A deposition model coupling Stokes' and Darcy's equations with nonlinear deposition

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Abstract

In this work we investigate a filtration process whereby particulate is deposited in the flow domain, causing the porosity of the region to decrease. The fluid flow is modeled as a coupled Stokes-Darcy flow problem and the deposition (in the Darcy domain) is modeled using a nonlinear equation for the porosity. Existence and uniqueness of a solution to the governing equations is established. Additionally, the nonnegativity and boundedness of the porosity is shown. A finite element approximation scheme that preserves the nonnegativity and boundedness of the porosity is investigated. Accompanying numerical experiments support the analytical findings.

Key words. Stokes equation, Darcy equation, filtration
AMS Mathematics subject classifications. 76505, 76D07, 35M10, 35Q35, 65M60, 65M55

1 Introduction

Applications of filtration abound in our everyday lives. From the routine activities such as: the preparation of espresso coffee in the morning [13], the water we drink from the faucet [29], and the car we drive to work [26]. To the less obvious but not less important such as: The absorption of nutrients in the small intestine [22], the cleansing of blood in the kidneys [21], and the prevention of postoperative infections [23]. All these phenomena rely on the separation of some solid from a fluid by means of a medium that is permeable to the fluid but (mostly) impermeable to the solid.

In this work we investigate a filtration process whereby particulate is deposited in the flow domain, causing the porosity of the region to decrease. The fluid flow is modeled as a coupled Stokes-Darcy flow problem and the deposition (in the Darcy domain) is modeled using a nonlinear equation for the porosity. (See Figure 1.) The model considered in this paper extends the work presented in [11] where the analysis was restricted to the filtration domain.

The addition of the upstream flow domain introduces into the model a different set of flow equations which must be suitable coupled across the interface (Γ) between the two subdomains. The coupling equations, given in (2.6) represent the conservation of mass across Γ (see (2.6a)) , the conservation of the normal component of stress across Γ (see (2.6b)) , and an equation (the Beavers-Joseph-Saffman condition (2.6c)) for the tangential component of the stress vector on the Stokes domain.

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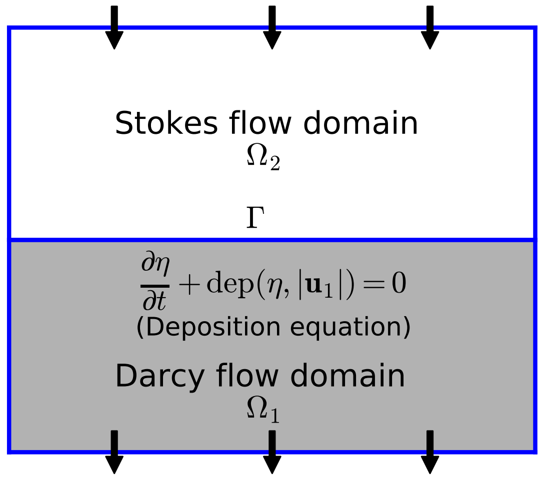


Figure 1: Coupled Stokes-Darcy flow domain. A fluid in the Stokes domain Ω_2 flows across the interface Γ towards the filtration (porous) region Ω_1, where the Darcy equations are applicable.

As a first step in analyzing this coupled model we use the quasi static Stokes equations (i.e., neglect the $\frac{\partial \mathbf{u}_2}{\partial t}$) to model the fluid flow in the upstream domain Ω_2. For the filtration process we assume that the rate of change of the porosity (η), caused by the particulate deposition, is only dependent upon the porosity and the magnitude of the fluid velocity, i.e.,

$$\frac{\partial \eta}{\partial t} = -\mathrm{dep}\,(\eta\,,\,|\mathbf{u}_1|)\,, \tag{1.1}$$

where $\mathbf{u}_1$ denotes the fluid velocity in the filter, Ω_1. As a first approximation, we assume that the deposition function is a separable function of $|\mathbf{u}_1|$ and η, i.e., $\mathrm{dep}\,(\eta\,,\,|\mathbf{u}_1|) = \mathrm{g}(|\mathbf{u}_1|)\mathrm{h}(\eta)$. (The specific assumptions we make on $g(\cdot)$ and $h(\cdot)$ are given by ***A1*** and ***A2*** below.

Following the seminal work of Discacciati, Miglio and Quarteroni in [8], and Layton, Schieweck and Yotov in [25], much work has been done on the numerical approximation of the coupled Stokes-Darcy fluid flow problem. These investigations have covered different variational formulations of the modeling equations, different discretization schemes, and different solution algorithms for the discretized system of equations. A recent overview of this work is summarized in the Introduction of [3]. The coupled Stokes-Darcy fluid flow system has been extended to model other challenging physical problems. In particular we mention the incorporation of a two phase fluid into the Stokes-Darcy setting to yield the Cahn-Hilliard-Stokes-Darcy system [18, 19, 5, 7], and the dual-porosity Stokes-Darcy model [20] where the Darcy domain is assumed to be made up of a matrix and microfracture components. Also related, and much studied, are fluid-structure interaction (FSI) problems.

In [16] the authors investigated a model for contaminate transport coupled to a Stokes-Darcy fluid flow problem. Different from the filtration problem, the contaminate transport equation is a convection-diffusion equation defined across the entire flow domain and with the property that the transported contaminate does not influence the fluid flow.

A similar model to that studied herein in the filtration domain, Ω_1, arises in the study of single-phase, miscible displacement of one fluid by another in a porous medium. For this problem η would denote a fluid concentration, and the hyperbolic deposition equation (1.1) is replaced by a parabolic transport equation. Existence and uniqueness for this problem has been investigated and established by Feng [14], Chen and Ewing [6], and Çeşmelioğlu and Rivière [4]. Because of the connection of this model to oil extraction, numerical approximation schemes for this problem have been well established. A summary of these methods is discussed in the recent papers by Bartels, Jensen and Müller [1], and Riviére and Walkington [27].

Following a discussion of the notation and assumptions in Section 2, the existence of a solution to the modeling equations is established in Sections 3 and 4. Subsequently, we show that the porosity function η is nonnegative and bounded. An approximation scheme for the filtration model is presented in Section 6. Also in Section 6 we show that the computed approximation for the discrete porosity is nonnegative and bounded. Three numerical experiments are presented whose results support the theoretical findings.

2 Notation and assumptions

Throughout this manuscript, the symbol C indicates a generic constant independent of the discretization parameters, whose value may change from line to line. The symbol μ denotes the dynamic viscosity in the fluid domain Ω_2. We let $\mathbf{u}$, p and η denote the velocity, pressure and porosity throughout Ω, respectively, and use a subscript $i = 1, 2$ to indicate if a variable corresponds to the Darcy ($i = 1$) or Stokes ($i = 2$) domain. Moreover, for a generic function f_i supported on Ω_i, we extend f_i to the whole Ω by setting $f_i \equiv 0$ on $\Omega \setminus \Omega_i$. We omit the subscript whenever it is clear over which region the function is evaluated. We partition the boundary of $\Omega = \Omega_1 \cup \Omega_2$ into three disjoint pieces: The interface or connecting boundary $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, the Darcy boundary $\Gamma_1 = \partial\Omega_1 \setminus \Gamma$ and the Stokes boundary $\Gamma_2 = \partial\Omega_2 \setminus \Gamma$. We use the notation

$$(f, g)_U = \int_U f(\mathbf{x}) \cdot g(\mathbf{x})\, d\mathbf{x}, \quad \|f\|_U^2 = (f, f)_U,$$

to denote the L^2 inner product and norm on $U \subset \Omega$, where the dot product is replaced with the Frobenius product in the case of tensors, and

$$\langle f, g\rangle_\Gamma = \int_\Gamma f \cdot g\, dS$$

to indicate either a surface integral along the interface Γ, or the duality pairing between f and g. We omit the subscript whenever it is clear from the context over which region we compute the integral. The function $|\cdot|$ represents the Euclidean norm for vectors, the Frobenius norm for tensors and the Lebesgue measure for sets. The relevant function spaces in the following derivations are: The Darcy and Stokes velocity spaces

$$\mathbf{X}_1 = \{\mathbf{v}_1 \in \mathbf{H}_{\text{div}}(\Omega_1) \,|\, \mathbf{v}_1 \cdot \mathbf{n} = 0 \text{ on } \Gamma_1\}, \quad \mathbf{X}_2 = \left\{\mathbf{v}_2 \in H^1(\Omega_2)^d \,|\, \mathbf{v}_2 = 0 \text{ on } \Gamma_2\right\},$$

the Darcy and Stokes pressure spaces, $Q_1 = L^2(\Omega_1)$ and $Q_2 = L^2(\Omega_2)$, the porosity space $L^2(\Omega_1)$, the space of Lagrange multipliers $\Lambda = H^{1/2}(\Gamma)$, and the space of continuous functions on Ω_1, $C^0(\Omega_1)$. The corresponding norms in the velocity spaces are, for $\mathbf{u}_1 \in \mathbf{X}_1$ and $\mathbf{u}_2 \in \mathbf{X}_2$:

$$\|\mathbf{u}_1\|_{\mathbf{X}_1} = \left(\|\mathbf{u}_1\|^2 + \|\nabla \cdot \mathbf{u}_1\|^2\right)^{1/2}, \quad \|\mathbf{u}_2\|_{\mathbf{X}_2} = \sqrt{2\mu}\, \|\mathbb{D}(\mathbf{u}_2)\|.$$

Furthermore, we introduce the spaces $Q = \left\{q \in L^2(\Omega) \,\middle|\, (q, 1)_\Omega = 0\right\}$,

$$\mathbf{X} = \left\{\mathbf{v} \in L^2(\Omega)^d \,\middle|\, \mathbf{v}_1 \in \mathbf{X}_1 \text{ and } \mathbf{v}_2 \in \mathbf{X}_2\right\},$$

its continuous dual $\mathbf{X}'$, and the norm

$$\|\mathbf{u}\|_{\mathbf{X}} = \left(\|\mathbf{u}\|_{\mathbf{X}_1}^2 + \|\mathbf{u}\|_{\mathbf{X}_2}^2\right)^{1/2}. \tag{2.1}$$

The following definition introduces the concept of a smoothing operator in Ω_1.

Definition 2.1. *We say that the linear operator $\mathcal{S} : L^2(\Omega_1) \to C^0(\Omega_1)$ is smoothing if:*

***S*1**: *There exists a constant $C_s = C_s(\Omega_1)$ such that $\|\mathcal{S}(u)\|_{L^\infty(\Omega_1)} \le C_s \|u\|_{L^2(\Omega_1)}$ for all $u \in L^2(\Omega_1)$, and*

***S*2**: *if $\{u_n\}_{n=1}^\infty \subset L^2(\Omega_1)$ such that $u_n \rightharpoonup u$ in $L^2(\Omega_1)$, then $\mathcal{S}(u_n) \to \mathcal{S}(u)$ in $L^\infty(\Omega_1)$, i.e., $\mathcal{S}$ transforms a weakly convergent sequence in $L^2(\Omega_1)$ into a strongly convergent sequence in $L^\infty(\Omega_1)$.*

Examples of smoothing operators can be found in [12]. For the rest of the article we adopt the convention of indicating the action of $\mathcal{S}$ on η as η^s.

The full system is now restated for ease of reference. For $T \in \mathbb{R}^+$ denoting the time horizon of the filtration process, we consider

$$\beta(\eta^s)\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega_1 \times (0,T),$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_1 \times (0,T), \tag{2.2}$$
$$\frac{\partial \eta}{\partial t} + g(|\mathbf{u}|)\, h(\eta) = 0 \text{ in } \Omega_1 \times (0,T), \tag{2.3}$$
$$\eta = \eta_0 \text{ in } \Omega_1 \times \{0\},$$
$$-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \text{ in } \Omega_2 \times (0,T),$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_2 \times (0,T), \tag{2.4}$$

where $\mathbb{T}(\mathbf{u}, p) = -2\mu\mathbb{D}(\mathbf{u}) + \mathbb{I}p$ is the stress tensor, $\mathbb{D}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ is the deformation tensor, and $\mathbb{I}$ is the identity tensor. Furthermore, system (2.2)-(2.4) is complemented by the boundary data

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_1 \times (0,T), \tag{2.5a}$$
$$\mathbf{u} = 0 \text{ on } \Gamma_2 \times (0,T), \tag{2.5b}$$

and the interface conditions

$$(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{n} = 0 \text{ on } \Gamma \times (0,T), \tag{2.6a}$$
$$\mathbf{n} \cdot \mathbb{T}(\mathbf{u}_2, p_2) \cdot \mathbf{n} = p_1 \text{ on } \Gamma \times (0,T), \tag{2.6b}$$
$$P_{\mathbf{t}}(\mathbb{T}(\mathbf{u}_2, p_2) \cdot \mathbf{n}) = \Psi(\eta)\, P_{\mathbf{t}}(\mathbf{u}_2) \text{ on } \Gamma \times (0,T), \tag{2.6c}$$

where $P_{\mathbf{t}}(\mathbf{v}(p))$ is the projection of $\mathbf{v}$ onto the tangent plane at the point p on Γ, and $\Psi(\eta)$ the proportionality function in the Beavers-Joseph-Saffman condition relating the tangential component of the normal stress in the Stokes domain to its tangential velocity.

We make the following assumptions:

***A*1**: The functions

$$\beta(\,\cdot\,) : \mathbb{R}^+ \to \mathbb{R}^+, \quad g(\,\cdot\,) : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\},$$
$$\Psi(\,\cdot\,) : \mathbb{R}^+ \to \mathbb{R}^+, \quad h(\,\cdot\,) : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\},$$

satisfy the following bounds:

$$0 < \beta_{\min} \le \beta(\,\cdot\,) \le \beta_{\max}, \quad g(\,\cdot\,) \le g_{\max},$$
$$0 < \Psi_{\min} \le \Psi(\,\cdot\,) \le \Psi_{\max}, \quad h(\,\cdot\,) \le h_{\max}.$$

A2: $\beta(\cdot)$, $g(\cdot)$, $h(\cdot)$, $\Psi(\cdot)$ are Lipschitz continuous with Lipschitz constants β_{Lip}, g_{Lip}, h_{Lip} and Ψ_{Lip}, respectively.

A3: $\eta_0 \in L^\infty(\Omega_1)$ and $\eta_0(\mathbf{x}) \geq 0$ for *a.e.* $\mathbf{x}$ in Ω_1.

A4: $\mathbf{f} \in C^0(0^-, T;\, L^2(\Omega))$ defined as $C^0(0-\delta, T;\, L^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ for some $\delta > 0$.

A discussion of the filtration model and assumptions is presented in [11].

