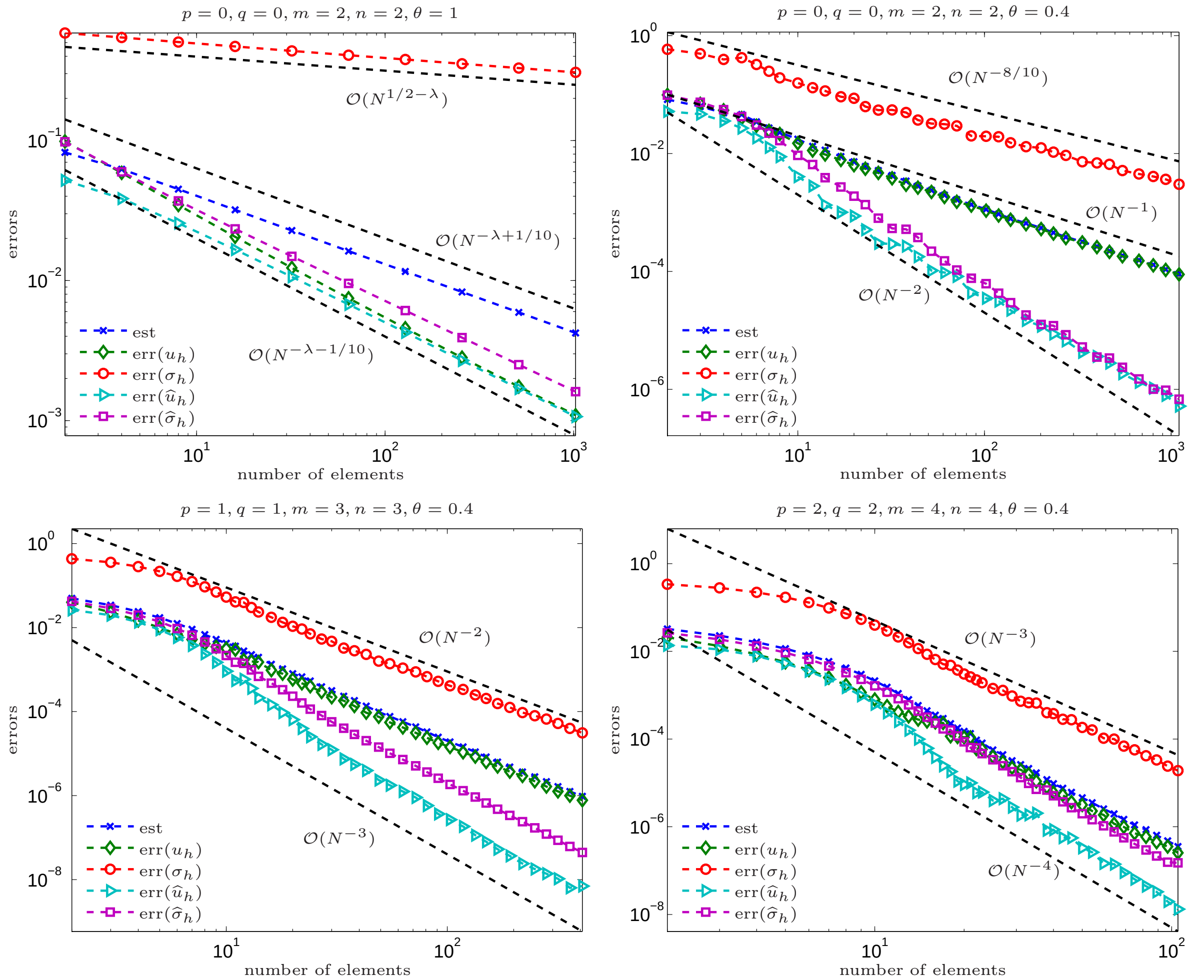

Figure 2: Experimental convergence rates for Example 2 from Section 4.3. Uniform mesh refinement (upper left) and adaptive mesh refinement (upper right and below).

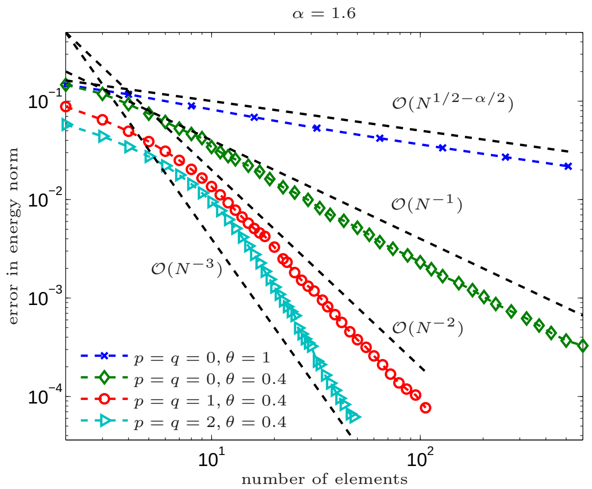

Figure 3: Experimental convergence rates for Example 3 from Section 4.4. The choice $\theta = 1$ refers to uniform mesh refinement, while $\theta = 0.4$ refers to adaptive mesh refinement.