Analysis of the Oseen-Viscoelastic Fluid Flow Problem

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Abstract. In this article we study the numerical approximation of an Oseen type model for viscoelastic fluid flow. Existence and uniqueness of the continuous and approximate solutions, under a *small data* assumption, are proved. Error estimates for the numerical approximations are also derived. Numerical experiments are presented which support the error estimates, and which demonstrate the relevance of the *small data* assumption for the solvability of the continuous and discrete systems.

Key words. viscoelasticity, finite element method, a priori error estimates, iteration

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1 Introduction

Newtonian fluids are characterized by the assumption that the extra stress tensor, $\boldsymbol{\tau}_{ex}$, is directly proportional to the deformation tensor, $\mathbf{D}(\mathbf{u})$, i.e. $\boldsymbol{\tau}_{ex} = 2\eta\mathbf{D}(\mathbf{u})$, where $\mathbf{u}$ and η denote the fluid velocity and its viscosity, respectively. For viscoelastic fluids the relationship between $\boldsymbol{\tau}_{ex}$ and $\mathbf{u}$ is necessarily more complicated, in order to account for the elastic property of the fluid. In the Elastic Viscous Split Stress (EVSS) [9] formulation $\boldsymbol{\tau}_{ex}$ is written as a sum of a Newtonian stress, $\boldsymbol{\tau}_N = 2\alpha\mathbf{D}(\mathbf{u})$, and an elastic stress tensor $\boldsymbol{\tau}$. A suitable constitutive model is then specified for $\boldsymbol{\tau}$ which may be purely algebraic (e.g. Power Law: $\boldsymbol{\tau} = c|\mathbf{D}(\mathbf{u})|^{1-\gamma}\mathbf{D}(\mathbf{u})$), differential (Oldroyd-B, Giesekus [2]), or integro-differential [2]. Our interest is with the general constitutive model (Johnson-Segalman) described by

$$\boldsymbol{\tau} + \lambda(\mathbf{u}\cdot\nabla\boldsymbol{\tau} + g_a(\boldsymbol{\tau},\nabla\mathbf{u})) - 2\alpha\mathbf{D}(\mathbf{u}) = \mathbf{0}\ , \tag{1.1}$$

where,

$$g_a(\boldsymbol{\tau},\nabla\mathbf{u})) = \frac{1-a}{2}\left((\nabla\mathbf{u})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla\mathbf{u})^T\right) - \frac{1+a}{2}\left(\boldsymbol{\tau}(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\boldsymbol{\tau}\right). \tag{1.2}$$

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Note: In (1.1) the choice $a = -1$ corresponds to the Oldroyd-B model.

Existence and uniqueness of the solution to viscoelastic fluid flow problems, governed by differential constitutive equations, is still an active area of research. Renardy [10], and Fernandez-Cara et al. [5] have shown, under small data assumptions, existence of a unique solution to the system of model equations.

The work in this paper is motivated by the *High Weissenberg Number Problem* in viscoelasticity. Numerically it has been observed that as λ increases the solution algorithm for the nonlinear system of approximating algebraic equations may fail to converge. Additionally, for large λ fixed, computed approximations may not converge under mesh refinement [1, 4, 8, 12].

In Newtonian fluid flow the Oseen equations may be *visually* obtained from the Navier-Stokes equations by fixing the velocity, $\mathbf{u} = \mathbf{b}$, in the nonlinear term in the conservation of momentum equation, thereby yielding a linear system of equations. (Physically the Oseen equations in Newtonian flow represent the linearization of the Navier-Stokes equations about a free stream velocity.) In the modeling of viscoelastic fluid flow, under a "creeping flow" assumption the inertia term $\mathbf{u} \cdot \nabla \mathbf{u}$ is neglected in the conservation of momentum equation. Hence the only nonlinearities in the modeling equations for viscoelastic fluid flow occur in the constitutive equation.

In order to gain additional insight into the *High Weissenberg Number Problem*, in this paper we introduce the Oseen-viscoelastic fluid flow equations, in which the unknown velocity $\mathbf{u}$ occurring in the nonlinear terms in the constitutive equation is replaced by a known velocity field $\mathbf{b}$. For the resulting system of equations we are able to explicitly describe the parameter space for α, λ, and $\|\nabla \mathbf{b}\|_\infty$ which guarantee existence and uniqueness (sufficiency condition) for the solution of the continuous problem and its numerical approximation. As an iterative solution method for the approximation of viscoelastic fluid flow equations, the solution of the Oseen-viscoelastic equations can be viewed as a step in a fixed point iteration method.

This paper is organized as follows. In the next section we state the Oseen-viscoelastic equations and investigate existence and uniqueness of its solution. Presented in Sections 2.1-2.3 is an analysis of a Finite Element approximation Method (FEM), including numerical experiments supporting the derived error estimates. In Sections 3 and 4 we investigate, analytically and numerically, the solvability of the Oseen-viscoelastic equations as a function of λ.

2 Modeling Equations

In this section we present the Oseen type model for viscoelastic fluid flow, and establish existence and uniqueness of the continuous solution and its Galerkin approximation.

The steady-state (Johnson-Segalman) modeling equations for viscoelasticity, assuming creeping flow, with homogeneous boundary conditions are given by:

$$\boldsymbol{\tau} + \lambda\left(\mathbf{u} \cdot \nabla \boldsymbol{\tau} + g_a(\boldsymbol{\tau}, \nabla \mathbf{u})\right) - 2\alpha \mathbf{D}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$-2(1-\alpha)\nabla \cdot \mathbf{D}(\mathbf{u}) - \nabla \cdot \boldsymbol{\tau} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.2}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where $\alpha \in (0,1)$ and, for $a \in [-1,1]$,

$$\begin{aligned} g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) &:= \boldsymbol{\sigma}\mathbf{W}(\mathbf{u}) - \mathbf{W}(\mathbf{u})\boldsymbol{\sigma} - a(\mathbf{D}(\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{D}(\mathbf{u})) \\ &= \frac{1-a}{2}\left((\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T\right) - \frac{1+a}{2}\left(\boldsymbol{\sigma}(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \boldsymbol{\sigma}\right). \end{aligned}$$

For the Oseen model we assume a given velocity field $\mathbf{b}(\mathbf{x})$ in the non-linear terms in the constitutive equation (2.1) to obtain:

$$\begin{aligned} \boldsymbol{\tau} + \lambda\left(\mathbf{b} \cdot \nabla \boldsymbol{\tau} + g_a(\boldsymbol{\tau}, \nabla \mathbf{b})\right) - 2\alpha \mathbf{D}(\mathbf{u}) &= \mathbf{0} \quad \text{in } \Omega, && (2.5) \\ -2(1-\alpha)\nabla \cdot \mathbf{D}(\mathbf{u}) - \nabla \cdot \boldsymbol{\tau} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, && (2.6) \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, && (2.7) \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. && (2.8) \end{aligned}$$

Inspecting (2.5)-(2.8) we note that for λ constant in Ω the product $\lambda \mathbf{b}$ is an invariant of (2.5)-(2.8). Thus we introduce $\widetilde{\mathbf{b}} := \lambda \mathbf{b}$ and consider the system of equations

$$\begin{aligned} \boldsymbol{\tau} + \widetilde{\mathbf{b}} \cdot \nabla \boldsymbol{\tau} + g_a(\boldsymbol{\tau}, \nabla \widetilde{\mathbf{b}}) - 2\alpha \mathbf{D}(\mathbf{u}) &= \mathbf{0} \quad \text{in } \Omega, && (2.9) \\ -2(1-\alpha)\nabla \cdot \mathbf{D}(\mathbf{u}) - \nabla \cdot \boldsymbol{\tau} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, && (2.10) \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, && (2.11) \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. && (2.12) \end{aligned}$$

The following notation will be used. Let $W^{l,q}(\Omega)$ represent the usual Sobolev spaces with norm and seminorm $|| \cdot ||_{l,q;\Omega}$ and $| \cdot |_{l,q;\Omega}$, respectively. Likewise, the $L^p(\Omega)$ norm is defined as $|| \cdot ||_{p;\Omega}$. In particular, let H^l represent the Sobolev space $W^{l,2}(\Omega)$ with norm and seminorm $|| \cdot ||_{l,2;\Omega}$ and $| \cdot |_{l,2;\Omega}$, respectively. For ease of notation we drop the domain from the norm and seminorm notations when the domain is obvious. The following function spaces are used in the analysis:

$$\begin{aligned} \text{Velocity Space} &: \quad X := \left(H_0^1(\Omega)\right)^d := \left\{\mathbf{u} \in \left(H_0^1(\Omega)\right)^d : \mathbf{u} = 0 \ \ on \ \ \partial\Omega\right\}, \\ \text{Pressure Space} &: \quad Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}, \\ \text{Divergence} - \text{free Space} &: \quad Z := \{\mathbf{v} \in X : \int_\Omega q(\nabla \cdot \mathbf{v}) \, dx = 0, \ \forall \, q \in Q\}. \end{aligned}$$

The modeling equations (2.1)-(2.3) are derived under a *creeping flow* assumption on the fluid velocity, i.e. $\mathbf{b} \sim O(1)$. Additionally we assume:

$$\textbf{Assumption A:} \quad \mathbf{b} \in \left(H_0^1(\Omega)\right)^d, \quad \nabla \cdot \mathbf{b} = 0, \ \|\mathbf{b}\|_\infty \le M, \quad \|\nabla \mathbf{b}\|_\infty \le M < \infty,$$

for some $M > 0$.