Remark 2.1. *Note that since η^s is the result of $\mathcal{S}(\eta)$, the Beavers-Joseph-Saffman condition* (2.6c) *is well-defined in view that $\eta^s \in C^0(\Omega_1)$.*

Additional notation that we need in the subsequent analysis is: For $\mathbf{u}, \mathbf{v} \in \mathbf{X}$, $q \in Q$, $\eta \in L^2(\Omega_1)$ and $\nu \in \Lambda$, define

$$
\begin{aligned}
a_1(\eta; \mathbf{u}, \mathbf{v}) &:= (\beta(\eta^s)\, \mathbf{u}, \mathbf{v})_{\Omega_1}, \quad a_2(\mathbf{u}, \mathbf{v}) := \Big(2\mu\, \mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v})\Big)_{\Omega_2}, \\
b(\mathbf{v}, q) &:= -(\nabla \cdot \mathbf{v}, q)_\Omega, \quad c(\mathbf{v}, \nu) := \langle (\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n}, \nu \rangle_\Gamma, \quad \ell(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_\Omega, \\
d(\eta; \mathbf{u}, \mathbf{v}) &:= \langle \Psi(\eta^s)\, P_{\mathbf{t}}(\mathbf{u}_2), P_{\mathbf{t}}(\mathbf{v}_2) \rangle_\Gamma .
\end{aligned} \tag{2.7}
$$

Remark 2.2. *Owing to assumption **A4**, the operator $\ell : \mathbf{X} \to \mathbb{R}$ introduced in* (2.7) *is a continuous linear functional. Thus, $\ell \in \mathbf{X}'$.*

In the next section we investigate the well-posedness of the weak form corresponding to system (2.2)-(2.5).

3 Existence and uniqueness of the solution

Multiplying (2.2)-(2.4) by the corresponding test functions and incorporating conditions (2.6)-(2.5), the resulting weak form is: *Given $\eta_0 \in L^2(\Omega_1)$ and $\mathbf{f} \in C^0(0^-, T;\, L^2(\Omega))$, find $\mathbf{u} \in L^2(0, T; \mathbf{X})$, $p \in L^2(0, T; Q)$, $\lambda \in L^2(0, T; \Lambda)$ and $\eta \in H^1(0, T;\, L^2(\Omega_1))$, satisfying $\eta(\cdot, 0) = \eta_0$ a.e. in Ω_1, and for a.e. $t \in (0, T)$*

$$
a_1(\eta; \mathbf{u}, \mathbf{v}) + a_2(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{v}, \lambda) + d(\eta; \mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}) \;\forall \mathbf{v} \in \mathbf{X}, \tag{3.1}
$$

$$
b(\mathbf{u}, q) = 0 \;\forall q \in Q, \tag{3.2}
$$

$$
c(\mathbf{u}, \nu) = 0 \;\forall \nu \in \Lambda, \tag{3.3}
$$

$$
(\frac{\partial \eta}{\partial t}, \xi)_{\Omega_1} + \Big(g(|\mathbf{u}|)\, h(\eta), \xi\Big)_{\Omega_1} = 0 \;\forall \xi \in L^2(\Omega_1). \tag{3.4}
$$

To simplify the analysis, we introduce the space

$$
\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{X} \,\middle|\, c(\mathbf{v}, \nu) = 0 \;\forall \nu \in \Lambda, \text{ and } \; b(\mathbf{v}, q) = 0 \;\forall q \in Q \right\}.
$$

In order to restrict the analysis to $\mathbf{V}$, we require the following inf-sup condition.

Lemma 3.1. *There exists a positive constant γ such that*

$$
\gamma < \inf_{\mathbf{0} \neq (q,\nu) \in Q \times \Lambda} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, q) + c(\mathbf{v}, \nu)}{\|\mathbf{v}\|_{\mathbf{X}}\, \|(q, \nu)\|_{Q \times \Lambda}}, \tag{3.5}
$$

where $\|(q, \nu)\|^2_{Q \times \Lambda} = \|q\|^2_{L^2(\Omega)} + \|\nu\|^2_\Lambda$.

Proof. The result follows from Proposition 4.7 and Remark 4.8 in [15]. □

There are two steps in establishing the existence and uniqueness of a solution to (3.1)-(3.4). The fist step, presented in Section 3.1, establishes that given η, (3.1)-(3.3) is uniquely solvable for $\mathbf{u}$, p, and λ (in terms of η). For η given, we denote the solution to $\mathbf{u}$ as $\mathbf{u}(t) = \mathcal{P}(\eta, t)$. Step 2, in Section 3.2, then considers (3.4) with $\mathbf{u}(t)$ replaced by $\mathcal{P}(\eta, t)$, thereby yielding a first order initial value problem for $\eta(t)$. Existence and uniqueness for $\eta(t)$ is obtained using the Picard-Lindelöf theorem.

3.1 Unique solvability of (3.1)-(3.3)

In view of the definition of $\mathbf{V}$ and Lemma 3.1, assuming $\eta \in L^2(\Omega_1)$ and $\mathbf{f} \in C^0(0^-, T; L^2(\Omega))$ are given, system (3.1)-(3.3) is equivalent to: Find $\mathbf{u} \in L^2(0, T; \mathbf{V})$ satisfying

$$a(\eta; \mathbf{u}, \mathbf{v}) := a_1(\eta; \mathbf{u}, \mathbf{v}) + a_2(\mathbf{u}, \mathbf{v}) + d(\eta; \mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}. \tag{3.6}$$

Lemma 3.2. *Let $\eta \in L^2(\Omega_1)$ be given. Then, the problem: Find $\mathbf{u} \in \mathbf{V}$ satisfying for all $\mathbf{v} \in \mathbf{V}$*

$$a(\eta; \mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}), \tag{3.7}$$

has a unique solution. We call $(\eta, \mathbf{u})$ a solution pair to (3.7).

Proof. In view of assumption ***A1*** and the continuity of the trace map with norm $C_{\mathcal{T}}$, it follows that

$$\begin{aligned} a(\eta; \mathbf{u}, \mathbf{v}) &\leq \beta_{\max} \|\mathbf{u}\|_{\mathbf{X}_1} \|\mathbf{v}\|_{\mathbf{X}_1} + 2\mu \|\mathbf{u}\|_{\mathbf{X}_2} \|\mathbf{v}\|_{\mathbf{X}_2} + \Psi_{\max} C_{\mathcal{T}}^2 \|\mathbf{u}\|_{\mathbf{X}_2} \|\mathbf{v}\|_{\mathbf{X}_2} \\ &\leq \max\left\{\beta_{\max},\, 2\mu,\, \Psi_{\max} C_{\mathcal{T}}^2\right\} \|\mathbf{u}\|_{\mathbf{X}} \|\mathbf{v}\|_{\mathbf{X}}. \end{aligned} \tag{3.8}$$

Now observe that owing to ***A1*** and the nonnegativity of $d(\eta, \mathbf{u}, \mathbf{u})$,

$$a(\eta; \mathbf{u}, \mathbf{u}) \geq \beta_{\min} \|\mathbf{u}\|_{\mathbf{X}_1}^2 + \|\mathbf{u}\|_{\mathbf{X}_2}^2 \geq \min\{\beta_{\min},\, 1\} \|\mathbf{u}\|_{\mathbf{X}}^2. \tag{3.9}$$

Finally, since $\|\mathbf{v}\|_{H^1(\Omega_2)} \leq C_K \|\mathbf{v}\|_{\mathbf{X}_2}$ for some positive constant C_K, we obtain

$$\begin{aligned} \ell(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\Omega_1} + (\mathbf{f}, \mathbf{v})_{\Omega_2} &\leq \|\mathbf{f}\|_{L^2(\Omega_1)} \|\mathbf{v}\|_{L^2(\Omega_1)} + \|\mathbf{f}\|_{H^{-1}(\Omega_2)} \|\mathbf{v}\|_{H^1(\Omega_2)} \\ &\leq \|\mathbf{f}\|_{L^2(\Omega_1)} \|\mathbf{v}\|_{L^2(\Omega_1)} + \|\mathbf{f}\|_{L^2(\Omega_2)} C_K \|\mathbf{v}\|_{\mathbf{X}_2} \leq \max\{1,\, C_K\} \|\mathbf{f}\| \|\mathbf{v}\|_{\mathbf{X}}. \end{aligned} \tag{3.10}$$

Consequently, from (3.8), (3.9) and (3.10), the existence of a unique solution to (3.7) follows by the Lax-Milgram lemma. □

The next corollary provides an estimate for the norm of the solution to problem (3.7).

Corollary 3.1. *Define*

$$C_\beta = \frac{1}{\min\{\beta_{\min},\, 1\}}, \quad C_b = \max\{1,\, C_K\}\, C_\beta, \quad C_{\mathbf{f}} = C_b \|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega))}, \tag{3.11}$$

and let $\mathbf{u} \in \mathbf{V}$ be the solution to the problem stated in Lemma 3.2. Then,

$$\|\mathbf{u}\|_{\mathbf{X}}(t) \leq C_{\mathbf{f}}. \tag{3.12}$$

Proof. This is a direct consequence of (3.9) and (3.10). □

The following result is related to the continuity of the solution $\mathbf{u}$ as a function of the porosity η.

Lemma 3.3. *Let* $(\eta^1, \mathbf{u}^1)$, $(\eta^2, \mathbf{u}^2) \in L^2(\Omega_1) \times \mathbf{V}$ *be solution pairs to problem* (3.7)*. Then, there exists a positive constant* C_{Lip} *such that*

$$\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}}(t) \le C_{\mathrm{Lip}}\,\|\eta_1 - \eta_2\|_{L^2(\Omega_1)}(t). \tag{3.13}$$

Proof. For clarity of exposition, we suppress the dependence of the functions on t. First note that

$$\begin{aligned} a_1(\eta^1;\mathbf{u}^1,\mathbf{v}) - a_1(\eta^2;\mathbf{u}^2,\mathbf{v}) = &\left(\beta(\eta^{1,s})\,(\mathbf{u}^1 - \mathbf{u}^2),\mathbf{v}\right)_{\Omega_1} \\ &+ \left(\left(\beta(\eta^{1,s}) - \beta(\eta^{2,s})\right)\,\mathbf{u}^2,\mathbf{v}\right)_{\Omega_1}. \end{aligned} \tag{3.14}$$

Similarly for $d(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ and $a_2(\,\cdot\,,\,\cdot\,)$,

$$\begin{aligned} d(\eta^1;\mathbf{u}^1,\mathbf{v}) - d(\eta^2;\mathbf{u}^2,\mathbf{v}) = &\left\langle \Psi(\eta^{1,s})\,P_{\mathbf{t}}(\mathbf{u}^1_2 - \mathbf{u}^2_2), P_{\mathbf{t}}(\mathbf{v}_2)\right\rangle_\Gamma \\ &+ \left\langle \left(\Psi(\eta^{1,s}) - \Psi(\eta^{2,s})\right)\,P_{\mathbf{t}}(\mathbf{u}^2_2), P_{\mathbf{t}}(\mathbf{v}_2)\right\rangle_\Gamma, \end{aligned} \tag{3.15}$$

$$a_2(\mathbf{u}^1,\mathbf{v}) - a_2(\mathbf{u}^2,\mathbf{v}) = \left(2\mu\,\mathbb{D}(\mathbf{u}^1 - \mathbf{u}^2),\mathbb{D}(\mathbf{v})\right)_{\Omega_2}. \tag{3.16}$$

Now observe that

$$\begin{aligned} a(\eta^1;\mathbf{u}^1,\mathbf{v}) - a(\eta^2;\mathbf{u}^2,\mathbf{v}) = &\, a_1(\eta^1;\mathbf{u}^1,\mathbf{v}) - a_1(\eta^2;\mathbf{u}^2,\mathbf{v}) \\ &+ a_2(\mathbf{u}^1,\mathbf{v}) - a_2(\mathbf{u}^2,\mathbf{v}) + d(\eta^1;\mathbf{u}^1,\mathbf{v}) - d(\eta^2;\mathbf{u}^2,\mathbf{v}) = 0. \end{aligned} \tag{3.17}$$

Adding (3.14) , (3.15) and (3.16), and using (3.17), yields

$$\begin{aligned} &\left(\beta(\eta^{1,s})\,(\mathbf{u}^2 - \mathbf{u}^1),\mathbf{v}\right)_{\Omega_1} + \left\langle \Psi(\eta^{1,s})\,P_{\mathbf{t}}(\mathbf{u}^2_2 - \mathbf{u}^1_2), P_{\mathbf{t}}(\mathbf{v}_2)\right\rangle_\Gamma \\ &+ \left(2\mu\,\mathbb{D}(\mathbf{u}^2 - \mathbf{u}^1),\mathbb{D}(\mathbf{v})\right)_{\Omega_2} = \left\langle \left(\Psi(\eta^{1,s}) - \Psi(\eta^{2,s})\right)\,P_{\mathbf{t}}(\mathbf{u}^2_2), P_{\mathbf{t}}(\mathbf{v}_2)\right\rangle_\Gamma \\ &+ \left(\left(\beta(\eta^{1,s}) - \beta(\eta^{2,s})\right)\,\mathbf{u}^2,\mathbf{v}\right)_{\Omega_1}. \end{aligned} \tag{3.18}$$

Setting $\mathbf{v} = \mathbf{u}_2 - \mathbf{u}_1$ in (3.18) and using assumptions ***A1***, ***A2***, and the trace theorem, we obtain

$$\begin{aligned} &\beta_{\min}\,\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} \\ &\qquad \le \Psi_{\mathrm{Lip}}\,\|\eta^{1,s} - \eta^{2,s}\|_{L^\infty(\Omega_1)}\,C^2_{\mathcal{T}}\,\|\mathbf{u}^2\|_{\mathbf{X}_2}\,\|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}_2} \\ &\qquad + \beta_{\mathrm{Lip}}\,\|\eta^{1,s} - \eta^{2,s}\|_{L^\infty(\Omega_1)}\,\|\mathbf{u}^2\|_{\mathbf{X}_1}\,\|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}_1}. \end{aligned} \tag{3.19}$$

Applying Corollary 3.1 to bound $\|\mathbf{u}\|_{\mathbf{X}_i}$ for $i = 1, 2$ and Young's inequality in (3.19), yields

$$\begin{aligned} \beta_{\min}\,\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} \le &\frac{1}{4\varepsilon_2}\Psi^2_{\mathrm{Lip}}\,\|\eta^{1,s} - \eta^{2,s}\|^2_{L^\infty(\Omega_1)}\,C^4_{\mathcal{T}}\,C^2_{\mathbf{f}} \\ + \varepsilon_2\,\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} + &\frac{1}{4\varepsilon_1}\beta^2_{\mathrm{Lip}}\,\|\eta^{1,s} - \eta^{2,s}\|^2_{L^\infty(\Omega_1)}\,C^2_{\mathbf{f}} + \varepsilon_1\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1}. \end{aligned} \tag{3.20}$$

Finally, setting $\varepsilon_1 = \frac{\beta_{\min}}{2}$ and $\varepsilon_2 = 1/2$ in (3.20), we obtain (using property ***S1*** of the smoother)

$$\begin{aligned} \beta_{\min}\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} &\le \|\eta^{1,s} - \eta^{2,s}\|^2_{L^\infty(\Omega_1)} C^2_{\mathbf{f}} \left(\Psi^2_{\mathrm{Lip}}\,C^4_{\mathcal{T}} + \frac{\beta^2_{\mathrm{Lip}}}{\beta_{\min}}\right) \\ &\le C^2_s\,\|\eta^1 - \eta^2\|^2_{L^2(\Omega_1)} C^2_{\mathbf{f}} \left(\Psi^2_{\mathrm{Lip}}\,C^4_{\mathcal{T}} + \frac{\beta^2_{\mathrm{Lip}}}{\beta_{\min}}\right). \end{aligned} \tag{3.21}$$

Estimate (3.13) follows from (3.21). □

The next lemma shows that for a given porosity η, the solution $\mathbf{u}$ to (3.7) depends continuously on the forcing term $\mathbf{f}$.

Lemma 3.4. *Let $\eta \in L^2(\Omega_1)$ be given and let $\mathbf{u}^1, \mathbf{u}^2$ be the solutions to (3.7) corresponding to the linear functionals ℓ^1 and ℓ^2, respectively. Then,*

$$\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{X}}(t) \leq C_\beta \, \|\ell^1 - \ell^2\|_{\mathbf{X}'}(t).$$

Proof. Consider the problems

$$a(\eta, \mathbf{u}^1, \mathbf{v}) = \ell^1(\mathbf{v}) \;\forall \mathbf{v} \in \mathbf{V}, \tag{3.22}$$
$$a(\eta, \mathbf{u}^2, \mathbf{v}) = \ell^2(\mathbf{v}) \;\forall \mathbf{v} \in \mathbf{V}. \tag{3.23}$$

Subtracting (3.22) from (3.23) and proceeding in a similar manner as in Lemma 3.3 (see (3.19)), we obtain the bound

$$\beta_{\min} \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} \leq \|\ell^2 - \ell^1\|_{\mathbf{X}'} \, \|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}}. \tag{3.24}$$

From (3.24), the result follows. □

The next corollary is a generalization of Lemma 3.3 and Lemma 3.4.

Corollary 3.2. *Let $(\eta^1, \mathbf{u}^1)$, $(\eta^2, \mathbf{u}^2) \in L^2(\Omega_1) \times \mathbf{V}$ be solution pairs to*

$$a(\eta^1, \mathbf{u}^1, \mathbf{v}) = \ell^1(\mathbf{v}) \;\forall \mathbf{v} \in \mathbf{V},$$
$$a(\eta^2, \mathbf{u}^2, \mathbf{v}) = \ell^2(\mathbf{v}) \;\forall \mathbf{v} \in \mathbf{V}.$$

Then,

$$\|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}}(t) \leq 2\, C_\beta \, C_{\text{Lip}} \, \|\eta^2 - \eta^1\|^2_{L^2(\Omega_1)}(t) + 4\, C_\beta^2 \, \|\ell^2 - \ell^1\|^2_{\mathbf{X}'}(t). \tag{3.25}$$

Proof. Following the same steps that lead to (3.20) and (3.24), and applying Young's inequality, yields

$$\begin{aligned}
\beta_{\min} \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} &\leq \frac{1}{4\varepsilon_2} \Psi^2_{\text{Lip}} \, \|\mathcal{S}(\eta^1 - \eta^2)\|^2_{L^\infty(\Omega_1)} \, C_{\mathcal{T}}^4 \, C_{\mathbf{f}}^2 \\
&+ \varepsilon_2 \, \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} + \frac{1}{4\varepsilon_1} \beta^2_{\text{Lip}} \, \|\mathcal{S}(\eta^1 - \eta^2)\|^2_{L^\infty(\Omega_1)} \, C_{\mathbf{f}}^2 + \varepsilon_1 \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} \\
&+ \varepsilon_3 \|\ell^2 - \ell^1\|^2_{\mathbf{X}'} + \frac{1}{4\varepsilon_3} \|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}}.
\end{aligned} \tag{3.26}$$

Setting $\varepsilon_1 = \frac{\beta_{\min}}{2}$, $\varepsilon_2 = 1/2$, $\varepsilon_3 = C_\beta$ in (3.26) and replacing the $L^\infty(\Omega)$ norm of the smoothed variables with the $L^2(\Omega_1)$ norm of the original variables, we obtain

$$\begin{aligned}
\beta_{\min} \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2} \leq C_{\text{Lip}} \, \|\eta^2 - \eta^1\|^2_{L^2(\Omega_1)}(t) + 2C_\beta \, \|\ell^2 - \ell^1\|^2_{\mathbf{X}'} \\
+ \frac{C_\beta^{-1}}{2} \|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}}.
\end{aligned} \tag{3.27}$$

From (2.1) and (3.11) we have

$$C_\beta^{-1} \, \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}} \leq \beta_{\min} \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_1} + \|\mathbf{u}^2 - \mathbf{u}^1\|^2_{\mathbf{X}_2},$$

which together with (3.27) leads to (3.25). □

3.2 Unique solvability of (3.4)

We proceed to define two operators. One outputs the Darcy velocity as a function of the porosity η and the other describes the deposition function in terms of η.

Definition 3.1. *Let $\mathcal{P} : L^2(\Omega_1) \times (0,T) \to \mathbf{V}$ be given by*

$$\mathcal{P}(\eta, t) = \mathbf{u}(t),$$

where $\mathbf{u}(t)$ is defined through the solution pair $(\eta, \mathbf{u})$ of the problem introduced in (3.7)*, and define $\mathcal{F} : L^2(\Omega_1) \times (0,T) \to L^2(\Omega_1)$ by*

$$\mathcal{F}(\eta, t) = g\left(|\mathcal{P}(\eta, t)|\right) h(\eta).$$

Remark 3.1. *The time dependency of $\mathcal{P}$ is due the forcing term $\mathbf{f}(t)$.*

The following properties of $\mathcal{P}$ and $\mathcal{F}$ are used in the main result of this section.

Lemma 3.5. *The operator $\mathcal{P}(\,\cdot\,, t)$ is Lipschitz continuous for every $t \in (0,T)$, and $\mathcal{P}(\eta, \,\cdot\,)$ is continuous for every $\eta \in L^2(\Omega_1)$.*

Proof. The Lipschitz continuity of $\mathcal{P}(\,\cdot\,, t)$ is a direct consequence of Lemma 3.3. To establish the continuity of $\mathcal{P}(\eta, \,\cdot\,)$, let $t \in (0,T)$, $\varepsilon > 0$, and $\eta \in L^2(\Omega_1)$ be given. With reference to Lemma 3.4, define the linear functionals $\ell^1(\mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_\Omega$ and $\ell^2(\mathbf{v}) = (\mathbf{f}(t+h), \mathbf{v})_\Omega$ for some $h \in \mathbb{R}$, and let $\mathbf{u}^1 = \mathcal{P}(\eta, t)$, $\mathbf{u}^2 = \mathcal{P}(\eta, t+h)$ be the corresponding solutions, respectively. Then, by Lemma 3.4,

$$\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{X}} \le C_\beta \,\|\mathbf{f}(t) - \mathbf{f}(t+h)\|. \tag{3.28}$$

Owing to assumption $\boldsymbol{A4}$, we can find an open ball $B \subset \mathbb{R}$ centered at zero of radius $\delta > 0$ such that for all $h \in B$,

$$\|\mathbf{f}(t) - \mathbf{f}(t+h)\| \le \varepsilon. \tag{3.29}$$

Hence, combining (3.28) and (3.29) the result follows. □

Lemma 3.6. *The operator $\mathcal{F}(\,\cdot\,, t)$ is Lipschitz continuous for every $t \in \mathbb{R}^+$, and $\mathcal{F}(\eta, \,\cdot\,)$ is continuous for every $\eta \in L^2(\Omega_1)$.*

Proof. In view that the composition of Lipschitz continuous functions is Lipschitz and the product of bounded Lipschitz continuous functions is Lipschitz, assumptions $\boldsymbol{A1}$ and $\boldsymbol{A2}$ together with Lemma 3.5 imply that $\mathcal{F}(\,\cdot\,, t)$ is Lipschitz continuous. Now note that owing to assumption $\boldsymbol{A2}$ and Lemma 3.5, the function $g\left(|\mathcal{P}(\eta, \cdot)|\right)$ is continuous. Hence, $\mathcal{F}(\eta, \,\cdot\,)$ is continuous. □

Remark 3.2. *In view of $\boldsymbol{A1}$, the operator $\mathcal{F}(\cdot, \cdot)$ is uniformly bounded in $L^2(\Omega_1) \times (0,T)$ by the constant $C_\mathcal{F} = g_{\max}\, h_{\max}$. Thus,*

$$\|\mathcal{F}(\cdot, \cdot)\|_{L^2(\Omega_1)} \le C_\mathcal{F}\, |\Omega_1|^{1/2}\,.$$

Remark 3.3. *With an additional assumption on $\mathbf{f}$, we can strengthen Lemma 3.5 to obtain Lipschitz continuity with respect to the variable t and, additionally, Lipschitz continuity on the whole domain $L^2(\Omega_1) \times (0,T)$. This idea is explored in Section 4.*

Before introducing the main theorem of this section, we restate the Picard-Lindelöf theorem.

Theorem 3.1 ([17], Theorem I.3.1)**.** *Let I denote a domain in $\mathbb{R}$ containing the point t_0, Y a Banach space and $f : Y \times \mathbb{R} \to Y$. Suppose that f is locally Lipschitz continuous in its first variable and continuous in its second variable. Then, there exists $\varepsilon > 0$ such that the initial value problem*

$$u' = f(u,t),$$
$$u(t_0) = u_0,$$

has a unique solution in $C^0(t_0 - \varepsilon, t_0 + \varepsilon; Y)$.

Theorem 3.2. *Under assumptions **A1**-**A4** and **S1**-**S2**, there exists a unique solution $\mathbf{u} \in L^2(0,T;\mathbf{V})$, $p \in L^2(0,T;Q)$, $\lambda \in L^2(0,T;\Lambda)$ and $\eta \in H^1(0,T;\, L^2(\Omega_1))$ satisfying* (3.1)-(3.4) *for a.e. $t \in (0,T)$.*

Proof. First, we focus on computing the porosity. In view of (3.4) and using Definition 3.1, we consider the problem: Find $\eta \in C^0(0,T)$ such that

$$\frac{\partial \eta}{\partial t} = -g(|\mathcal{P}(\eta,t)|)\, h(\eta) = -\mathcal{F}(\eta,t)\ \forall t \in (0,T). \tag{3.30}$$

Let $t \in (0,T)$ be given. Owing to Theorem 3.1, Lemma 3.5 and Lemma 3.6, there exists $\varepsilon > 0$ and an interval $(t-\varepsilon, t+\varepsilon)$ where the existence of a unique η is guaranteed. From Lemma 3.3 and Remark 3.2, the Lipschitz constant C_{Lip} and the bound for $\|\mathcal{F}(\,\cdot\,,\,\cdot\,)\|_{L^2(\Omega_1)}$ are independent of η_0 and t. Hence, one can extend the solution to the whole interval $(0,T)$. Next, we use η in Lemma 3.2 to obtain the velocity $\mathbf{u}$. Finally, owing to the inf-sup condition in Lemma 3.1, the existence and uniqueness of p and λ follow. □

A simple consequence of Theorem 3.2 is the next corollary, which upgrades the regularity of η.

Corollary 3.3. *The porosity function η given by Theorem 3.2 is Lipschitz continuous on $(0,T)$.*

Proof. In view of Lemma 3.5 and Lemma 3.6, $\mathcal{F}(\,\cdot\,,\,\cdot\,)$ is continuous. Furthermore, owing to Theorem 3.2, the obtained solution η is continuous on $(0,T)$. Thus, from (3.30), it follows that $\dfrac{\partial \eta}{\partial t}$ is continuous. Consequently η is $C^1(0,T)$ and therefore Lipschitz continuous on the same interval. □

Notation 3.1. *We denote the Lipschitz constant of η in Corollary 3.3 by η_{Lip}.*

In the next section we aim to extend the regularity of η and $\mathbf{u}$ in Theorem 3.2 by upgrading the regularity of $\mathbf{f}$.

4 Additional regularity of the solution

The key ingredient in the subsequent derivations is to assume that the function $\mathbf{f}$ in (2.2) and (2.4) is Lipschitz continuous. At the end of this section we conclude that $\mathbf{u}$ is Lipschitz continuous and $\eta \in H^2(0,T;L^2(\Omega_1))$.

Notation 4.1. *We denote the Lipschitz constant of $\mathbf{f}$ by $\mathbf{f}_{\mathrm{Lip}}$.*

Lemma 4.1. *Assume $\mathbf{f} : (0,T) \to L^2(\Omega)$ is Lipschitz continuous. Then, the operator $\mathcal{P}(\,\cdot\,,\,\cdot\,)$ is Lipschitz continuous on $L^2(\Omega_1) \times (0,T)$.*

Proof. Let $\eta^1, \eta^2 \in L^2(\Omega_1)$ be given. Using similar notation to that introduced in the proof of Lemma 3.5, and owing to Corollary 3.2, we obtain the bound

$$\|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}}^2 \leq 2C_{\text{Lip}} \|\eta^2 - \eta^1\|_{L^2(\Omega_1)}^2 + 4C_\beta \|\mathbf{f}(t) - \mathbf{f}(t+h)\|^2, \tag{4.1}$$

where $\mathbf{u}^1 = \mathcal{P}(\eta^1, t)$, $\mathbf{u}^2 = \mathcal{P}(\eta^2, t+h)$. Define $t_1 = t$ and $t_2 = t + h$. Making use of the Lipschitz continuity of $\mathbf{f}$ in (4.1), yields

$$\|\mathcal{P}(\eta^2, t_2) - \mathcal{P}(\eta^1, t_1)\|_{\mathbf{X}}^2 = \|\mathbf{u}^2 - \mathbf{u}^1\|_{\mathbf{X}}^2 \leq 2C_{\text{Lip}} \|\eta^2 - \eta^1\|_{L^2(\Omega_1)}^2 + 4C_\beta \, \mathbf{f}_{\text{Lip}}^2 \, |t_2 - t_1|^2 \, |\Omega| \,. \tag{4.2}$$

Hence, the Lipschitz continuity of $\mathcal{P}(\,\cdot\,,\,\cdot\,)$ follows from (4.2). □

Similar to Lemma 4.1, we can upgrade the regularity of $\mathcal{F}$ by means of the additional regularity of $\mathbf{f}$.

Corollary 4.1. *Assume* $\mathbf{f} : (0,T) \to L^2(\Omega)$ *is Lipschitz continuous. Then, the operator* $\mathcal{F}(\,\cdot\,,\,\cdot\,)$ *is Lipschitz continuous on* $L^2(\Omega_1) \times (0,T)$.

Proof. This is a direct consequence of Lemma 4.1 and the arguments given in the proof of Lemma 3.6. □

To close this section, we prove two results that improve the regularity of $\mathbf{u}$ and $\dfrac{\partial \eta}{\partial t}$.

Corollary 4.2. *Assume* $\mathbf{f} : (0,T) \to L^2(\Omega)$ *is Lipschitz continuous. Then, the velocity* $\mathbf{u}$ *given by Theorem 3.2 is Lipschitz continuous on* $(0,T)$.