Note that Assumption A is consistent with the existence results which have been established for viscoelasticity, see [5, 11].

For the stress space we use

$$S := \left\{\boldsymbol{\tau} | \boldsymbol{\tau} \in (L^2(\Omega))^{d \times d}, \boldsymbol{\tau}_{ij} = \boldsymbol{\tau}_{ji}, \widetilde{\mathbf{b}} \cdot \nabla \boldsymbol{\tau} \in (L^2(\Omega))^{d \times d}\right\}.$$

Define the norm $\|\cdot\|_{\mathbf{b}}$ as:

$$\|\boldsymbol{\tau}\|_{\mathbf{b}} \ := \ \left(\|\boldsymbol{\tau}\|_{0,2}^2 + \lambda^2 \|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2}^2\right)^{1/2} \ = \ \left(\|\boldsymbol{\tau}\|_{0,2}^2 + \|\widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\tau}\|_{0,2}^2\right)^{1/2} \ .$$

Note that S is a Hilbert space with associated inner product

$$\langle\boldsymbol{\tau},\boldsymbol{\sigma}\rangle_{\widetilde{\mathbf{b}}} = (\boldsymbol{\tau},\boldsymbol{\sigma}) + (\widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\tau}, \widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\sigma}) \ .$$

Taking the inner product of (2.9)-(2.11) with a stress test function (using Streamline Upwind Petrov Galerkin (SUPG) stabilization), a velocity test function, and a pressure test function respectively, we obtain the variational formulation

$$\left(\boldsymbol{\tau} + \widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\tau} + g_a(\boldsymbol{\tau},\nabla\widetilde{\mathbf{b}}) - 2\alpha\mathbf{D}(\mathbf{u}), \boldsymbol{\sigma} + \delta\widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\sigma}\right) = 0, \qquad \forall\, \boldsymbol{\sigma}\in S, \tag{2.13}$$

$$(2(1-\alpha)\mathbf{D}(\mathbf{u}) + \boldsymbol{\tau}, \mathbf{D}(\mathbf{v})) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}), \qquad \forall\, \mathbf{v}\in X, \tag{2.14}$$

$$(\nabla\cdot\mathbf{u}, q) = 0, \qquad \forall\, q\in Q, \tag{2.15}$$

where $\delta > 0$ is a constant. The space Z is the space of weakly divergence free functions. Note that the condition

$$(\nabla\cdot\mathbf{u}, q) = 0, \ \ \forall\, q\in Q, \ \mathbf{u}\in X,$$

is equivalent in a "distributional" sense to

$$(\mathbf{u}, \nabla q) = 0, \ \ \forall\, q\in Q, \ \mathbf{u}\in X, \tag{2.16}$$

where in (2.16), $(\cdot,\cdot)$ denotes the duality pairing between H^{-1} and H_0^1 functions. In addition, note that the velocity and pressure spaces, X and Q, satisfy the *inf-sup* condition

$$\inf_{q\in Q}\ \sup_{\mathbf{v}\in X} \frac{(q,\nabla\cdot\mathbf{v})}{\|q\|_{0,2}\,\|\mathbf{v}\|_{1,2}} \geq \beta > 0. \tag{2.17}$$

Since the inf-sup condition (2.17) holds, an equivalent variational formulation to (2.13)-(2.15) is: *Find* $(\mathbf{u},\boldsymbol{\tau})\in Z\times S$ *such that*

$$\left(\boldsymbol{\tau} + \widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\tau} + g_a(\boldsymbol{\tau},\nabla\widetilde{\mathbf{b}}) - 2\alpha\mathbf{D}(\mathbf{u}), \boldsymbol{\sigma} + \delta\widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\sigma}\right) = 0, \qquad \forall\, \boldsymbol{\sigma}\in S, \tag{2.18}$$

$$(2(1-\alpha)\mathbf{D}(\mathbf{u}) + \boldsymbol{\tau}, \mathbf{D}(\mathbf{v})) = (\mathbf{f},\mathbf{v}), \qquad \forall\, \mathbf{v}\in Z. \tag{2.19}$$

In order to establish existence and uniqueness of $(\mathbf{u},\boldsymbol{\tau},p)$ satisfying (2.13)-(2.15) we require the λ, M, α and $\delta > 0$ satisfy

$$1 \ - \ 2\lambda M d \ - \ \epsilon_1\delta\lambda M d \ > \ 0\,, \tag{2.20}$$

$$1 \ - \ \lambda M d/\epsilon_1 \ - \ \epsilon_2 \ > \ 0\,, \tag{2.21}$$

$$4\alpha(1-\alpha) \ - \ \alpha^2\delta/\epsilon_2 \ > \ 0\,, \tag{2.22}$$

for ϵ_1, $\epsilon_2 > 0$.
A careful analysis presented in Section 3 establishes that (2.20)-(2.22) implies

$$0 < \delta \leq \frac{4(1-\alpha)}{\alpha} \quad \text{ and } 0 < \lambda M d < \frac{\left(\frac{(1-\alpha)\delta}{4\alpha} - 1\right) + \sqrt{\left(\frac{(1-\alpha)\delta}{4\alpha} - 1\right)^2 + \delta\left(\frac{(1-\alpha)\delta}{4\alpha} - 1\right)}}{\delta} \ . \tag{2.23}$$

Lemma 2.1 *Given* $\mathbf{f} \in H^{-1}(\Omega)$*, and* $0 < \alpha < 1, \lambda, M, \delta$ *satisfying (2.23), there exists a unique solution* $(\mathbf{u}, \boldsymbol{\tau}) \in Z \times S$ *satisfying (2.18),(2.19). In addition,*

$$\|\mathbf{u}\|_{1,2} + \|\boldsymbol{\tau}\|_{0,2} + \sqrt{\delta}\lambda\|\mathbf{b} \cdot \nabla\boldsymbol{\tau}\|_{0,2} \le c_1\|\mathbf{f}\|_{-1,2}. \tag{2.24}$$

Proof:
Introduce the bilinear form $A(\cdot,\cdot) : (Z \times S) \times (Z \times S) \to \mathbb{R}$ as

$$\begin{aligned} A((\mathbf{u},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\sigma})) \;&:=\; \left(\boldsymbol{\tau} + \widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau} + g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}) - 2\alpha\mathbf{D}(\mathbf{u}), \boldsymbol{\sigma} + \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\sigma}\right) \\ &\quad + \left(4\alpha(1-\alpha)\mathbf{D}(\mathbf{u}) + 2\alpha\boldsymbol{\tau}, \mathbf{D}(\mathbf{v})\right), \end{aligned} \tag{2.25}$$

and the linear functional $F(\cdot) : (Z \times S) \to \mathbb{R}$ as

$$F((\mathbf{v},\boldsymbol{\sigma})) := 2\alpha(\mathbf{f}, \mathbf{v}). \tag{2.26}$$

Observe that

$$\begin{aligned} A((\mathbf{u},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\sigma})) \;&\le\; \|\boldsymbol{\tau} + \widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau} + g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}) - 2\alpha\mathbf{D}(\mathbf{u})\|_{0,2}\, \|\boldsymbol{\sigma} + \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\sigma}\|_{0,2} \\ &\quad + \|4\alpha(1-\alpha)\mathbf{D}(\mathbf{u}) + 2\alpha\boldsymbol{\tau}\|_{0,2}\, \|\mathbf{D}(\mathbf{v})\|_{0,2} \\ &\le \left(\|\boldsymbol{\tau}\|_{0,2} + \|\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\|_{0,2} + 4\,\lambda\, M\, d\, \|\boldsymbol{\tau}\|_{0,2} + 2\alpha\|\mathbf{D}(\mathbf{u})\|_{0,2}\right) \|\boldsymbol{\sigma} + \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\sigma}\|_{0,2} \\ &\quad + \left(4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u})\|_{0,2} + 2\alpha\|\boldsymbol{\tau}\|_{0,2}\right) \|\mathbf{D}(\mathbf{v})\|_{0,2} \\ &\le C_1 \left((1+2\alpha)\|\boldsymbol{\tau}\|_{0,2} + \|\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\|_{0,2} + 4\,\lambda\, M\, d\, \|\boldsymbol{\tau}\|_{0,2} + 6\alpha\|\mathbf{D}(\mathbf{u})\|_{0,2}\right) \|(\mathbf{v},\boldsymbol{\sigma})\|_{X\times S} \\ &\le C_2\, \|(\mathbf{u},\boldsymbol{\tau})\|_{X\times S}\, \|(\mathbf{v},\boldsymbol{\sigma})\|_{X\times S}, \end{aligned} \tag{2.27}$$

where

$$\|(\mathbf{u},\boldsymbol{\tau})\|_{X\times S} = \left(\|\mathbf{u}\|_{1,2}^2 + \|\boldsymbol{\tau}\|_{\mathbf{b}}^2\right)^{1/2}.$$

Hence $A(\cdot,\cdot)$ is bounded on $(Z \times S) \times (Z \times S)$.

Also,

$$\begin{aligned} |F((\mathbf{v},\boldsymbol{\sigma}))| = |2\alpha(\mathbf{f},\mathbf{v})| &\le 2\alpha\|\mathbf{f}\|_{-1,2}\|\mathbf{v}\|_{1,2} \\ &\le 2\alpha\|\mathbf{f}\|_{-1,2}\|(\mathbf{v},\boldsymbol{\sigma})\|_{X\times S}. \end{aligned} \tag{2.28}$$

Thus $F(\cdot)$ is bounded on $(Z \times S)$.

Next we show $A(\cdot,\cdot)$ is coercive on $(Z \times S) \times (Z \times S)$. Note that as $\mathbf{b} = \mathbf{0}$ on $\partial\Omega$ and $\nabla \cdot \mathbf{b} = 0$, then on integrating by parts one can show

$$(\boldsymbol{\tau}, \mathbf{b} \cdot \nabla\boldsymbol{\tau}) \;=\; -(\boldsymbol{\tau}, \mathbf{b} \cdot \nabla\boldsymbol{\tau}) \quad\Rightarrow\quad (\boldsymbol{\tau}, \mathbf{b} \cdot \nabla\boldsymbol{\tau}) = 0. \tag{2.29}$$

Thus,

$$\begin{aligned} A((\mathbf{u},\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\tau})) \;&=\; \left(\boldsymbol{\tau} + \widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau} + g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}) - 2\alpha\mathbf{D}(\mathbf{u}), \boldsymbol{\tau} + \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\right) \\ &\quad + \left(4\alpha(1-\alpha)\mathbf{D}(\mathbf{u}) + 2\alpha\boldsymbol{\tau}, \mathbf{D}(\mathbf{u})\right) \\ &= \|\boldsymbol{\tau}\|_{0,2}^2 + \delta\lambda^2\|\mathbf{b} \cdot \nabla\boldsymbol{\tau}\|_{0,2}^2 + \left(g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}), \boldsymbol{\tau} + \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\right) \\ &\quad -2\alpha\left(\mathbf{D}(\mathbf{u}), \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\right) + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u})\|_{0,2}^2. \end{aligned}$$

A straight forward computation shows that

$$|(\boldsymbol{\tau}\nabla\widetilde{\mathbf{b}}, \boldsymbol{\tau})| \;\leq\; \lambda M\, d\, \|\boldsymbol{\tau}\|_{0,2}^2 \,.$$

Similarly,

$$|(\nabla\widetilde{\mathbf{b}}^T\boldsymbol{\tau}, \boldsymbol{\tau})|,\; |(\nabla\widetilde{\mathbf{b}}\boldsymbol{\tau}, \boldsymbol{\tau})|, |\;(\boldsymbol{\tau}\nabla\widetilde{\mathbf{b}}^T, \boldsymbol{\tau})| \;\leq\; \lambda M\, d\, \|\boldsymbol{\tau}\|_{0,2}^2 \,,$$

Hence

$$\begin{aligned} |\left(g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}), \boldsymbol{\tau}\right)| &\leq 2\,\lambda\, M\, d\, \|\boldsymbol{\tau}\|_{0,2}^2 \,, \\ |\left(g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}), \delta\widetilde{\mathbf{b}}\cdot\nabla\boldsymbol{\tau}\right)| &\leq 2\,\lambda\, M\, d\, \|\boldsymbol{\tau}\|_{0,2}\delta\,\lambda\, \|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2} \,. \end{aligned} \tag{2.30}$$

Therefore, using Young's inequality,

$$\begin{aligned} A((\mathbf{u},\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\tau})) &\geq \|\boldsymbol{\tau}\|_{0,2}^2 + \delta\lambda^2\|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2}^2 + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u})\|_{0,2}^2 - 2\,\lambda\, M\, d\, \|\boldsymbol{\tau}\|_{0,2}^2 \\ &\quad -2\,\lambda\, M\, d\, \|\boldsymbol{\tau}\|_{0,2}\, \delta\lambda\|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2} - 2\alpha\|\mathbf{D}(\mathbf{u})\|_{0,2}\, \delta\,\lambda\|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2} \\ &\geq (1 - 2\lambda M d - \epsilon_1\delta\lambda M d)\, \|\boldsymbol{\tau}\|_{0,2}^2 \\ &\quad + \left(1 - \frac{\lambda M d}{\epsilon_1} - \epsilon_2\right) \delta\,\lambda^2\|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2}^2 \\ &\quad + \left(4\alpha(1-\alpha) - \frac{\alpha^2\,\delta}{\epsilon_2}\right) \|\mathbf{D}(\mathbf{u})\|_{0,2}^2 \\ &\geq c\|(\mathbf{u},\boldsymbol{\tau})\|_{M\times S}^2 \,, \end{aligned} \tag{2.31}$$

provided λ, M and α are such that δ, ϵ_1, $\epsilon_2 > 0$ can be chosen in order that

$$\min\{(1 - 2\lambda M d - \epsilon_1\delta\lambda M d)\,,\, (\lambda - \lambda M d/\epsilon_1 - \epsilon_2)\,,\, \left(4\alpha(1-\alpha) - \alpha^2\delta/\epsilon_2\right)\} > 0\,. \tag{2.32}$$

Applying the Lax-Milgram theorem we have that there exists a unique solution $(\mathbf{u},\boldsymbol{\tau}) \in (Z\times S)$ satisfying $A((\mathbf{u},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\sigma})) = F((\mathbf{v},\boldsymbol{\sigma}))$, $\forall\;\; (\mathbf{v},\boldsymbol{\sigma}) \in (Z\times S)$. Finally, combining the coercivity estimates (2.31) with (2.28) we obtain (2.24).