Proof. Using the same notation introduced in Lemma 4.1, let $\mathbf{u}^1 = \mathbf{u}(t_1)$, $\mathbf{u}^2 = \mathbf{u}(t_2)$ and $\eta^1 = \eta(t_1)$, $\eta^2 = \eta(t_2)$. Then, owing to the Lipschitz continuity of $\mathbf{f}$ and (4.2), we obtain

$$\|\mathbf{u}(t_2) - \mathbf{u}(t_1)\|_{\mathbf{X}}^2 \leq 2C_{\text{Lip}} \|\eta(t_2) - \eta(t_2)\|_{L^2(\Omega_1)}^2 + 4C_\beta \, \mathbf{f}_{\text{Lip}}^2 \, |t_2 - t_1|^2 \, |\Omega| \,. \tag{4.3}$$

Finally, a direct application of Corollary 3.3 in (4.3) yields

$$\|\mathbf{u}(t_2) - \mathbf{u}(t_1)\|_{\mathbf{X}}^2 \leq 2C_{\text{Lip}} \, \eta_{\text{Lip}}^2 \, |t_2 - t_1|^2 \, |\Omega_1| + 4C_\beta \, \mathbf{f}_{\text{Lip}}^2 \, |t_2 - t_1|^2 \, |\Omega| \,,$$

proving the claim. □

Corollary 4.3. *Assume* $\mathbf{f} : (0,T) \to L^2(\Omega)$ *is Lipschitz continuous. Then, the function* $\dfrac{\partial \eta}{\partial t}$ *given in* (3.30) *is Lipschitz continuous on* $(0,T)$. *In particular* $\dfrac{\partial^2 \eta}{\partial t^2} \in L^\infty(0,T; L^2(\Omega_1))$.

Proof. Combining Corollary 4.1 and Corollary 3.3, it follows that the right hand side of (3.30) is Lipschitz continuous. Hence, $\dfrac{\partial \eta}{\partial t}$ is Lipschitz and consequently, by Rademacher's theorem, differentiable almost everywhere. Moreover, the Lipschitz continuity of $\dfrac{\partial \eta}{\partial t} : (0,T) \to L^2(\Omega_1)$ implies that $\|\dfrac{\partial^2 \eta}{\partial t^2}\|_{L^2(\Omega_1)}(t)$ is bounded in $(0,T)$. Hence $\dfrac{\partial^2 \eta}{\partial t^2} \in L^\infty(0,T; L^2(\Omega_1))$, concluding the proof. □

We summarize the last two propositions in the following remark.

Remark 4.1. *Under the additional regularity assumption*

A5*: The forcing term* $\mathbf{f} : (0,T) \to L^2(\Omega)$ *is Lipschitz continuous,*

it follows that $\mathbf{u} : (0,T) \to \mathbf{X}$ *is Lipschitz continuous and* $\dfrac{\partial^2 \eta}{\partial t^2} \in L^\infty(0,T;\, L^2(\Omega_1))$. *In particular* $\eta \in H^2(0,T; L^2(\Omega_1))$.

The next section further extends the properties of η and shows that η is a nonnegative bounded function. This is relevant in view that, physically, the porosity is always between zero and one.

5 Nonnegativity of the porosity

Having established the existence and uniqueness of a solution to (3.1)-(3.4), in this section we show that the porosity function η remains nonnegative a.e. in Ω_1. A brief outline of how this is achieved follows. First, we introduce some notation and discretize in time the deposition equation (3.30). Subsequently, we show that the discretized problem is well-posed and exhibit the boundedness and nonnegativity of the discrete porosity function. Then, we proceed to construct sequences of functions that approximate the continuous porosity and show that they converge to a (weak) common limit. Finally, we prove that under assumption ***A*5**, the discrete porosity converges to the continuous porosity as the time step goes to zero.

First, we discretize the interval $[0, T]$ into $M+1$ uniformly spaced times $t_k = k\,\Delta t$, $k = 0, 1, \ldots, M$, where $\Delta t = T/M$ and consider the following problem.

Lemma 5.1. *Define $\mathbf{u}_k = \mathbf{u}(t_k) \in \mathbf{X}$, where $\mathbf{u}$ is the solution obtained in Theorem 3.2. Then, the problem: Given $\eta_0 \in L^2(\Omega_1)$, find $\eta_k \in L^2(\Omega_1)$, for $k = 1, \ldots, M$ such that for all $\xi \in L^2(\Omega_1)$*

$$\left(\frac{\eta_k - \eta_{k-1}}{\Delta t}, \xi\right)_{\Omega_1} + \Big(g(|\mathbf{u}_k|)h(\eta_k), \xi\Big)_{\Omega_1} = 0 \tag{5.1}$$

has a unique solution, provided $\Delta t < g_{\max}\, h_{\mathrm{Lip}}$.

Proof. Assume $\eta_0, \ldots, \eta_{k-1}$ have already been computed. Define the operator $A : L^2(\Omega_1) \to L^2(\Omega_1)$ by $y = Ax$, where y satisfies

$$\left(\frac{y - \eta_{k-1}}{\Delta t}, \xi\right)_{\Omega_1} + \Big(g(|\mathbf{u}_k|)h(x), \xi\Big)_{\Omega_1} = 0 \quad \forall \xi \in L^2(\Omega_1). \tag{5.2}$$

Let $x_1, x_2 \in L^2(\Omega_1)$ and define $y_1 = Ax_1$, $y_2 = Ax_2$. Then, from (5.2), it follows that

$$(y_1 - y_2, \xi)_{\Omega_1} = -\Delta t \Big(g(|\mathbf{u}_k|)\ (h(x_1) - h(x_2))\,, \xi\Big)_{\Omega_1} \quad \forall \xi \in L^2(\Omega_1). \tag{5.3}$$

Thus, setting $\xi = y_1 - y_2$ in (5.3), using Cauchy-Schwarz and assumptions ***A*1** and ***A*2**, yields

$$\|y_1 - y_2\|_{L^2(\Omega_1)} \leq \Delta t\, g_{\max}\, h_{\mathrm{Lip}}\, \|x_1 - x_2\|_{L^2(\Omega_1)},$$

implying that, as $\Delta t < g_{\max}\, h_{\mathrm{Lip}}$, A is a contraction in $L^2(\Omega_1)$. Consequently, owing to Banach's fixed point theorem, it follows that A has a unique fixed point, proving the existence of a unique solution to (5.1). The result follows by induction. □

Definition 5.1. *We use the notation $\boldsymbol{\eta}^M$ to denote the tuple $(\eta_0, \eta_1, \ldots, \eta_M)$, where the η_k, $k = 1, \ldots, M$ are defined through Lemma 5.1.*

Lemma 5.2. *Let $m \in \mathbb{Z}^+$ be given with $m \leq M$. The solution η_k given in Lemma 5.1 satisfies the estimate*

$$\|\eta_m\|^2_{L^2(\Omega_1)} + \sum_{k=1}^{m} \|\eta_k - \eta_{k-1}\|^2_{L^2(\Omega_1)} \leq \exp\left(4T\, g_{\max}\, h_{\max}\, |\Omega_1|\right) \|\eta_0\|^2_{L^2(\Omega_1)}, \tag{5.4}$$

provided

$$\Delta t < \frac{1}{4\, g_{\max}\, h_{\max}\, |\Omega_1|}.$$

Proof. Let $\xi = \eta_k$ in (5.1) and use assumption $\boldsymbol{A1}$, to obtain

$$\|\eta_k\|^2_{L^2(\Omega_1)} - \|\eta_{k-1}\|^2_{L^2(\Omega_1)} + \|\eta_k - \eta_{k-1}\|^2_{L^2(\Omega_1)} \leq 2\Delta t\, g_{\max}\, h_{\max}\, |\Omega_1|\, \|\eta_k\|_{L^2(\Omega_1)}. \tag{5.5}$$

Summing (5.5) from $k = 1$ to $k = m$, yields

$$\|\eta_m\|^2_{L^2(\Omega_1)} + \sum_{k=1}^{m} \|\eta_k - \eta_{k-1}\|^2_{L^2(\Omega_1)} \leq \|\eta_0\|^2_{L^2(\Omega_1)} + C\,\Delta t \sum_{k=1}^{m} \|\eta_k\|_{L^2(\Omega_1)}, \tag{5.6}$$

where $C = 2\, g_{\max}\, h_{\max}\, |\Omega_1|$. Finally, from a discrete version of Gronwall's lemma (see [24] pg. 167), we obtain

$$\|\eta_m\|^2_{L^2(\Omega_1)} + \sum_{k=1}^{m} \|\eta_k - \eta_{k-1}\|^2_{L^2(\Omega_1)} \leq \exp\left(\frac{m\,C\,\Delta t}{1 - C\,\Delta t}\right) \|\eta_0\|^2_{L^2(\Omega_1)}.$$

Bounding $m\,\Delta t$ with $M\,\Delta t = T$, and observing that $1 - C\,\Delta t < 1/2$, estimate (5.4) now follows. □

Lemma 5.3. *Let $\eta_{k-1} \in L^2(\Omega_1)$ be given with $\eta_{k-1} \geq 0$ a.e. in Ω_1. Then, the solution η_k given in Lemma 5.1 is nonnegative a.e. in Ω_1.*

Proof. Define $\eta_k^- = \max\{0, -\eta_k\}$, i.e., the negative part of η_k. Let $\xi = -\eta_k^-$ in (5.1) and define $h(\,\cdot\,)$ to be zero for negative arguments. Thus, we obtain

$$\|\eta_k^-\|^2_{L^2(\Omega_1)} = -\left(\eta_{k-1}, \eta_k^-\right)_{\Omega_1} + \Delta t\left(g(|\mathbf{u}_k|)h(\eta_k), \eta_k^-\right)_{\Omega_1} = -\left(\eta_{k-1}, \eta_k^-\right)_{\Omega_1} \leq 0.$$

Consequently, η_k^- is zero a.e. in Ω_1, implying the nonnegativity of η_k. □

Lemma 5.4. *Let $\eta_{k-1} \in L^2(\Omega_1)$ be given. Then, the solution η_k given in Lemma 5.1 satisfies $\eta_k \leq \eta_{k-1}$ a.e. in Ω_1. Equivalently stated, η_k is monotonically decreasing.*

Proof. Define $(\eta_k - \eta_{k-1})^+ = \max\{0, \eta_k - \eta_{k-1}\}$. Setting $\xi = (\eta_k - \eta_{k-1})^+$ in (5.1), yields

$$\begin{aligned}\|(\eta_k - \eta_{k-1})^+\|^2_{L^2(\Omega_1)} &= \left(\eta_k - \eta_{k-1}, (\eta_k - \eta_{k-1})^+\right)_{\Omega_1} \\ &= -\Delta t\left(g(|\mathbf{u}_k|)h(\eta_k), (\eta_k - \eta_{k-1})^+\right)_{\Omega_1} \leq 0.\end{aligned}$$

Therefore, $(\eta_k - \eta_{k-1})^+$ is zero a.e. in Ω_1, implying that $\eta_k \leq \eta_{k-1}$ a.e. in Ω_1. □

Corollary 5.1. *Let $\eta_{k-1} \in L^2(\Omega_1)$ be given with $0 \leq \eta_{k-1} \leq \eta_0$ a.e. in Ω_1. Then, the solution η_k given in Lemma 5.1 is also bounded above by η_0 a.e. in Ω_1.*

Proof. This is a direct consequence of Lemma 5.4. □

Remark 5.1. *In view of assumption $\boldsymbol{A3}$, Lemma 5.3 and Corollary 5.1, it follows that $\eta_k \in L^\infty(\Omega_1)$ for $k = 0, \ldots, M$.*

Now that we have established some relevant properties of the functions in the vector $\boldsymbol{\eta}^M$, it remains to exhibit that the continuous solution η possesses the same properties. We achieve this by constructing a sequence of interpolants using $\boldsymbol{\eta}^M$ and showing that they converge to η.

Definition 5.2. *Let $\Delta t > 0$ and $\eta_0 \in L^2(\Omega_1)$ be given. Then, the piecewise constant and piecewise linear interpolants $\eta^*_{\Delta t}, \eta^{**}_{\Delta t} : [0,T] \to L^2(\Omega_1)$ are given by*

$$\eta^*_{\Delta t}(t) = \begin{cases} \eta_0, & t = 0, \\ \eta_k, & (k-1)\,\Delta t < t \le k\,\Delta t \end{cases}$$

$$\eta^{**}_{\Delta t}(t) = \left(\frac{\eta_k - \eta_{k-1}}{\Delta t}\right)(t - (k-1)\Delta t) + \eta_{k-1}, \quad (k-1)\,\Delta t \le t \le k\,\Delta t,$$

for $k = 1, \ldots, M$. Similarly, we define $\mathbf{u}^*_{\Delta t} : [0,T] \to \mathbf{X}$ by

$$\mathbf{u}^*_{\Delta t}(t) = \begin{cases} \mathbf{u}(0), & t = 0, \\ \mathbf{u}(t_k), & (k-1)\,\Delta t < t \le k\,\Delta t \end{cases}$$

for $k = 1, \ldots, M$, where $\mathbf{u}$ is the solution found in Theorem 3.2.

Remark 5.2. *In view of Definition 5.2 and Lemma 5.1, $\eta^{**}_{\Delta t}$, $\eta^*_{\Delta t}$ and $\mathbf{u}^*_{\Delta t}$ satisfy for all $\xi \in L^2(\Omega_1)$ and for all $t \in (0,T)$*

$$\left(\frac{\partial \eta^{**}_{\Delta t}}{\partial t}, \xi\right)_{\Omega_1} + \Big(g(|\mathbf{u}^*_{\Delta t}|)h(\eta^*_{\Delta t}), \xi\Big)_{\Omega_1} = 0. \tag{5.7}$$

The following lemma establishes norm estimates that we use to extract weakly convergent subsequences.