■

Theorem 2.1 *Given $f \in H^{-1}$, and $0 < \alpha < 1, \lambda, M, \delta$ satisfying (2.23), there exists a unique solution $(\mathbf{u},\boldsymbol{\tau},p) \in X\times Z\times Q$ satisfying (2.13)-(2.15). In addition,*

$$\|\mathbf{u}\|_{1,2} + \|\boldsymbol{\tau}\|_{0,2} + \sqrt{\delta}\lambda\|\mathbf{b}\cdot\nabla\boldsymbol{\tau}\|_{0,2} + \|p\|_{0,2} \leq c\, \|\mathbf{f}\|_{-1,2}. \tag{2.33}$$

Proof:
Note that as Z is a closed subspace of the Hilbert space X, we can write

$$X = Z \oplus Z^{\perp}. \tag{2.34}$$

In view of Lemma 2.1, what remains to be shown is the existence and uniqueness of $p \in Q$ satisfying

$$b(\mathbf{v},p) := (\nabla\cdot\mathbf{v}, p) = \widetilde{F}(\mathbf{v}) := -(\mathbf{f},\mathbf{v}) + (2(1-\alpha)\mathbf{D}(\mathbf{u}) + \boldsymbol{\tau}, \mathbf{D}(\mathbf{v})). \tag{2.35}$$

Note that for $b : (X \times Q) \to \mathbb{R}$,

$$\begin{aligned} |b(\mathbf{v}, p)| &\leq \|p\|_{0,2} \|\nabla \cdot \mathbf{v}\|_{0,2} \\ &\leq d^{1/2} \|p\|_{0,2} \|\mathbf{v}\|_{1,2}, \end{aligned} \tag{2.36}$$

i.e. $b(\cdot)$ is a bounded linear operator on $X \times Q$.

From the inf-sup condition (2.17) we have

$$\begin{aligned} \beta \|p\|_{0,2} &\leq \sup_{\mathbf{v} \in H_0^1} \frac{b(\mathbf{v}, p)}{\|\mathbf{v}\|_{1,2}} = \sup_{\mathbf{v} \in H_0^1} \frac{\widetilde{F}(\mathbf{v})}{\|\mathbf{v}\|_{1,2}} \\ &\leq \sup_{\mathbf{v} \in H_0^1} \frac{(\|\mathbf{f}\|_{-1,2} + 2(1-\alpha)\|\mathbf{u}\|_{1,2} + \|\boldsymbol{\tau}\|_{0,2})\|\mathbf{v}\|_{1,2}}{\|\mathbf{v}\|_{1,2}} \\ &\leq \|\mathbf{f}\|_{-1,2} + 2(1-\alpha)\|\mathbf{u}\|_{1,2} + \|\boldsymbol{\tau}\|_{0,2}. \end{aligned} \tag{2.37}$$

Estimate (2.33) then follows from (2.37) and (2.24). ■

Remark: For convenience, in Assumption A, M is used as a bound for both $\|\mathbf{b}\|_\infty$ and $\|\nabla \mathbf{b}\|_\infty$. In (2.32) the M occuring in the coercivity condition arises only from the $\|\nabla \mathbf{b}\|_\infty$ term. Thus, it is the magnitude of the gradient of $\mathbf{b}$ and not the magnitude of $\mathbf{b}$ which plays a fundamental role in the solvability of (2.13)-(2.15).

2.1 Finite Element Approximation

In this section we describe the finite element approximation to (2.5)-(2.8). We begin by describing the mathematical framework and the approximation properties.

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a polygonal domain and let Π_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrals (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\Omega = \bigcup T; \quad T \in \Pi_h.$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \leq h_T \leq c_2 \rho_T$$

where h_T is the diameter of triangle (tetrahedral) T, ρ_T is the diameter of the greatest ball (sphere) included in T, and $h = \max_{T \in \Pi_h} h_T$. Let $\mathbb{P}_k(A)$ denote the space of polynomials on A of degree no greater than k. Then we define the finite element spaces as follows.

$$\begin{aligned} X_h &:= \left\{ \mathbf{v} \in X \cap C(\bar{\Omega})^d : \mathbf{v}|_T \in \mathbb{P}_k(T), \ \forall \, T \in \Pi_h \right\}, \\ S_h &:= \left\{ \boldsymbol{\sigma} \in S \cap C(\bar{\Omega})^{d \times d} : \boldsymbol{\sigma}|_T \in \mathbb{P}_m(T), \ \forall \, T \in \Pi_h \right\}, \\ Q_h &:= \left\{ q \in Q \cap C(\bar{\Omega}) : q|_T \in \mathbb{P}_r(T), \ \forall \, T \in \Pi_h \right\}, \\ Z_h &:= \left\{ \mathbf{v} \in X_h : (q, \nabla \cdot \mathbf{v}) = 0, \ \forall \, q \in Q_h \right\}, \end{aligned}$$

where $C(\bar{\Omega})^d$ denotes a vector valued function with d components continuous on $\bar{\Omega}$. Analogous to the continuous spaces, we assume that X_h and Q_h satisfy the discrete *inf-sup* condition

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\|_{0,2} \|\mathbf{v}\|_{1,2}} \geq \beta > 0 . \tag{2.38}$$

We summarize several properties of finite element spaces and Sobolev spaces which we will use in our subsequent analysis. For $(\mathbf{u}, p) \in \left(H^{k+1}(\Omega)\right)^d \times H^{r+1}(\Omega)$ we have (see [6]) that there exists $(\mathcal{U}, \mathcal{P}) \in Z_h \times Q_h$ such that

$$\|\mathbf{u} - \mathcal{U}\|_{0,2} \leq C_I h^{k+1} \|\mathbf{u}\|_{k+1,2} , \tag{2.39}$$
$$\|\mathbf{u} - \mathcal{U}\|_{1,2} \leq C_I h^{k} \|\mathbf{u}\|_{k+1,2} , \tag{2.40}$$
$$\|p - \mathcal{P}\|_{0,2} \leq C_I h^{r+1} \|p\|_{r+1,2} . \tag{2.41}$$

Let $\mathcal{T} \in S_h$ be a $\mathbb{P}_1$ continuous interpolant of $\boldsymbol{\tau}$. For $\boldsymbol{\tau} \in H^{m+1}(\Omega)^{d \times d}$ we have (see [6]) that

$$\|\boldsymbol{\tau} - \mathcal{T}\|_{0,2} + h|\boldsymbol{\tau} - \mathcal{T}|_{1,2} \leq C_I h^{m+1} \|\boldsymbol{\tau}\|_{m+1,2} . \tag{2.42}$$

Finite Element Approximation:

Given $\mathbf{f} \in H^{-1}(\Omega)$, find $\mathbf{u}_h \in X_h$, $\boldsymbol{\tau}_h \in S_h$, and $p_h \in Q_h$ such that

$$(\boldsymbol{\tau}_h + \lambda (\mathbf{b} \cdot \nabla \boldsymbol{\tau}_h + g_a(\boldsymbol{\tau}_h, \nabla \mathbf{b})) - 2\alpha \mathbf{D}(\mathbf{u}_h), \boldsymbol{\sigma} + \delta\lambda \mathbf{b} \cdot \nabla \boldsymbol{\sigma}) = 0, \qquad \forall\, \boldsymbol{\sigma} \in S_h, \tag{2.43}$$
$$(2(1-\alpha)\mathbf{D}(\mathbf{u}_h) + \boldsymbol{\tau}_h, \mathbf{D}(\mathbf{v})) - (p_h, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall\, \mathbf{v} \in X_h, \tag{2.44}$$
$$(\nabla \cdot \mathbf{u}_h, q) = 0, \qquad \forall\, q \in Q_h. \tag{2.45}$$

Equivalently, given $f \in H^{-1}(\Omega)$, find $\mathbf{u}_h \in Z_h$, $\boldsymbol{\tau}_h \in S_h$, such that

$$(\boldsymbol{\tau}_h + \lambda (\mathbf{b} \cdot \nabla \boldsymbol{\tau}_h + g_a(\boldsymbol{\tau}_h, \nabla \mathbf{b})) - 2\alpha \mathbf{D}(\mathbf{u}_h), \boldsymbol{\sigma} + \delta\lambda \mathbf{b} \cdot \nabla \boldsymbol{\sigma}) = 0, \qquad \forall\, \boldsymbol{\sigma} \in S_h, \tag{2.46}$$
$$(2(1-\alpha)\mathbf{D}(\mathbf{u}_h) + \boldsymbol{\tau}_h, \mathbf{D}(\mathbf{v})) = (\mathbf{f}, \mathbf{v}), \qquad \forall\, \mathbf{v} \in Z_h. \tag{2.47}$$

Theorem 2.2 *Given $\mathbf{f} \in H^{-1}(\Omega)$, and $0 < \alpha < 1, \lambda, M, \delta$ satisfying (2.23), there exists a unique solution $(\mathbf{u}_h, \boldsymbol{\tau}_h, p_h) \in X_h \times S_h \times Q_h$ satisfying (2.43),(2.45), and.*

$$\|\mathbf{u}_h\|_{1,2} + \|\boldsymbol{\tau}_h\|_{0,2} + \sqrt{\delta}\lambda \|\mathbf{b} \cdot \nabla \boldsymbol{\tau}_h\|_{0,2} + \|p_h\|_{0,2} \leq c\|\mathbf{f}\|_{-1,2}. \tag{2.48}$$

Proof:
The proof is analogous to that for the continuous problem.

■

2.2 A Priori Error Estimates

For the finite element approximation $(\mathbf{u}_h, \boldsymbol{\tau}_h)$ defined in (2.46),(2.47) we have the following a priori error estimate.

Theorem 2.3 *For* $(\mathbf{u}_h, \boldsymbol{\tau}_h)$ *satisfying (2.46)-(2.47), and* $(\mathbf{u}, \boldsymbol{\tau}, p) \in (H^{k+1}(\Omega) \times H^{m+1}(\Omega) \times Q)$ *satisfying (2.13)-(2.14), we have the following error estimate*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,2} + \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{0,2} + \sqrt{\delta}\lambda\|\mathbf{b}\cdot\nabla(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\|_{0,2} \;&\leq\; c_1\, h^k \|\mathbf{u}\|_{k+1,2} + c_2\, h^m \|\boldsymbol{\tau}\|_{m+1,2} \\ &\quad + \, c_3 \inf_{q\in Q_h} \|p - q\|_{0,2}, \end{aligned} \tag{2.49}$$

where k *and* m *are the polynomial degrees of* $\mathbf{u}_h$ *and* $\boldsymbol{\tau}_h$ *respectively.*

Proof:
Let

$$\begin{aligned} \boldsymbol{\Lambda} &= \mathbf{u} - \mathcal{U}, & \mathbf{E} &= \mathcal{U} - \mathbf{u}_h, \\ \boldsymbol{\Gamma} &= \boldsymbol{\tau} - \mathcal{T}, & \mathbf{F} &= \mathcal{T} - \boldsymbol{\tau}_h, \\ \boldsymbol{\epsilon}_{\mathbf{u}} &= \mathbf{u} - \mathbf{u}_h, & \boldsymbol{\epsilon}_{\boldsymbol{\tau}} &= \boldsymbol{\tau} - \boldsymbol{\tau}_h. \end{aligned}$$

With $A(\cdot,\cdot)$ defined as in (2.25), from (2.13)-(2.14) and (2.46)-(2.47), we have

$$A((\boldsymbol{\epsilon}_{\mathbf{u}}, \boldsymbol{\epsilon}_{\boldsymbol{\tau}}), (\mathbf{v}, \boldsymbol{\sigma})) = (p, \nabla\cdot\mathbf{v}), \qquad \forall\, (\mathbf{v}, \boldsymbol{\sigma}) \in Z_h \times S_h,$$

and using the orthogonality of $\nabla\cdot\mathbf{v}$ to all $q \in Q_h$,

$$A((\mathbf{E}, \mathbf{F}), (\mathbf{v}, \boldsymbol{\sigma})) = (p - q, \nabla\cdot\mathbf{v}) - A((\boldsymbol{\Lambda}, \boldsymbol{\Gamma}), (\mathbf{v}, \boldsymbol{\sigma})), \qquad \forall\, (\mathbf{v}, \boldsymbol{\sigma}, q) \in Z_h \times S_h \times Q_h.$$

With the choice $(\mathbf{v}, \boldsymbol{\sigma}) = (\mathbf{E}, \mathbf{F})$, in view of (2.31) and (2.27) we have

$$c\,\|(\mathbf{E}, \mathbf{F})\|^2_{X\times S} \;\leq\; \|p - q\|_{0,2}\,\|\nabla\cdot\mathbf{E}\|_{0,2} \;+\; C_2\,\|(\mathbf{E}, \mathbf{F})\|_{X\times S}\,\|(\boldsymbol{\Lambda}, \boldsymbol{\Gamma})\|_{X\times S}\;.$$

Using Korn's inequality [3] with constant C_k,

$$\begin{aligned} \|p - q\|_{0,2}\,\|\nabla\cdot\mathbf{E}\|_{0,2} \;&\leq\; \|p - q\|_{0,2}\, d^{1/2}\|\nabla\mathbf{E}\|_{0,2} \;\leq\; d^{1/2}\|p - q\|_{0,2}\, C_k\|\mathbf{D}(\mathbf{E})\|_{0,2} \\ &\leq\; d^{1/2} C_k\,\|p - q\|_{0,2}\,\|(\mathbf{E}, \mathbf{F})\|_{X\times S}. \end{aligned} \tag{2.50}$$

Thus we have the estimate

$$\|(\mathbf{E}, \mathbf{F})\|_{X\times S} \;\leq\; \frac{C_2}{c}\|(\boldsymbol{\Lambda}, \boldsymbol{\Gamma})\|_{X\times S} \;+\; \frac{d^{1/2}C_k}{c}\inf_{q\in Q_h}\|p - q\|_{0,2}\;.$$

Finally, using the triangle inequality

$$\begin{aligned} \|(\boldsymbol{\epsilon}_{\mathbf{u}}, \boldsymbol{\epsilon}_{\boldsymbol{\tau}})\|_{X\times S} \;&\leq\; \|(\mathbf{E}, \mathbf{F})\|_{X\times S} \;+\; \|(\boldsymbol{\Lambda}, \boldsymbol{\Gamma})\|_{X\times S} \\ &\leq\; \left(1 + \frac{C_2}{c}\right)\|(\boldsymbol{\Lambda}, \boldsymbol{\Gamma})\|_{X\times S} \;+\; \frac{d^{1/2}C_k}{c}\inf_{q\in Q_h}\|p - q\|_{0,2}\,, \end{aligned}$$

from which (2.49) follows using the approximation properties (2.40) and (2.42). ■

Corollary 2.1 *Given* $(\mathbf{u}, \boldsymbol{\tau}, p) \in X\times S\times Q$ *satisfying (2.13)-(2.15) and* $(\mathbf{u}_h, \boldsymbol{\tau}_h, p_h) \in X_h\times S_h\times Q_h$ *satisfying (2.43)-(2.45), with* $(\mathbf{u}_h, \boldsymbol{\tau}_h)$ *satisfying the error estimate of Theorem 2.3 and* $p \in H^{r+1}(\Omega)$*, we have the error estimate*

$$\|p - p_h\|_{0,2} \leq c_4\, h^{r+1}\|p\|_{r+1,2} + c_5\, h^k\|\mathbf{u}\|_{k+1,2} + c_6\, h^m\|\boldsymbol{\tau}\|_{m+1,2}. \tag{2.51}$$

where r *is the polynomial degree of* p_h*.*

Proof:
For $\mathbf{v} \in X$, $p \in Q$, and the bilinear form $b(\cdot,\cdot) : X \times Q \to \mathrm{I\!R}$,

$$b(\mathbf{v}, p) = (\nabla \cdot \mathbf{v}, p), \tag{2.52}$$

then for any $\widetilde{p} \in Q_h$, we have from (2.17) and (2.14) that

$$\begin{aligned}
\beta \|\widetilde{p} - p_h\|_{0,2} &\leq \sup_{\mathbf{v} \in X_h} \frac{|b(\mathbf{v}, \widetilde{p} - p_h)|}{\|\mathbf{v}\|_{1,2}} \\
&\leq \sup_{\mathbf{v} \in X_h} \frac{|b(\mathbf{v}, p - p_h) + b(\mathbf{v}, \widetilde{p} - p)|}{\|\mathbf{v}\|_{1,2}} \\
&\leq \sup_{\mathbf{v} \in X_h} \frac{|(2(1-\alpha)\mathbf{D}(u - \mathbf{u}_h) + (\boldsymbol{\tau} - \boldsymbol{\tau}_h), \mathbf{D}(\mathbf{v})) + b(\mathbf{v}, \widetilde{p} - p)|}{\|\mathbf{v}\|_{1,2}} \\
&\leq C\,(2(1-\alpha)\|\mathbf{u} - \mathbf{u}_h\|_{1,2} + \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{0,2} + \|\widetilde{p} - p\|_{0,2}).
\end{aligned} \tag{2.53}$$