Lemma 5.5. *There exists a positive constant C, independent of Δt, such that:*

$$\|\frac{\partial \eta^{**}_{\Delta t}}{\partial t}\|_{L^2(0,T;L^2(\Omega_1))} \le C, \tag{5.8}$$

$$\|\eta^*_{\Delta t}\|_{L^\infty((0,T)\times\Omega_1)}, \|\eta^{**}_{\Delta t}\|_{L^\infty((0,T)\times\Omega_1)} \le C, \tag{5.9}$$

$$\|\eta^*_{\Delta t} - \eta^{**}_{\Delta t}\|_{L^2(0,T;L^2(\Omega_1))} \le C\sqrt{\Delta t}, \tag{5.10}$$

$$\|\eta^{**}_{\Delta t}\|_{H^1(0,T;L^2(\Omega_1))} \le C. \tag{5.11}$$

Proof. First note that

$$\begin{aligned} \|\frac{\partial \eta^{**}_{\Delta t}}{\partial t}\|^2_{L^2(0,T;L^2(\Omega_1))} &= \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \|\frac{\partial \eta^{**}_{\Delta t}}{\partial t}(s)\|^2_{L^2(\Omega_1)}\, ds \\ &= \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \|\frac{\eta_k - \eta_{k-1}}{\Delta t}\|^2_{L^2(\Omega_1)}\, ds = \frac{1}{\Delta t}\sum_{k=1}^{M} \|\eta_k - \eta_{k-1}\|^2_{L^2(\Omega_1)}. \end{aligned} \tag{5.12}$$

Setting $\xi = \eta_k - \eta_{k-1}$ in (5.1), we obtain

$$\frac{1}{\Delta t}\|\eta_k - \eta_{k-1}\|_{L^2(\Omega_1)} \le \|g(|\mathbf{u}_k|)h(\eta_k)\|_{L^2(\Omega_1)} \le |\Omega_1|^{1/2}\, g_{\max}\, h_{\max}. \tag{5.13}$$

Thus, squaring (5.13), multiplying by Δt and summing from $k = 1$ to $k = M$, yields

$$\frac{1}{\Delta t}\sum_{k=1}^{M} \|\eta_k - \eta_{k-1}\|^2_{L^2(\Omega_1)} \le |\Omega_1|\; g^2_{\max}\, h^2_{\max}\, T. \tag{5.14}$$

Combining (5.12) and (5.14), (5.8) follows. Estimate (5.9) is a direct consequence of Definition 5.2 and Remark 5.1. Also owing to Definition 5.2 and Lemma 5.2,

$$\begin{aligned}\|\eta^*_{\Delta t}-\eta^{**}_{\Delta t}\|^2_{L^2(0,T;L^2(\Omega_1))} &= \sum_{k=1}^{M}\int_{(k-1)\,\Delta t}^{k\,\Delta t}\|\eta^*_{\Delta t}(t)-\eta^{**}_{\Delta t}(t)\|^2_{L^2(\Omega_1)}\,dt\\ &= \sum_{k=1}^{M}\|\eta_k-\eta_{k-1}\|^2_{L^2(\Omega_1)}\int_{(k-1)\,\Delta t}^{k\,\Delta t}\left(\frac{t-k\Delta t}{\Delta t}\right)^2 dt \le \frac{\Delta t}{3}C,\end{aligned}$$

where C is given by the right hand side of (5.4). Hence (5.10) holds. Finally, (5.11) is a direct consequence of (5.8) and (5.9). □

With the aim of letting $\Delta t \to 0$, we consider the sequence of functions $\{\eta^{**}_n\}_{n=1}^{\infty}$ and $\{\eta^*_n\}_{n=1}^{\infty}$, where $\eta^{**}_n = \eta^{**}_{\Delta t}$, $\eta^*_n = \eta^*_{\Delta t}$ and $\Delta t = 1/n$.

Lemma 5.6. *Assume $\eta^*_n \to \eta^1$ weak-$*$ in $L^\infty((0,T)\times\Omega_1)$ and $\eta^{**}_n \to \eta^2$ weak-$*$ in $L^\infty((0,T)\times\Omega_1)$. Then, $\eta^1=\eta^2$ almost everywhere.*

Proof. Let $\phi \in L^2(0,T;L^2(\Omega_1)) \subset L^1((0,T)\times\Omega_1)$. Then,

$$\langle \eta^1-\eta^2,\phi\rangle = \langle \eta^1-\eta^*_n,\phi\rangle + \langle \eta^*_n-\eta^{**}_n,\phi\rangle + \langle \eta^{**}_n-\eta^2,\phi\rangle \tag{5.15}$$

Taking the limit $n\to\infty$ in (5.15), using the weak-$*$ convergence of η^*_n, η^{**}_n, and Cauchy-Schwarz, we obtain

$$\left|\langle \eta^1-\eta^2,\phi\rangle\right| \le \lim_{n\to\infty}\|\eta^*_n-\eta^{**}_n\|_{L^2(0,T;L^2(\Omega_1))}\,\|\phi\|_{L^2(0,T;L^2(\Omega_1))}. \tag{5.16}$$

Owing to (5.10) and (5.16), it follows that $\langle \eta^1-\eta^2,\phi\rangle = 0$ for all $\phi\in L^2(0,T;L^2(\Omega_1))$. Hence $\|\eta^1-\eta^2\|_{L^2(0,T;L^2(\Omega_1))}=0$, implying that $\eta^1=\eta^2$ almost everywhere. □

The following lemma is used in the analysis below.

Lemma 5.7. *Let X and Y be normed vector spaces with X' and Y' its corresponding duals. Let $T: X\to Y$ be a bounded linear operator and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_n \rightharpoonup x$. Then, $T(x_n)\rightharpoonup T(x)$.*

Proof. Let $y^*\in Y'$. Define the bounded linear functional $f\in X'$ by $f = y^*\circ T$. Owing to the fact that $x_n\rightharpoonup x$, it follows that $f(x_n)\to f(x)$, i.e., $y^*(T(x_n))\to y^*(T(x))$. Observing that y^* is arbitrary, the proposition follows. □

We are now in position to show that η^*_n and η^{**}_n weak-$*$ converge to a common limit.

Lemma 5.8. *There exists a subsequence of $\left\{\boldsymbol{\eta}^M\right\}_{M\ge 1}$ and a function $\eta^*\in L^\infty((0,T)\times\Omega_1)$, such that*

$$\eta^*_n \to \eta^* \text{ weak-}* \text{ in } L^\infty((0,T)\times\Omega_1), \tag{5.17}$$
$$\eta^{**}_n \to \eta^* \text{ weak-}* \text{ in } L^\infty((0,T)\times\Omega_1), \tag{5.18}$$
$$\eta^{**}_n \to \eta^* \text{ weakly in } H^1(0,T;L^2(\Omega_1)), \tag{5.19}$$
$$\frac{\partial \eta^{**}_n}{\partial t} \to \frac{\partial \eta^*}{\partial t} \text{ weakly in } L^2(0,T;L^2(\Omega_1)). \tag{5.20}$$

Proof. For the sake of simplicity, all the subsequences that we derive in the analysis are labeled as the original sequences. In view of (5.9), it follows from the Banach-Alaoglu theorem the existence of a subsequence of $\{\boldsymbol{\eta}^M\}$ and a function $\eta^* \in L^\infty((0,T)\times\Omega_1)$ such that (5.17) is satisfied. By the same token, there exists a subsequence of η_n^{**} and a function $\widetilde{\eta} \in L^\infty((0,T)\times\Omega_1)$ such that $\eta_n^{**} \to \widetilde{\eta}$ weak-$*$ in $L^\infty((0,T)\times\Omega_1)$. From Lemma 5.6 it follows that $\widetilde{\eta} = \eta^*$, establishing (5.18). Now observe that (5.11) and the Banach-Alaoglu theorem yield a further subsequence and a function $\widehat{\eta} \in H^1(0,T;L^2(\Omega_1))$, such that

$$\eta_n^{**} \to \widehat{\eta} \text{ weakly in } H^1(0,T;L^2(\Omega_1)). \tag{5.21}$$

From the continuous embedding $H^1(0,T) \hookrightarrow L^\infty(0,T)$, it follows that

$$\eta_n^{**} \to \widehat{\eta} \text{ weak-}* \text{ in } L^\infty(0,T;L^2(\Omega_1)). \tag{5.22}$$

Moreover, owing to (5.18) and the fact that $L^\infty((0,T)\times\Omega_1) \subset L^\infty(0,T;L^2(\Omega_1))$, we obtain

$$\eta_n^{**} \to \eta^* \text{ weak-}* \text{ in } L^\infty(0,T;L^2(\Omega_1)). \tag{5.23}$$

Thus, in view of (5.22), (5.23) and the uniqueness of weak-$*$ limits, it follows that $\widehat{\eta} = \eta^*$. This establishes (5.19). Finally, in view of Lemma 5.7 and the fact that the time derivative is a bounded linear operator from $H^1(0,T)$ to $L^2(0,T)$, expression (5.20) follows from (5.19). □

The following lemma gives an error estimate in the L^2 norm for a first order approximation of the time derivative. We use this result in the next proposition, where we establish that the L^2 difference between the continuous function η and its discrete analog η_k is proportional to Δt. Its proof is easily established using Taylor's theorem.

Lemma 5.9. *Let $f \in H^2(0,T;L^2(\Omega))$ and let $\Delta t > 0$ be given. Then,*

$$\|\frac{\partial f}{\partial t}(t_k) - \frac{f(t_k)-f(t_{k-1})}{\Delta t}\|^2_{L^2(\Omega)} \le \frac{\Delta t}{3}\int_{t_{k-1}}^{t_k} \|\frac{\partial^2 f}{\partial t^2}(s)\|^2_{L^2(\Omega)}\, ds,$$

for $(t_{k-1},t_k) \subset (0,T)$.

Lemma 5.10. *Assume $\dfrac{\partial^2\eta}{\partial t^2} \in L^2(0,T;L^2(\Omega_1))$. Let $M,n \in \mathbb{Z}^+$ and $\Delta t \in \mathbb{R}^+$ be such that $\Delta t = T/M = 1/n$ and $\Delta t < \frac{1}{2}(1+2\,g_{\max}\,h_{\text{Lip}})$. Define $e_k = \eta(t_k) - \eta_k$ for $k = 0,\dots,M$, where $\eta(t)$ is the solution found in Theorem 3.2 and η_k is an element of $\boldsymbol{\eta}^M$ (see Definition 5.1). Then,*

$$\|e_k\|_{L^2(\Omega_1)} \le \mathfrak{C}_1\,\Delta t, \text{ where} \tag{5.24}$$

$$\mathfrak{C}_1^2 = \exp\Big(2\,T\,(1+2\,g_{\max}\,h_{\text{Lip}})\Big)\left(\frac{1}{3}\|\frac{\partial^2\eta}{\partial t^2}\|^2_{L^2(0,T;L^2(\Omega_1))}\right),$$

and

$$\|\eta - \eta_n^*\|_{L^2(0,T;\Omega_1)} \le \mathfrak{C}_2\,\Delta t \quad\text{for}\quad \mathfrak{C}_2^2 = 2\|\frac{\partial\eta}{\partial t}\|^2_{L^2(0,T;\Omega_1)} + 2T\,\mathfrak{C}_1^2. \tag{5.25}$$

Proof. From (3.4), we obtain

$$\begin{aligned}\left(\frac{\eta(t_k)-\eta(t_{k-1})}{\Delta t},\xi\right)_{\Omega_1} &+ \Big(g(|\mathbf{u}(t_k)|)\,h(\eta(t_k)),\xi\Big)_{\Omega_1}\\ &= \left(\frac{\eta(t_k)-\eta(t_{k-1})}{\Delta t} - \frac{\partial\eta}{\partial t}(t_k),\xi\right)_{\Omega_1}\end{aligned} \tag{5.26}$$

Subtracting (5.1) from (5.26), yields

$$\left(\frac{e_k - e_{k-1}}{\Delta t}, \xi\right)_{\Omega_1} + \Big(g(|\mathbf{u}(t_k)|)\,\big(h(\eta(t_k)) - h(\eta_k)\big), \xi\Big)_{\Omega_1} = \left(\frac{\eta(t_k) - \eta(t_{k-1})}{\Delta t} - \frac{\partial \eta}{\partial t}(t_k), \xi\right)_{\Omega_1}. \tag{5.27}$$

Setting $\xi = e_k$ in (5.27), and using assumptions $\boldsymbol{A1}$ and $\boldsymbol{A2}$, we obtain

$$\frac{1}{2\Delta t}\big(\|e_k\|^2_{L^2(\Omega_1)} - \|e_{k-1}\|^2_{L^2(\Omega_1)} + \|e_k - e_{k-1}\|^2_{L^2(\Omega_1)}\big) \leq g_{\max}\, h_{\mathrm{Lip}}\, \|e_k\|^2_{L^2(\Omega_1)} + \left(\frac{\eta(t_k) - \eta(t_{k-1})}{\Delta t} - \frac{\partial \eta}{\partial t}(t_k), e_k\right)_{\Omega_1}. \tag{5.28}$$

Applying Young's inequality and Lemma 5.9 to (5.28), yields

$$\|e_k\|^2_{L^2(\Omega_1)} - \|e_{k-1}\|^2_{L^2(\Omega_1)} + \|e_k - e_{k-1}\|^2_{L^2(\Omega_1)} \leq \Delta t\,(1 + 2\, g_{\max}\, h_{\mathrm{Lip}})\, \|e_k\|^2_{L^2(\Omega_1)} + \frac{(\Delta t)^2}{3}\int_{t_{k-1}}^{t_k} \|\frac{\partial^2 \eta}{\partial t^2}(s)\|^2_{L^2(\Omega_1)}\, dt. \tag{5.29}$$

Let $m \in \mathbb{Z}^+$, with $m \leq M$. Summing (5.29) from $k = 1$ to $k = m$, and using a discrete version of Gronwall's lemma (see [24] pg. 167), we obtain

$$\|e_m\|^2_{L^2(\Omega_1)} + \sum_{k=1}^{m} \|e_k - e_{k-1}\|^2_{L^2(\Omega_1)} \leq \exp\Big(2\,T\,(1 + 2\, g_{\max}\, h_{\mathrm{Lip}})\Big)\left(\frac{(\Delta t)^2}{3}\|\frac{\partial^2 \eta}{\partial t^2}\|^2_{L^2(0,T;L^2(\Omega_1))}\right). \tag{5.30}$$

Statement (5.24) follows from (5.30).
To prove (5.25), recall that η_n^* is a piecewise constant interpolant such that $\eta_n^*(s) = \eta_k$ for $s \in ((k-1)\Delta t, k\Delta t]$. Hence, owing to (5.24),

$$\begin{aligned}\int_0^T \|\eta - \eta_n^*\|^2_{L^2(\Omega_1)}(t)\, dt &= \sum_{k=1}^{M}\int_{(k-1)\Delta t}^{k\Delta t} \|\eta(t) - \eta_k\|^2_{L^2(\Omega_1)}\, dt \\ &\leq 2\sum_{k=1}^{M}\int_{(k-1)\Delta t}^{k\Delta t} \|\eta(t) - \eta(t_k)\|^2_{L^2(\Omega_1)} + \|\eta(t_k) - \eta_k\|^2_{L^2(\Omega_1)}\, dt \\ &\leq 2\sum_{k=1}^{M}\int_{(k-1)\Delta t}^{k\Delta t} \|\eta(t) - \eta(t_k)\|^2_{L^2(\Omega_1)} + \mathfrak{C}_1^2\,(\Delta t)^2\, dt. \end{aligned} \tag{5.31}$$

Now note that for $t \in ((k-1)\Delta t, k\Delta t]$ and Cauchy-Schwarz

$$\begin{aligned}\|\eta(t_k) - \eta(t)\|^2_{L^2(\Omega_1)} &= \int_{\Omega_1}\left(\int_t^{t_k} \frac{\partial \eta}{\partial t}(s)\, ds\right)^2 d\Omega_1 \\ &\leq \int_{\Omega_1}\left(\int_t^{t_k}\left(\frac{\partial \eta}{\partial t}(s)\right)^2 ds \int_t^{t_k} 1\, ds\right) d\Omega_1 \leq \Delta t\int_{t_{k-1}}^{t_k} \|\frac{\partial \eta}{\partial t}\|^2_{L^2(\Omega_1)}\, ds. \end{aligned} \tag{5.32}$$

Thus, substituting (5.32) in (5.31), we obtain

$$\int_0^T \|\eta - \eta_n^*\|_{L^2(\Omega_1)}^2(t)\,dt \le 2\sum_{k=1}^{M}\int_{(k-1)\Delta t}^{k\Delta t}\left(\Delta t\int_{t_{k-1}}^{t_k}\|\frac{\partial\eta}{\partial t}(s)\|_{L^2(\Omega_1)}^2\,ds + \mathfrak{C}_1^2\,(\Delta t)^2\right)dt$$
$$= 2(\Delta t)^2\int_0^T\|\frac{\partial\eta}{\partial t}(s)\|_{L^2(\Omega_1)}^2\,ds + 2T\,\mathfrak{C}_1^2\,(\Delta t)^2$$
$$= (\Delta t)^2\left(2\|\frac{\partial\eta}{\partial t}\|_{L^2(0,T;\Omega_1)}^2 + 2T\,\mathfrak{C}_1^2\right). \tag{5.33}$$

Statement (5.25) follows from (5.33). □

We now state the main result of this section.