Using the triangle inequality, (2.53), (2.41) and (2.49) we obtain

$$\begin{aligned}
\|p - p_h\|_{0,2} &\leq \|p - \mathcal{P}\|_{0,2} + \|\mathcal{P} - p_h\|_{0,2} \\
&\leq (1 + \frac{C}{\beta})\|p - \mathcal{P}\|_{0,2} + \frac{C}{\beta}(2(1-\alpha)\|\mathbf{u} - \mathbf{u}_h\|_{1,2} + \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{0,2}) \\
&\leq c_I(1 + \frac{C}{\beta})h^{r+1}\|p\|_{r+1,2} + \frac{C}{\beta}\widetilde{C}(c_1 h^k \|\mathbf{u}\|_{k+1,2} + c_2 h^m \|\boldsymbol{\tau}\|_{m+1,2}) \\
&\leq c_4\, h^{r+1}\|p\|_{r+1,2} + c_5\, h^k \|\mathbf{u}\|_{k+1,2} + c_6\, h^m \|\boldsymbol{\tau}\|_{m+1,2}.
\end{aligned} \tag{2.54}$$

■

2.3 Numerical Experiments

In this section we present numerical results for the Oseen-viscoelasticity problem. For the numerical approximations we used as the approximation spaces for the velocity and pressure the Taylor-Hood pair; continuous piecewise quadratics ($k = 2$) and continuous piecewise linears ($r = 1$), respectively. For the polymetric stress tensor computations were performed for the approximation spaces comprised of (i) continuous piecewise linear elements ($m = 1$), and (ii) continuous piecewise quadratic elements ($m = 2$). The numerical results were computed for $\lambda = 0.1$ and $\lambda = 0.5$ using five different meshes ($\Pi_1, \Pi_2, \ldots, \Pi_5$), with spatial mesh sizes $h, h/2, \ldots, h/16$, respectively. Associated with the approximations $\mathbf{x}_h$ and $\mathbf{x}_{h/2}$ of the variable $\mathbf{x} \in X$, the experimental rate of convergence ER_h is given by

$$ER_h := \log(\|\mathbf{x} - \mathbf{x}_h\|_X \,/ \|\mathbf{x} - \mathbf{x}_{h/2}\|_X) / \log 2\,.$$

In the examples the fixed flow field $\mathbf{b}$ was chosen to be the true velocity field u. In all the computations $\delta = 1/2$, and $\alpha = 0.41$ (for the Boger Fluid) were used .

Example 1.
The unit square, $\Omega = (0,1) \times (0,1)$ was used as the computational domain. The true velocity, polymetric stress, and pressure used were

$$\mathbf{u}(x,y) := \frac{1}{4}\begin{bmatrix} x^3 + x^2 - 2xy + x \\ -3x^2y + y^2 - 2xy - y \end{bmatrix}, \qquad \boldsymbol{\tau}(x,y) := 2\alpha\,\mathbf{D}(u), \qquad p(x,y) := x^2 + y^2.$$

	P/w linear approx. for Stress ($m = 1$)				P/w quadratic approx. for Stress ($m = 2$)			
	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 0.1$		$\lambda = 0.5$	
h	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}
h	0.9552	–	0.9696	–	0.9554	–	0.9567	–
$h/2$	0.2317	2.0	0.2365	2.0	0.2311	2.0	0.2319	2.0
$h/4$	0.0575	2.0	0.0585	2.0	0.0570	2.0	0.0572	2.0
$h/8$	0.0144	2.0	0.0146	2.0	0.0142	2.0	0.0142	2.0
$h/16$	0.0036	2.0	0.0036	2.0	0.0035	2.0	0.0035	2.0
Predicted		≥ 1.0				≥ 2.0		

Table 2.1: Example 1: Velocity Error, and Experimental Convergence Rate (where $\|\mathbf{u} - \mathbf{u}_h\|_{H^1} = 10^{-2} * E(\mathbf{u}_h)_{H^1}$).

	P/w linear approx. for Stress ($m = 1$)				P/w quadratic approx. for Stress ($m = 2$)			
	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 0.1$		$\lambda = 0.5$	
h	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}
h	0.9454	–	2.4370	–	0.2654	–	0.2784	–
$h/2$	0.3005	1.7	1.1223	1.1	0.0739	1.8	0.0679	2.0
$h/4$	0.1194	1.3	0.5433	1.0	0.0186	2.0	0.0160	2.1
$h/8$	0.0550	1.1	0.2679	1.0	0.0045	2.1	0.0037	2.1
$h/16$	0.0268	1.0	0.1331	1.0	0.0010	2.1	0.0009	2.1
Predicted		≥ 1.0				≥ 2.0		

Table 2.2: Example 1: Stress Error, and Experimental Convergence Rate (where $\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{b}} = 10^{-2} * E(\boldsymbol{\tau}_h)_{\mathbf{b}}$).

Numerical results are presented in Tables 2.1-2.3.

Example 2.

For this example Ω an L-shaped domain given by $\Omega = (-1, 1) \times (-1, 1) - (0, 1) \times (0, 1)$ was used. The true velocity, polymetric stress, and pressure were

$$\mathbf{u}(x, y) := \frac{\sqrt{2}}{10} \begin{bmatrix} \frac{(y-0.1)}{[(x-0.1)^2+(y-0.1)^2]^{1/2}} \\ \\ \frac{(0.1-x)}{[(x-0.1)^2+(y-0.1)^2]^{1/2}} \end{bmatrix}, \qquad \boldsymbol{\tau} := 2\alpha\, \mathbf{D}(u), \qquad p(x, y) := (2 - x - y)^{1/2}.$$

Numerical results are presented in Tables 2.4-2.6.

From Theorem 2.3 and Corollary 2.1 we have that the velocity, stress and pressure errors are bounded by $Ch^{\min\{k,m,r+1\}}$. Hence (asymptotically) $ER_h \sim \min\{k, m, r + 1\}$. In Example 1, where the solution for the velocity, stress, and pressure are polynomials, the predicted error estimate is clearly demonstrated. For Example 2 the results are consistent with those predicted. For this example, due to the influence of the singular point of the solution which lies just outside the domain, the asymptotic behavior of the error is not completely dominant on the computational meshes used.

For both examples the case $\lambda = 0.5$ lies outside the parameter space (3.1)-(3.3) for which the solution is guaranteed to exist and the convergence rates apply.

	P/w linear approx. for Stress ($m = 1$)				P/w quadratic approx. for Stress ($m = 2$)			
	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 0.1$		$\lambda = 0.5$	
h	$E(p_h)_{L^2}$	ER_{h_i}	$E(p_h)_{L^2}$	ER_{h_i}	$E(p_h)_{L^2}$	ER_{h_i}	$E(p_h)_{L^2}$	ER_{h_i}
h	3.7596	–	3.2317	–	4.5995	–	4.5115	–
$h/2$	0.8974	2.1	0.7868	2.0	1.1452	2.0	1.1138	2.0
$h/4$	0.2220	2.0	0.2084	1.9	0.2839	2.0	0.2777	2.0
$h/8$	0.0559	2.0	0.0539	1.9	0.0704	2.0	0.0694	2.0
$h/16$	0.0139	2.0	0.0137	2.0	0.0175	2.0	0.0173	2.0
Predicted		≥ 1.0				≥ 2.0		

Table 2.3: Example 1: Pressure Error, and Experimental Convergence Rate (where $\|p - p_h\|_{L^2} = 10^{-2} * E(p_h)_{L^2}$).

	P/w linear approx. for Stress ($m = 1$)				P/w quadratic approx. for Stress ($m = 2$)			
	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 0.1$		$\lambda = 0.5$	
h	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}	$E(\mathbf{u}_h)_{H^1}$	ER_{h_i}
h	1.0104	–	1.0114	–	0.9061	–	0.9135	–
$h/2$	0.3838	1.4	0.3935	1.4	0.3329	1.4	0.3348	1.4
$h/4$	0.1090	1.8	0.1262	1.6	0.0994	1.7	0.1009	1.7
$h/8$	0.0289	1.9	0.0393	1.7	0.0262	1.9	0.0335	1.6
$h/16$	0.0077	1.9	0.0200	1.0	0.0069	1.9	0.0156	1.1
Predicted		≥ 1.0				≥ 2.0		

Table 2.4: Example 2: Velocity Error, and Experimental Convergence Rate (where $\|\mathbf{u} - \mathbf{u}_h\|_{H^1} = 10^{-1} * E(\mathbf{u}_h)_{H^1}$).