Lemma 5.11. *Let assumption **A5** hold. Then, $\eta = \eta^*$, where η is the solution found in Theorem 3.2 and η^* is the weak limit introduced in Lemma 5.8.*

Proof. Remark 4.1 readily yields $\frac{\partial^2\eta}{\partial t^2} \in L^2(0,T;L^2(\Omega_1))$. Let $\phi \in L^2(0,T;\Omega_1) \subset L^1(0,T;\Omega_1)$. Then, owing to Cauchy-Schwarz,

$$\langle \eta - \eta^*, \phi\rangle = \langle \eta - \eta_n^*, \phi\rangle + \langle \eta_n^* - \eta^*, \phi\rangle$$
$$\le \|\eta - \eta_n^*\|_{L^2(0,T;\Omega_1)}\|\phi\|_{L^2(0,T;\Omega_1)} + \langle \eta_n^* - \eta^*, \phi\rangle\,. \tag{5.34}$$

Hence, in view of the second statement of Lemma 5.10, the weak-$*$ convergence of η_n^* to η^* (see (5.17)), and the fact that $1/n = \Delta t$, it follows from (5.34) that

$$\langle \eta - \eta^*, \phi\rangle \le \lim_{n\to\infty}\|\eta - \eta_n^*\|_{L^2(0,T;\Omega_1)}\|\phi\|_{L^2(0,T;\Omega_1)} + \lim_{n\to\infty}\langle \eta_n^* - \eta^*, \phi\rangle = 0. \tag{5.35}$$

We conclude that $\eta = \eta^*$ almost everywhere in $(0,T)\times\Omega_1$. □

We summarize the results of the last two sections in the next theorem.

Theorem 5.1. *Let the assumptions **A1**-**A5** and **S1**-**S2** hold. Let $\eta_0 \in L^2(\Omega_1)$ be given with η_0 nonnegative and bounded a.e. in Ω_1. Then, there exist a unique solution $\mathbf{u} \in H^1(0,T;\mathbf{X})$, $p \in H^1(0,T;Q)$, $\lambda \in H^1(0,T;\Lambda)$ and $\eta \in H^2(0,T;L^2(\Omega_1))$, that satisfy the system (3.1)-(3.4) for a.e. $t \in (0,T)$ with $\eta(0) = \eta_0$. Moreover, $0 \le \eta(\mathbf{x},t) \le \eta_0(\mathbf{x})$ for a.e. $(t,\mathbf{x}) \in (0,T)\times\Omega_1$.*

6 Finite element approximation

In this section we investigate the finite element approximation to (3.1)-(3.4), for the specific choice $h(\eta) = \eta$. This simple choice enables us to illustrate the approximation method, while avoiding the additional (minor) complication of handling a nonlinear $h(\eta)$.

Remark 6.1. *With minor modifications, the following discussion can be extended to include partitions composed of triangles in 2D or tetrahedra in 3D.*

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote convex domains where Ω_1 denotes the Darcy or porous domain, and Ω_2 denotes the Stokes or fluid domain. Moreover, let $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2 \subset \mathbb{R}^{d-1}$ be the interface that connects them. We decompose Ω_i, $i = 1,2$, into either quadrilaterals in 2D or hexahedra in 3D to obtain a partition $\mathcal{T}_h^i$, where h is defined below. Hence, the computational domain is

$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, with $\Omega_i = \bigcup_{K \in \mathcal{T}_h^i} K$. We assume that for each partition there exist constants c_1, c_2 such that $c_1 h \leq h_K \leq c_2 \rho_K$, where h_K is the diameter of the cell K, ρ_K is the diameter of the largest sphere included in K, and $h = \max_{K \in \mathcal{T}_h^i} h_k$. Moreover, we assume that the partitions match along Γ, i.e., if F is a face on Γ of a cell in the Stokes domain, then it is also a face of a cell in the Darcy domain. For $k \in \mathbb{N}$, K a cell of a triangulation, and $\mathbf{F}_K : \widehat{K} \to K$ a $C^1(\widehat{K})$ mapping from the reference cell $\widehat{K}$ (the unit square in 2D or the unit cube in 3D) to the cell K, define the $(k+1)^d$-dimensional polynomial space

$$\mathbb{P}_k^Q = \text{span}\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d} : 0 \leq \alpha_1, \alpha_2, \ldots, \alpha_d \leq k\},$$

and let $P_k(K) = \{f : f|_K = \widehat{f} \circ \mathbf{F}^{-1}, \text{ where } \widehat{f} \in \mathbb{P}_k^Q\}$. For F a face of K, the space $P_k(F)$ is defined similarly. The notation $RT_k(\mathcal{T}_h^1)$ is used the denote the Raviart-Thomas space of order k on Ω_1 [2]. We use the following finite element spaces:

$$\begin{aligned}
\mathbf{X}_{1,h} &= \mathbf{X}_1 \cap RT_k(\mathcal{T}_h^1), \quad Q_{i,h} = \left\{q \in L^2(\Omega_i) : q|_K \in P_k(K), \forall K \in \mathcal{T}_h^i\right\}, \\
\mathbf{X}_{2,h} &= \mathbf{X}_2 \cap P_{k+1}(K)^d, \quad Z_h = \{\mathbf{v} \in \mathbf{X}_{1,h} : (q, \nabla \cdot \mathbf{v})_{\Omega_1} = 0, \forall q \in Q_{1,h}\}, \\
\mathbf{X}_h &= \mathbf{X}_{1,h} \times \mathbf{X}_{2,h}, \quad Q_{i,h}^0 = \{q \in Q_{i,h} : (q, 1)_{\Omega_i} = 0\}, \\
Q_h &= \{(q_1, q_2) \in Q_{1,h} \times Q_{2,h} : (q_1, 1)_{\Omega_1} + (q_2, 1)_{\Omega_2} = 0\}, \\
\Lambda_h &= \{f \in \Lambda : f|_F \in P_{k+1}(F), \forall F \in \Gamma\}, \\
R_h &= \left\{r \in L^2(\Omega_1) : r|_K \in P_m(K), \forall K \in \mathcal{T}_h^1\right\}, \\
R_{h,0} &= \left\{r \in L^2(\Omega_1) : r|_K \in P_0(K), \forall K \in \mathcal{T}_h^1\right\}, \\
R_h^s &= \left\{r \in C^0(\Omega_1) : r|_K \in P_{\max\{1,m\}}(K), \forall K \in \mathcal{T}_h^1\right\}.
\end{aligned} \tag{6.1}$$

Notation 6.1. *For F_i a face on Γ, let $W_{h,i}$ be a local partition of F_i, i.e., $F_i = \cup_{S \in W_{h,i}} S$. Note that in 2D, each of the F_i is divided into line segments while in 3D it is divided into quadrilaterals. The collection of all these partitions is denoted by W_h. Furthermore, define the space*

$$\Lambda_{h,0} = \left\{f \in L^2(\Gamma) : f \in P_0(S), \forall S \in W_h\right\}.$$

Notation 6.2. *For U denoting either $\mathcal{T}_h^1$, $\mathcal{T}_h^2$, Γ, or W_h, we define the space of continuous nodal Lagrange elements*

$$Q_k(U) = \left\{f \in C^0(U) : f \in P_k(K), \forall K \in U\right\},$$

and the space of discontinuous nodal Lagrange elements

$$\text{disc}Q_k(U) = \left\{f \in L^2(U) : f \in P_k(K), \forall K \in U\right\}.$$

We omit the dependency on U whenever it is clear from the context.

Remark 6.2. *Note that as $\nabla \cdot \mathbf{X}_{1,h} = Q_{1,h}$, for $\mathbf{v} \in Z_h$ we have that $\|\nabla \cdot \mathbf{v}\|_{L^2(\Omega_1)} = 0$. Thus, $\|\mathbf{v}\|_{\mathbf{X}_1} = \|\mathbf{v}\|_{L^2(\Omega_1)}$.*

For N given, let $\Delta t = T/N$, and $t_n = n\Delta t$, $n = 0, 1, \ldots, N$. Additionally, define

$$d_t f^n = \frac{f^n - f^{n-1}}{\Delta t}, \quad \overline{f}^n = \frac{f^n + f^{n-1}}{2}, \quad \widetilde{f}^n = f^{n-1} + \frac{1}{2} f^{n-2} - \frac{1}{2} f^{n-3}.$$

Initialization of the approximation scheme

The approximation scheme described and analyzed below is a three-level scheme. To initialize the procedure suitable approximations are required for $\mathbf{u}_h^1$, $\mathbf{u}_h^2$ and η_h^2. Here we state our assumptions on these initial approximates. (An initialization procedure is discussed in [11].)

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{X}}^2 + \|\eta^n - \eta_h^n\|_{L^2(\Omega_1)}^2 \leq C(\Delta t)^4 + C\left(h^{2k+2} + h^{2m+2}\right) \text{ for } n = 0, 1, 2. \tag{6.2}$$

Approximation scheme

The approximation scheme we investigate is: *Given* $\eta_0 \in R_h$, *for* $n = 3, \ldots, N$, *determine* $(\mathbf{u}_h^n,\ p_h^n,\ \lambda_h^n,\ \eta_h^n) \in \mathbf{X}_h \times Q_h \times \Lambda_h \times R_h$ *satisfying*:

Scheme 6.1.

$$a_1(\mathbf{u}_h^n, \mathbf{v}) + b(p_h^n, \mathbf{v}) + \langle \lambda_h^n, \mathbf{v} \cdot \mathbf{n}_1 \rangle = (\mathbf{f}^n, \mathbf{v})\ \forall \mathbf{v} \in \mathbf{X}_{1,h}, \tag{6.3}$$

$$b(q, \mathbf{u}_h^n) = 0\ \forall q \in Q_{1,h}, \tag{6.4}$$

$$a_2(\mathbf{u}_h^n, \mathbf{v}) + b(p_h^n, \mathbf{v}) + d(\mathbf{u}_h^n, \mathbf{v}) + \langle \lambda_h^n, \mathbf{v} \cdot \mathbf{n}_2 \rangle = (\mathbf{f}^n, \mathbf{v})\ \forall \mathbf{v} \in \mathbf{X}_{2,h}, \tag{6.5}$$

$$b(q, \mathbf{u}_h^n) = 0\ \forall q \in Q_{2,h}, \tag{6.6}$$

$$\langle \mathbf{u}_1 \cdot \mathbf{n}_1 + \mathbf{u}_2 \cdot \mathbf{n}_2, \nu \rangle = 0\ \forall \nu \in \Lambda_h, \tag{6.7}$$

$$(d_t \eta_h^n, r) + (g(|\widetilde{\mathbf{u}}_h^n|)\overline{\eta}_h^n, r) = 0\ \forall r \in R_h. \tag{6.8}$$

As mentioned in [11], regarding $\eta_h^{n,s}$, note that applying a smoother, $\mathcal{S}$, to a function $\eta_h^n \in R_h$ (typically) does not result in $\eta_h^{n,s} \in R_h^s$. Therefore, we let $\mathcal{S}(\eta_h^n) \in H^{m+1}(\Omega) \cap C^0(\Omega)$ denote the result of the smoother applied to η_h^n, and define

$$\eta_h^{n,s}(x) \ = \ I_h \mathcal{S}(\eta_h^n)(x)\,, \tag{6.9}$$

where $I_h\, :\, C^0(\Omega) \longrightarrow R_h^s$ denotes an interpolation operator.

We assume that the smoothed porosity $\mathcal{S}(\eta_h^n)$ is sufficiently regular such that there exists a constant dependent on $\mathcal{S}(\,\cdot\,)$, $C_{\mathcal{S}}$ such that

$$\|\mathcal{S}(\eta_h^n) \,-\, I_h \mathcal{S}(\eta_h^n)\|_{L^\infty(\Omega)} \ = \ \|\mathcal{S}(\eta_h^n) \,-\, \eta_h^{n,s}\|_{L^\infty(\Omega)} \ \le \ C_{\mathcal{S}} h^{m+1}\,. \tag{6.10}$$

The precise dependence of $C_{\mathcal{S}}$ on $\mathcal{S}(\,\cdot\,)$ will depend on the particular smoother used. Now we focus on the computability of Scheme 6.1. First, we prove that the porosity η_h^n can be computed from previous approximations and derive some of its properties.

Lemma 6.1. *Given* η_h^{n-1}, $\mathbf{u}_h^{n-3}$, $\mathbf{u}_h^{n-2}$ *and* $\mathbf{u}_h^{n-1}$, *there exists a unique solution* $(\mathbf{u}_h^n, p_h^n, \lambda_h^n, \eta_h^n) \in \mathbf{X}_h \times Q_h \times \Lambda_h \times R_h$ *satisfying* (6.3)-(6.8).

Proof. Firstly, the existence and uniqueness of η_h^n follows directly from Lemma 4.1 in [11]. Then, the existence and uniqueness of $\mathbf{u}_h^n$, p_h^n, and λ_h^n can be established as done in [25] (see also [10]). □

Nonnegativity of the discrete porosity

In this subsection we give conditions under which the discrete porosity η_h is nonnegative and bounded above by the initial porosity η_0. These results are consistent with the properties derived for the continuous porosity and mirror the propositions derived in Section 5. The key assumption in this section is that the finite element space for the discrete porosity is composed of piecewise constant functions, i.e., we replace R_h in (6.8) with $R_{h,0}$.

Lemma 6.2. *Let* $\eta_h^{n-1} \in R_{h,0}$ *be given with* $\eta_h^{n-1} \geq 0$ *in* Ω_1. *Furthermore, assume* $\Delta t < 2/g_{\max}$. *Then, the solution* $\eta_h^n \in R_{h,0}$ *given in Lemma 6.1 is nonnegative in* Ω_1.

Proof. Assume there exists $K \in \mathcal{T}_h^1$ such that $\eta_h^n|_K < 0$. Substituting $r = \eta_h^n\, \chi_K \in R_{h,0}$ into (6.8), where χ_K is the indicator function of the cell K yields

$$(\eta_h^n, \eta_h^n)_K + \frac{\Delta t}{2}\, (g(|\widetilde{\mathbf{u}}_h^n|)\, \eta_h^n, \eta_h^n)_K = \left(\eta_h^{n-1} \left(1 - \frac{\Delta t}{2} g(|\widetilde{\mathbf{u}}_h^n|) \right), \eta_h^n \right)_K. \tag{6.11}$$

Owing to the nonnegativity of $g(\,\cdot\,)$, and the nonnegativity of η_h^{n-1}, it follows that the left hand side of (6.11) is positive while the right hand side is nonpositive. This is a contradiction. Hence, no such cell K can exist. We conclude that η_h^n is nonnegative in Ω_1. $\square$

Lemma 6.3. *Let $\eta_h^{n-1} \in R_{h,0}$ be given, with $\eta_h^{n-1} \geq 0$ in Ω_1. Furthermore, assume $\Delta t < 2/g_{\max}$. Then, the solution $\eta_h^n \in R_{h,0}$ given in Lemma 6.1 satisfies $\eta_h^n \leq \eta_h^{n-1}$ in Ω_1.*

Proof. Assume there exists $K \in \mathcal{T}_h^1$ such that $\left(\eta_h^n - \eta_h^{n-1}\right)\big|_K > 0$. Substituting $r = \left(\eta_h^n - \eta_h^{n-1}\right)\chi_K \in R_{h,0}$ into (6.8), we obtain

$$\left(\eta_h^n - \eta_h^{n-1}, \eta_h^n - \eta_h^{n-1}\right)_K = -\frac{\Delta t}{2}\left(g(|\widetilde{\mathbf{u}}_h^n|)\left(\eta_h^n + \eta_h^{n-1}\right), \eta_h^n - \eta_h^{n-1}\right)_K. \tag{6.12}$$

Observe that the left hand side of (6.12) is positive. Moreover, owing to Lemma 6.2, the nonnegativity of $g(\,\cdot\,)$ and the nonnegativity of η_h^{n-1}, the right hand side of (6.12) is nonpositive. This is a contradiction. Thus, no such cell K exists, and $\eta_h^n \leq \eta_h^{n-1}$ in Ω_1. $\square$

Corollary 6.1. *Let $\eta_h^{n-1} \in R_{h,0}$ be given with $0 \leq \eta_h^{n-1} \leq \eta_0$ in Ω_1. Furthermore, assume $\Delta t < 2/g_{\max}$. Then, the solution $\eta_h^n \in R_{h,0}$ given in Lemma 6.1 is also bounded above by η_0 in Ω_1.*

Proof. This is a direct consequence of Lemma 6.3. $\square$

Although the porosity η_h^n can be computed, the Stokes and Darcy problems are still coupled through the interfacial pressure λ_h^n. Our objective is to decouple them in order to take advantage of the robust and efficient solvers that are available for each individual problem. We achieve the decoupling by treating λ as a given quantity, and proceed to demonstrate that we can systematically compute a sequence of approximations $\{\lambda_k\}_{k=1}^{\infty} \subset \Lambda_h$, such that (6.7) is satisfied. In the discussion that follows we assume that η_h is known.

Decoupling the problems

As mentioned in the previous paragraph, once the porosity η_h^n has been computed, we can decouple the Stokes and Darcy problems by assuming an interfacial pressure λ. However, this choice for λ may not satisfy the flux condition (6.7). Hence, we treat the Stokes and Darcy velocities as functions of λ and restate (6.7) as a least squares problem. Concretely, we define the operator $G : \Lambda_h \to \Lambda_{h,0} \times \Lambda_h$ given by

$$G(\lambda) = \begin{pmatrix} \rho\left(\mathbf{u}_{1,h}(\lambda); \mathbf{u}_{2,h}(\lambda)\right) \\ \delta\lambda \end{pmatrix},$$

where $\rho : \mathbf{X}_1 \times \mathbf{X}_2 \times \Gamma \to \mathbb{R}$ is the piecewise constant function

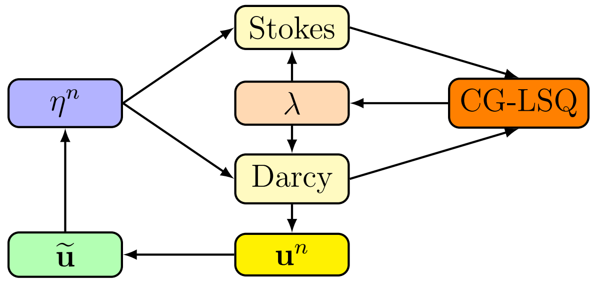
$$\rho\left(\mathbf{u}_{1,h}(\lambda); \mathbf{u}_{2,h}(\lambda)\right)(x) = \sum_{S \in W_h} \frac{\chi_S(x)}{|S|} \int_S \mathbf{u}_{1,h}(\lambda) \cdot \mathbf{n}_1 + \mathbf{u}_{2,h}(\lambda) \cdot \mathbf{n}_2 \, d\Gamma, \tag{6.13}$$

χ_S is the indicator function of the set S, $\delta \in (0,1)$ is a penalization parameter, and consider the minimization problem

$$\min_{\lambda \in \Lambda_h} \|G(\lambda)\|^2_{L^2(\Gamma)\times\Lambda}. \tag{6.14}$$

A thorough discussion of (6.14) and a computational algorithm (CG-LSQ) to approximate the minimum can be found in [9]. We now describe a typical iteration of Scheme 6.1.

Algorithm 6.1 One iteration of Scheme 6.1



1. Use the previously computed approximations $\mathbf{u}_h^{n-3}$, $\mathbf{u}_h^{n-2}$, and $\mathbf{u}_h^{n-1}$ to obtain $\widetilde{\mathbf{u}}_h^n$.
2. Use $\widetilde{\mathbf{u}}_h^n$ and η_h^{n-1} to compute the porosity η_h^n.
3. Smooth η_h^n to obtain $\eta_h^{n,s}$.
4. Use $\eta_h^{n,s}$ and Algorithm 4.4 in [9] to obtain λ_h^n.
5. Compute $\mathbf{u}_{i,h}^n(\lambda_h^n)$ and $p_{i,h}^n(\lambda_h^n)$.

We conclude this section stating an a priori error estimate for Scheme 6.1.

Conjecture 6.1. *Let* $(\mathbf{u}, p, \eta)$ *satisfy* (3.1)-(3.4) *with* $h(\eta) = \eta$ *and* $(\mathbf{u}_h^n, p_h^n, \eta_h^n)$ *satisfy* (6.3)-(6.8). *Furthermore, assume* $C_{\mathcal{S}(\eta_h^n)}$ *given in* (6.10) *is bounded by* $C_{\mathcal{S}}\|\eta^n\|_{m+1}$. *Then, for* Δt *sufficiently small there exists* $C > 0$ *independent of* h *and* Δt, *such that for* $n = 1, 2, \ldots, N$,

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{X}} + \|p^n - p_h^n\| + \|\eta^n - \eta_h^n\| \leq C\left((\Delta t)^2 + h^{k+1} + h^{m+1}\right). \tag{6.15}$$

In particular, for $k = m = 1$, *we obtain second order convergence in space and time.*

In the next section we give a numerical study of Scheme 6.1.

7 Numerical experiments

This section is divided into three experiments. Experiment 7.1 and Experiment 7.2 illustrate that the convergence properties of Scheme 6.1 are consistent with Conjecture 6.1. Finally, Experiment 7.3 shows that the numerical approximations for the porosity remain nonnegative if the conditions of Lemma 6.2 are satisfied.

Experiment 7.1 Consider the physical parameters

$$\beta(\eta) = \eta^2 + 0.1, \quad \Psi(\eta) = 1, \quad \mu = 1/2, \quad g(|\mathbf{u}|) = |\mathbf{u}|^2 + 1, \tag{7.1}$$

and let $\Omega_1 = (0,1) \times (0,1)$, $\Omega_2 = (0,1) \times (1,2)$ and $\Gamma = (0,1) \times \{1\}$. We partition Ω_i into square cells. On Ω_2 we use the Taylor-Hood element pair, i.e., Q_2 elements for the velocity and Q_1 elements for the pressure. On Ω_1 we use the Raviart-Thomas element of degree 1, RT_1 for the velocity and discQ_1 elements for the pressure. The interfacial pressure λ is approximated on Γ using Q_2 elements and the porosity η is approximated using discQ_1 elements.

Remark 7.1. *To compute* $\rho(\cdot, \cdot)$ *(see* (6.13) *and Notation 6.1), every face* F_i *on the interface* Γ *is uniformly refined twice so that* $|W_{h,i}| = 16$.

The boundary conditions for the Darcy problem are imposed weakly, i.e., through the weak form, while the boundary conditions for the Stokes problem are imposed strongly. We consider successively finer meshes $\mathcal{T}_h$ and smaller time-steps Δt, $\Delta t \propto h$, and compute the error between the exact solution

$$\begin{aligned}
\eta &= 0.8 - \delta\,\pi^2\,t^2 \sin(\pi x)\sin(\pi y) - \frac{1}{2}\,t^2 \sin(\pi x)\sin(\pi y)\,,\\
\mathbf{u}_1 &= \big(-x(\sin(y)\exp(1) + 2(y-1)), -\cos(y)\exp(1) + (y-1)^2\big)^T \cos(t),\\
p_1 &= \big(-\sin(y)\exp(1) + \cos(x)\exp(y) + y^2 - 2y + 1\big)\,\cos(t),\\
\mathbf{u}_2 &= \big((y-1)^2 x^3, -\exp(1)\cos(y)\big)^T \cos(t),\\
p_2 &= \big(\cos(x)\exp(y) + y^2 2y + 1\big)\,\cos(t),
\end{aligned}$$

and the numerical approximations under different discrete norms for a time horizon $T = 0.5$. The smoothed porosity is approximated using Q_1 elements by solving the boundary value problem: Find $\eta_h^s \in R_h^s$ satisfying

$$\begin{aligned}
(\delta\,\nabla\eta_h^s, \nabla\varphi) + (\eta_h^s, \varphi) &= (\eta_h, \varphi) \quad \forall \varphi \in R_h^s,\\
\eta_h^s &= \eta_h \text{ on } \partial\Omega_1,
\end{aligned} \tag{7.2}$$

where $\delta = 0.05$. The corresponding exact smoothed porosity is

$$\eta^s(\mathbf{x}, t) = 0.8 - \frac{1}{2}\,t^2 \sin(\pi x)\sin(\pi y)\,.$$

For $\|\cdot\|_U$ a norm on some generic space U and f_h^n a generic numerical approximation to $f(t_n)$, $n = 0, \ldots, N$, the discrete norm under consideration is defined as follows,

$$|||f - f_h|||_{L^2(0,T;U)} = \left(\sum_{k=0}^{N} \|f(t_k) - f_h^k\|_U^2\,\Delta t\right)^{1/2}.$$

Moreover, for h_1 and h_2 two different mesh parameters, the numerical convergence rates are computed using the formula

$$\log\left(\frac{|||f - f_{h_1}|||_{L^2(0,T;U)}}{|||f - f_{h_2}|||_{L^2(0,T;U)}}\right) / \log\left(\frac{h_1}{h_2}\right).$$

The results are summarized in Table 1. A graphical depiction of Table 1 can be found in Figure 2. The average number of iterations of Algorithm 6.1 as a function of Δt for two different meshes is shown in Figure 3. Note that the numerical convergence rates are in agreement with Conjecture 6.1.

Experiment 7.2 For this experiment we validate our numerical implementation in 3D. In view that Experiment 7.1 supports the correctness of the time-stepping scheme and the spatial discretization in 2D, it remains to verify the implementation of the spatial discretization in 3D. Let $\Omega_1 = (0,1) \times (0,1) \times (0,1)$, $\Omega_2 = (0,1) \times (0,1) \times (1,2)$ and $\Gamma = (0,1) \times (0,1) \times \{1\}$. We consider the same finite element spaces of Experiment 7.1 and the spatial parameters given in (7.1). The domains

Table 1: Numerical results for Experiment 7.1.

Stokes domain

h	Δt	$\lVert\mathbf{u} - \mathbf{u}_h\rVert_{L^2(0,T;H^1(\Omega_2))}$	Rate	$\lVert p - p_h\rVert_{L^2(0,T;L^2(\Omega_2))}$	Rate
0.354	$5.00 \cdot 10^{-2}$	$6.38 \cdot 10^{-3}$	–	$1.23 \cdot 10^{-2}$	–
0.177	$2.50 \cdot 10^{-2}$	$1.54 \cdot 10^{-3}$	2.05	$3.00 \cdot 10^{-3}$	2.03
0.088	$1.25 \cdot 10^{-2}$	$3.78 \cdot 10^{-4}$	2.02	$7.42 \cdot 10^{-4}$	2.01
0.044	$6.25 \cdot 10^{-3}$	$9.37 \cdot 10^{-5}$	2.01	$1.85 \cdot 10^{-4}$	2.00
0.022	$3.13 \cdot 10^{-3}$	$2.33 \cdot 10^{-5}$	2.01	$4.62 \cdot 10^{-5}$	2.00
Expected rate			2		2

Darcy domain

h	Δt	$\lVert\mathbf{u} - \mathbf{u}_h\rVert_{L^2(0,T;\mathbf{H}_{\mathrm{div}}(\Omega_1))}$	Rate	$\lVert p - p_h\rVert_{L^2(0,T;L^2(\Omega_1))}$	Rate
0.354	$5.00 \cdot 10^{-2}$	$8.67 \cdot 10^{-2}$	–	$8.55 \cdot 10^{-3}$	–
0.177	$2.50 \cdot 10^{-2}$	$2.60 \cdot 10^{-2}$	1.74	$2.11 \cdot 10^{-3}$	2.02
0.088	$1.25 \cdot 10^{-2}$	$7.27 \cdot 10^{-3}$	1.84	$5.28 \cdot 10^{-4}$	2.00
0.044	$6.25 \cdot 10^{-3}$	$1.97 \cdot 10^{-3}$	1.88	$1.33 \cdot 10^{-4}$	1.99
0.022	$3.13 \cdot 10^{-3}$	$5.19 \cdot 10^{-4}$	1.93	$3.34 \cdot 10^{-5}$	1.99
Expected rate			2		2

		Porosity		Divergence in Ω_1
h	Δt	$\lVert\eta - \eta_h\rVert_{L^2(0,T;L^2(\Omega_1))}$	Rate	$\lVert\nabla \cdot \mathbf{u}_h\rVert_{L^2(0,T;L^2(\Omega_1))}$
0.354	$5.00 \cdot 10^{-2}$	$1.68 \cdot 10^{-2}$	–	$2.01 \cdot 10^{-11}$
0.177	$2.50 \cdot 10^{-2}$	$5.64 \cdot 10^{-3}$	1.58	$4.57 \cdot 10^{-11}$
0.088	$1.25 \cdot 10^{-2}$	$1.66 \cdot 10^{-3}$	1.77	$1.40 \cdot 10^{-10}$
0.044	$6.25 \cdot 10^{-3}$	$4.56 \cdot 10^{-4}$	1.86	$3.83 \cdot 10^{-10}$
0.022	$3.13 \cdot 10^{-3}$	$1.20 \cdot 10^{-4}$	1.92	$1.09 \cdot 10^{-9}$
Expected rate			2	