3 Coercivity of $A(\cdot,\cdot) : (Z \times S) \times (Z \times S) \rightarrow \mathbb{R}$

In establishing the existence and uniqueness of the variational solution of (2.18),(2.19), we require, from (2.32), that λ, M, α and δ satisfy

$$1 - 2\widetilde{M} - \epsilon_1 \delta \widetilde{M} \quad > \quad 0\,, \tag{3.1}$$

$$1 - \widetilde{M}/\epsilon_1 - \epsilon_2 \quad > \quad 0\,, \tag{3.2}$$

$$4\alpha(1-\alpha) - \alpha^2 \delta/\epsilon_2 \quad > \quad 0\,, \tag{3.3}$$

where $\widetilde{M} = \lambda\, M\, d$, and $\delta, \epsilon_1, \epsilon_2 > 0$.

From (3.1) and (3.3), respectively, we obtain

$$0 < \quad \epsilon_1 \quad < \frac{1 - 2\widetilde{M}}{\delta \widetilde{M}}, \tag{3.4}$$

$$0 < \quad \tfrac{\alpha\,\delta}{4(1-\alpha)} \quad < \epsilon_2\ . \tag{3.5}$$

Substituting (3.4),(3.5) into (3.2) and assuming that $1 - 2\widetilde{M} > 0$, we obtain the condition

$$-\delta \widetilde{M}^2 + 2(\widetilde{\alpha}\frac{\delta}{4} - 1)\widetilde{M} + (1 - \widetilde{\alpha}\frac{\delta}{4}) > 0, \tag{3.6}$$

	P/w linear approx. for Stress ($m=1$)				P/w quadratic approx. for Stress ($m=2$)			
	$\lambda=0.1$		$\lambda=0.5$		$\lambda=0.1$		$\lambda=0.5$	
h	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}	$E(\boldsymbol{\tau}_h)_{\mathbf{b}}$	ER_{h_i}
h	1.1359	–	1.5018	–	0.4852	–	0.8418	–
$h/2$	0.4024	1.5	0.8504	0.82	0.1578	1.6	0.3882	1.1
$h/4$	0.1443	1.5	0.4750	0.84	0.0456	1.8	0.1448	1.4
$h/8$	0.0584	1.3	0.2682	0.82	0.0123	1.9	0.0962	0.59
$h/16$	0.0268	1.1	0.1667	0.68	0.0044	1.5	0.0686	0.49
Predicted		≥ 1.0				≥ 2.0		

Table 2.5: Example 2: Stress Error, and Experimental Convergence Rate (where $\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{\mathbf{b}} = 10^{-1} * E(\boldsymbol{\tau}_h)_{\mathbf{b}}$).

	P/w linear approx. for Stress ($m=1$)				P/w quadratic approx. for Stress ($m=2$)			
	$\lambda=0.1$		$\lambda=0.5$		$\lambda=0.1$		$\lambda=0.5$	
h	$E(p_h)_{L^2}$	ER_{h_i}	$E(p_h)_{L^2}$	ER_{h_i}	$E(p_h)_{L^2}$	ER_{h_i}	$E(p_h)_{L^2}$	ER_{h_i}
h	0.5005	–	0.4843	–	0.3922	–	0.3485	–
$h/2$	0.1721	1.5	0.2056	1.2	0.1230	1.7	0.1131	1.6
$h/4$	0.0360	2.3	0.0621	1.7	0.0474	1.4	0.0387	1.5
$h/8$	0.0090	2.0	0.0308	1.0	0.0094	2.3	0.0185	1.1
$h/16$	0.0033	1.5	0.0257	0.26	0.0028	1.8	0.0118	0.64
Predicted		≥ 1.0				≥ 2.0		

Table 2.6: Example 2: Pressure Error, and Experimental Convergence Rate (where $\|p-p_h\|_{L^2} = 10^{-1} * E(p_h)_{L^2}$).

where

$$\widetilde{\alpha} = \frac{\alpha}{(1-\alpha)}.$$

Physically, $\widetilde{\alpha}$ represents the ratio of the elastic component of the fluid viscosity to the Newtonian component of the fluid viscosity, i.e.

$$\widetilde{\alpha} = \frac{\eta_e}{\eta_s}, \tag{3.7}$$

where η_e and η_s represent the elastic and Newtonian components of the fluid viscosity, respectively.

For the corresponding quadratic equation to (3.6)

$$-\delta\widetilde{M}^2 + 2(\widetilde{\alpha}\frac{\delta}{4} - 1)\widetilde{M} + (1 - \widetilde{\alpha}\frac{\delta}{4}) = 0, \tag{3.8}$$

the roots are given by

$$\widetilde{M}_{1,2} = \frac{(\widetilde{\alpha}\frac{\delta}{4} - 1) \pm \sqrt{(\widetilde{\alpha}\frac{\delta}{4} - 1)^2 + \delta(1 - \widetilde{\alpha}\frac{\delta}{4})}}{\delta} \tag{3.9}$$

As the quadratic coefficient in (3.6) is negative, there exists non-negative values of λ, M, α, δ satisfying (3.1)-(3.3) provided the discriminant ≥ 0, i.e.

$$\frac{\widetilde{\alpha}}{4}(\frac{\widetilde{\alpha}}{4} - 1)\delta^2 \, + \, (1 - \frac{\widetilde{\alpha}}{2})\delta + 1 \geq 0. \tag{3.10}$$

For the corresponding quadratic equation

$$\frac{\widetilde{\alpha}}{4}(\frac{\widetilde{\alpha}}{4} - 1)\delta^2 \, + \, (1 - \frac{\widetilde{\alpha}}{2})\delta + 1 = 0, \tag{3.11}$$

the roots for δ are given by

$$\delta_1 = \frac{4}{\widetilde{\alpha}} \quad \text{and} \quad \delta_2 = \frac{4}{\widetilde{\alpha} - 4}. \tag{3.12}$$

Note that the sign of the coefficient of δ^2 in (3.11) and the sign of δ_2 both change at $\widetilde{\alpha} = 4$. Also, as $\widetilde{\alpha} = \alpha/(1-\alpha)$ and $0 < \alpha < 1$, we have that $0 < \widetilde{\alpha} < \infty$. Furthermore, from (3.2) we have $\epsilon_2 < 1$ which together with (3.5) implies $\delta < 4/\widetilde{\alpha} \, = \, \delta_1$.

Thus, combining (3.12) and (3.10) physically relevant values for $\widetilde{\alpha}$, δ and $\widetilde{M}$ satisfying (3.1)-(3.3) are:

$$0 < \widetilde{\alpha} \leq 4, \qquad 0 \leq \delta \leq \frac{4}{\widetilde{\alpha}}, \qquad \widetilde{M_1} < \widetilde{M} < \widetilde{M_2}$$

where,

$$\widetilde{M_1} = \max\left\{0\,, \, \frac{(\widetilde{\alpha}\frac{\delta}{4} - 1) - \sqrt{(\widetilde{\alpha}\frac{\delta}{4} - 1)^2 \, + \, \delta(1 - \widetilde{\alpha}\frac{\delta}{4})}}{\delta}\right\} \, = \, 0, \tag{3.13}$$

$$\widetilde{M_2} = \min\left\{\frac{1}{2}\,, \, \frac{(\widetilde{\alpha}\frac{\delta}{4} - 1) + \sqrt{(\widetilde{\alpha}\frac{\delta}{4} - 1)^2 \, + \, \delta(1 - \widetilde{\alpha}\frac{\delta}{4})}}{\delta}\right\} \, = \, \frac{(\widetilde{\alpha}\frac{\delta}{4} - 1) + \sqrt{(\widetilde{\alpha}\frac{\delta}{4} - 1)^2 \, + \, \delta(1 - \widetilde{\alpha}\frac{\delta}{4})}}{\delta}. \tag{3.14}$$

Thus for

$$0 < \widetilde{\alpha} < \infty, \qquad 0 < \delta \leq \frac{4}{\widetilde{\alpha}}, \qquad 0 < \widetilde{M} < \frac{(\widetilde{\alpha}\frac{\delta}{4} - 1) + \sqrt{(\widetilde{\alpha}\frac{\delta}{4} - 1)^2 \, + \, \delta(1 - \widetilde{\alpha}\frac{\delta}{4})}}{\delta} \tag{3.15}$$

equations (3.1)-(3.3) are satisfied, guaranteing existence and uniqueness of the solutions to (2.18)-(2.19) and (2.46)-(2.47).