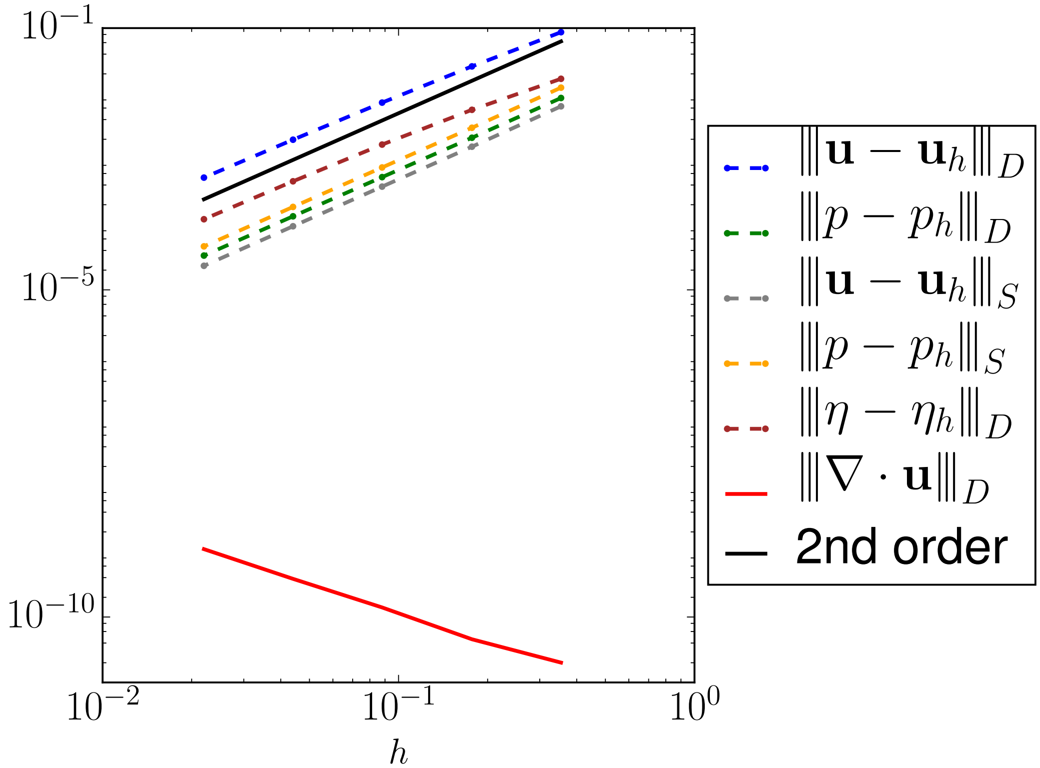


Figure 2: Convergence rates for Experiment 7.1. For vector arguments, the norm $|||\cdot|||_D$ corresponds to the discrete $L^2(0,T;\mathbf{H}_{\rm div}(\Omega_1))$ norm. For scalar inputs the norm $|||\cdot|||_D$ is the discrete $L^2(0,T;L^2(\Omega_1))$ norm. Similarly, the norm $|||\cdot|||_S$ is the $L^2(0,T;H^1(\Omega_2))$ for vector arguments and the $L^2(0,T;L^2(\Omega_2))$ for scalar inputs.

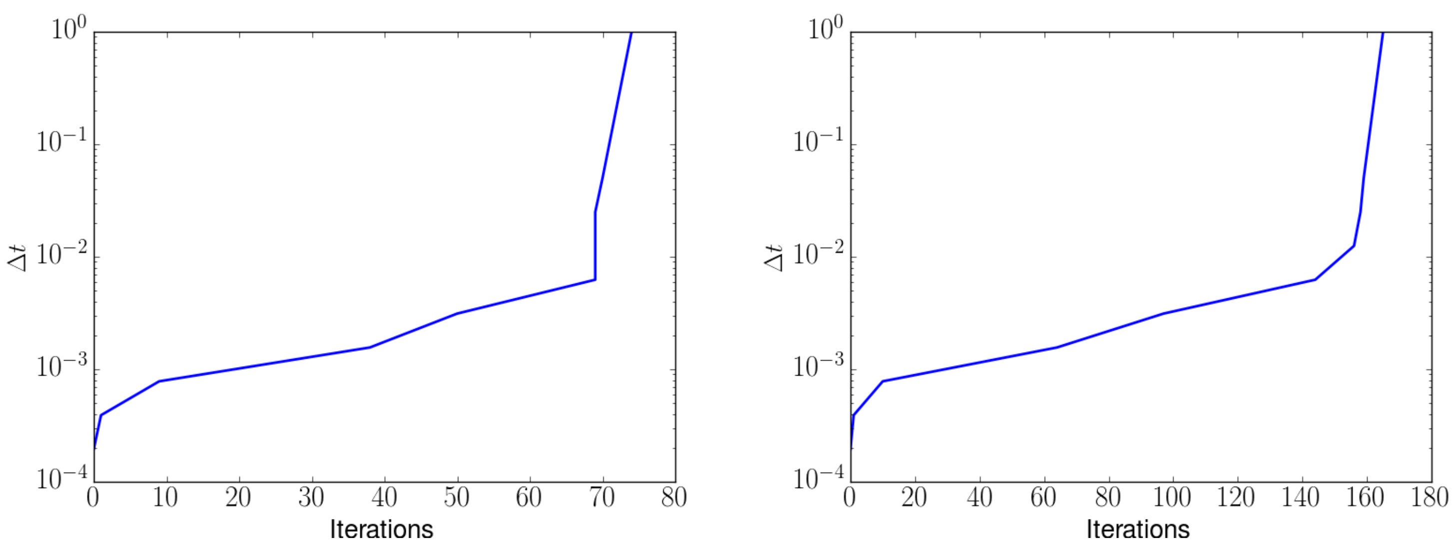


Figure 3: Average number of iterations of Algorithm 6.1 in Experiment 7.1. The number of iterations depend upon the step-size Δt, and the mesh refinement level. Mesh parameter $h = 0.044$ (left) and $h = 0.022$ (right).

Table 2: Numerical results for Experiment 7.2.

Stokes domain						
h	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega_2)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega_2)}$	Rate	$\|p-p_h\|_{L^2(\Omega_2)}$	Rate
0.433	$8.00 \cdot 10^{-1}$	–	$3.09 \cdot 10^{-2}$	–	$3.94 \cdot 10^{-2}$	–
0.217	$2.02 \cdot 10^{-1}$	1.99	$3.89 \cdot 10^{-3}$	2.99	$4.07 \cdot 10^{-3}$	3.27
0.108	$5.06 \cdot 10^{-2}$	2.00	$4.88 \cdot 10^{-4}$	3.00	$7.97 \cdot 10^{-4}$	2.35
	Expected rate	2		3		2

Darcy domain					
h	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}_{\text{div}}(\Omega_1)}$	Rate	$\|p-p_h\|_{L^2(\Omega_1)}$	Rate	$\|\nabla\cdot\mathbf{u}\|_{L^2(\Omega_1)}$
0.433	$1.65 \cdot 10^{-2}$	–	$1.51 \cdot 10^{-2}$	–	$1.63 \cdot 10^{-10}$
0.217	$3.94 \cdot 10^{-3}$	2.07	$2.82 \cdot 10^{-3}$	2.42	$3.67 \cdot 10^{-10}$
0.108	$9.68 \cdot 10^{-4}$	2.03	$6.85 \cdot 10^{-4}$	2.04	$1.39 \cdot 10^{-9}$
	Expected rate	2		2	

Ω_i are partitioned into hexahedra. To compute the convergence rates we use

$$\begin{aligned}
\eta^s &= 0.8 - \cos(y)\exp(x)\sin(z),\\
\mathbf{u}_1 &= \begin{pmatrix} -x(\sin(y)\exp(z) + 2(y-1)) \\ y^2 - 2yz - \cos(y)\exp(z) + \exp(x) \\ -\exp(y)x + (z-1)^2 \end{pmatrix},\\
p_1 &= (y-1)^2 + \exp(y)\cos(x) - \exp(z)\sin(y),\\
\mathbf{u}_2 &= \begin{pmatrix} \exp(2z)\cos(y)\sin(x) - 0.5\exp(y)z - 0.25\exp(y) \\ -0.5\exp(y)xz - 0.25\exp(y)x + \cos(x)\exp(y+2z) \\ \exp(y)xz \end{pmatrix},\\
p_2 &= \exp(y)z\cos(x) - \exp(y)x + (y-1)^2 - \exp(1)\sin(y).
\end{aligned}$$

The results are summarized in Table 2. Observe that the numerical convergence rates are consistent with expectations.

Experiment 7.3 Let $\Omega_1 = (0,1)\times(0,1)$ denote the Darcy domain, $\Omega_2 = (0,1)\times(1,2)$ denote the Stokes domain, and $\Gamma = (0,1)\times\{1\}$ their common interface. Furthermore, define the inflow boundary $\Gamma_{\text{in}} = (0,1)\times\{2\}$, the outflow boundary $\Gamma_{\text{out}} = (0,1)\times\{0\}$, the Darcy boundary $\Gamma_1 = \partial\Omega_1 \setminus (\Gamma\cup\Gamma_{\text{out}})$, and the Stokes boundary $\Gamma_2 = \partial\Omega_2 \setminus (\Gamma\cup\Gamma_{\text{in}})$. A graphical depiction of the computational domain is given in Figure 1. We partition Ω_i into 256 square cells and use, save for the porosity, the same finite element spaces as in Experiment 7.1. The smooth porosity is computed with the differential filter (7.2) using Q_1 elements and $\delta = 10^{-8}$.

The physical parameters are given by:

$$\beta(\eta) = C_\beta \frac{(1-\eta)^2}{\eta^3}, \quad \Psi(\eta) = 1, \quad \mu = 1/2, \quad g(|\mathbf{u}|) = |\mathbf{u}|\,. \tag{7.3}$$

In (7.3), the expression for $\beta(\,\cdot\,)$ arises from the well-known Kozeny-Carman equation [28], applicable for laminar flow through a bed packed with spherical particles. The constant C_β typically depends

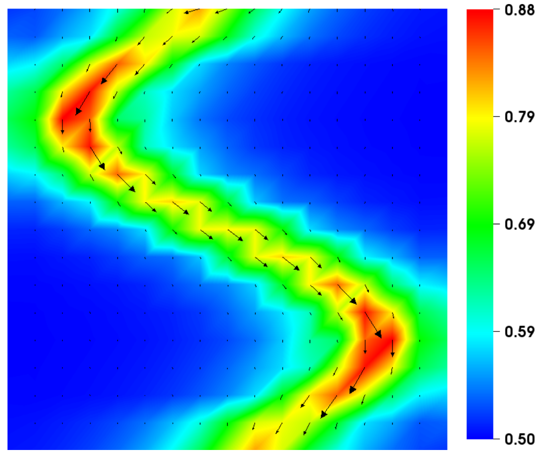


Figure 4: Initial profile of the smoothed porosity in Experiment 7.3. The arrows indicate the direction of the flow in the porous domain Ω_1. The colorbar denotes the magnitude of the porosity.

upon the viscosity of the fluid and the mean diameter of the particles in the porous media. In this experiment we assume that $C_\beta = 1$ and the viscosity in the fluid (Stokes) domain is $\mu = 0.5$. For the initial porosity profile, we consider a vertical channel in the porous medium given by

$$\text{channel}(y) = 0.5 + 0.35 \sin(2\pi\, y),$$

and assume the porosity attains a maximum $\eta_{\max} = 0.9$ along the channel and decreases at an exponential rate $\tau = 7.5$, proportional to the horizontal distance to the channel. We further assume the minimum porosity is given by $\eta_{\min} = 0.5$. The expression for η_0 satisfying the aforementioned characteristics is

$$\eta_0(x, y) = \eta_{\min} + (\eta_{\max} - \eta_{\min}) \exp\left(-\tau \left|\text{channel}(y) - x\right|\right).$$

A graphical depiction of η_0 is given in Figure 4. We impose zero flux boundary conditions on Γ_1 and homogeneous Dirichlet boundary conditions on Γ_2. On the inflow boundary Γ_{in} we impose the parabolic profile $\mathbf{u}_{\text{in}} = (0,\, 4x(x-1))^T$, and enforce weakly the condition $p = 0$ on the outflow boundary Γ_{out}. We set the computational parameters $T = 20$, $\Delta t = 0.045$, and approximate the porosity using both discQ_0 and discQ_1 elements.

In both cases, throughout the time horizon $T = 20$, the magnitude of the Darcy velocity $\mathbf{u}_1$ never exceeds the value of 4. Hence, $g(|\mathbf{u}_1(t)|) \leq g_{\max} := 4$ and $\Delta t < 2/g_{\max}$ for $t \leq 20$. This implies that the bound for Δt stated in Lemma 6.2 is satisfied. The results are summarized in Figure 5. We note that in agreement with Lemma 6.2, when the porosity is computed using discQ_0 elements the numerical approximation remain nonnegative. Moreover, observe that the results suggest that Lemma 6.2 can not be trivially extended to include discQ_1 elements.

Conclusion

In this work we considered a generalization of the filtration model discussed in [11]. The new system of equations coupled the previously analyzed augmented nonlinear Darcy problem to the Stokes equations. Well-posedness of the coupled system with appropriate boundary and interfacial conditions

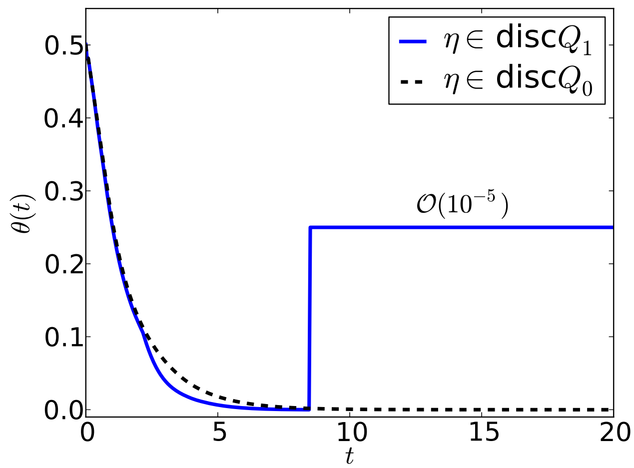


Figure 5: Comparison of the temporal evolution of the minimum of the porosity in Experiment 7.3 using discQ_1 and discQ_0 elements. The function θ is given by $\theta(t) = \max\{z(t), -0.25 \text{ sign}(z(t))\}$, where $z(t) = \min_{\mathbf{x}\in\Omega_1}\{\eta_h(\mathbf{x}, t)\}$. Note that, in agreement with Lemma 6.2, the porosity remains nonnegative for $\eta_h \in R_{h,0}$. In the case where $\eta_h \in$ discQ_1, the porosity becomes negative at $t = 8.505$. The negative values are of the order of 10^{-5}.

was established. Moreover, consistent with the physics, the porosity was shown to be nonnegative and bounded above by the initial porosity. Thereafter, we introduced a numerical scheme for the coupled problem capable of preserving the nonnegativity and boundedness of the porosity. Numerical experiments followed, showing that the numerical approximations for the porosity are indeed nonnegative under the derived conditions. Furthermore, the computations showed that the numerical convergence rates in space and time for the coupled problem are in agreement with the conjectured estimates.

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