The parameter space for $\widetilde{M}$, $\widetilde{\alpha}$, and δ defined by (3.15) is represented as the region between the two surfaces in Figure 3.1.

The SUPG term $\delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\sigma}$ provides both a stabilizing and a destabilizing effect in the analysis. The stabilizing part comes from the inner product $\left(\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\,, \, \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\sigma}\right)$, and the destabilizing part from $\left(g_a(\boldsymbol{\tau}, \nabla\widetilde{\mathbf{b}}), \delta\widetilde{\mathbf{b}} \cdot \nabla\boldsymbol{\tau}\right)$. In the analysis the destabilizing term must be bounded from above (see (2.30)). Consequently, as M increases the range of possible values for δ decreases.

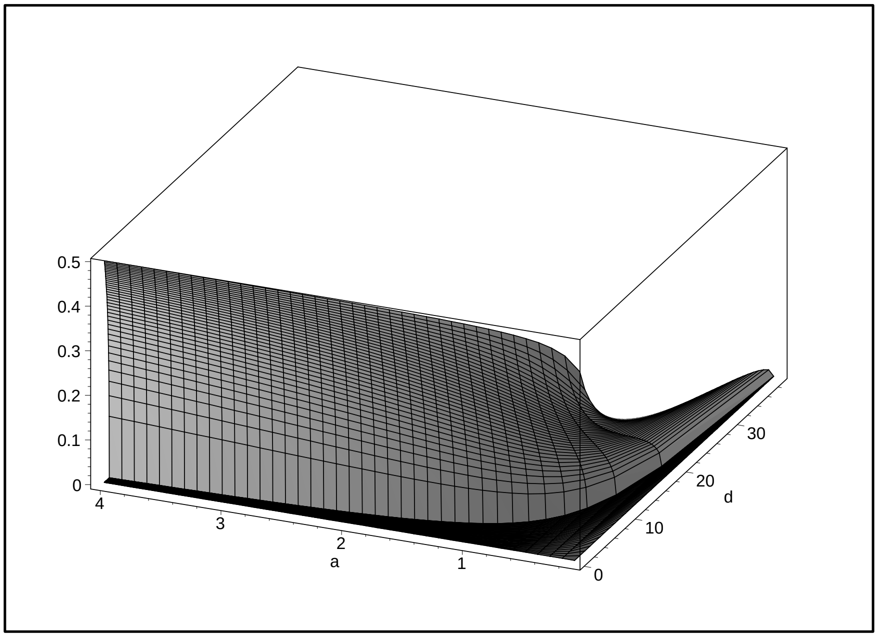

Figure 3.1: Parameter space for $\widetilde{M}$, $\widetilde{\alpha}$, and δ

Example: MIT Boger Fluid [12].
Corresponding to the Boger fluid we have $\alpha = 0.41$. Thus, $\widetilde{\alpha} = \alpha/(1-\alpha) = 0.695$, and from (3.15) we obtain $0 \le \delta \le 5.756$, and $\widetilde{M}$ must satisfy

$$0 < \widetilde{M} < \frac{(0.174\delta - 1) + \sqrt{(1 - 0.174\delta)(1 + 0.826\delta)}}{\delta}. \qquad (3.16)$$

The constraint (3.16) is satisfied for $\widetilde{M}$ lying below the curve in Figure 3.2.

4 Numerical Investigation of the Solvability of the Oseen-Viscoelastic Problem

In this section we numerically investigate the solvability of the approximating linear system as a function of the parameter λ. In viscoelasticity the difficulty of computing an accurate approximation as λ increases is well known and referred to as the *High Weissenberg Number Problem.* For the Oseen-viscoelastic problem sufficient conditions for the solvability of the continuous problem and the numerical approximation are given by (3.1)-(3.3). Below, as an indicator of solvability of the approximating linear system, $A\mathbf{x} = \mathbf{c}$, we investigate the condition number of the approximating linear system, $\kappa(A)$, (measured in the 1-norm) as a function of λ.

We choose for our model problem, the prototypal example from viscoelasticity, channel flow past a cylinder. At the inflow and outflow of the domain a parabolic profile is specified for the boundary condition for the velocity, together with zero velocity (no slip condition) along the other boundaries. At the inflow the stress (corresponding to parabolic channel flow) is specified.

Computations were performed on three meshes. The second and third meshes were successive,

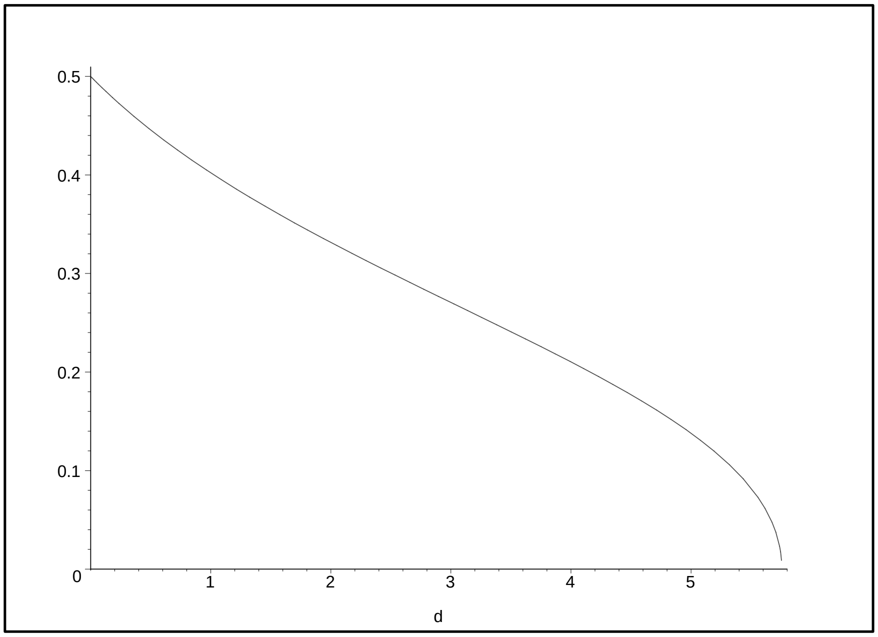

Figure 3.2: Parameter space for $\widetilde{M}$ for the Boger Fluid

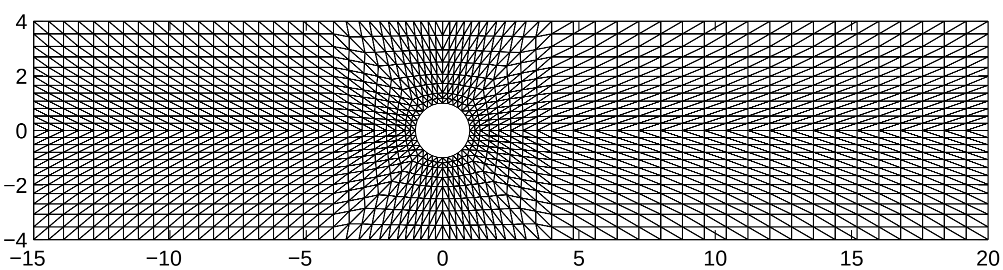

Figure 4.3: The second of three meshes used.

uniform refinements, of the initial mesh. The uniform refinement divided each triangle on the current mesh into four triangles by joining its mid-edges. On each mesh, the function **b** used was that computed for the velocity field for the Newtonian flow problem ($\lambda = 0$). In all the computations $\delta = 0.5$ was used. The algorithm described in [7] was used to estimate $\kappa(A)$. The algorithm is reported to give an estimate for $\kappa(A)$ to within a factor of 2.

Presented in Figures 4.4 and 4.5 is the condition number as a function of λ for $M = 1.0$ and $M = 2.67$.

As observed in the analysis, the product λM plays an important role. The relevance of the *product* is clearly demonstrated by the scaling of the graphs relative to M in Figures 4.4 and 4.5.

For the computations on the three meshes, and both values of M, the condition number was almost constant as a function of λ until a critical value λ^* was reached.

Illustrated in Figures 4.6-4.8 are the velocity-pressure plots, computed using Mesh 2, for λ values 1.0, 3.0, and 4.5, respectively, for a subdomain of the channel. Of interest to note is the **different**

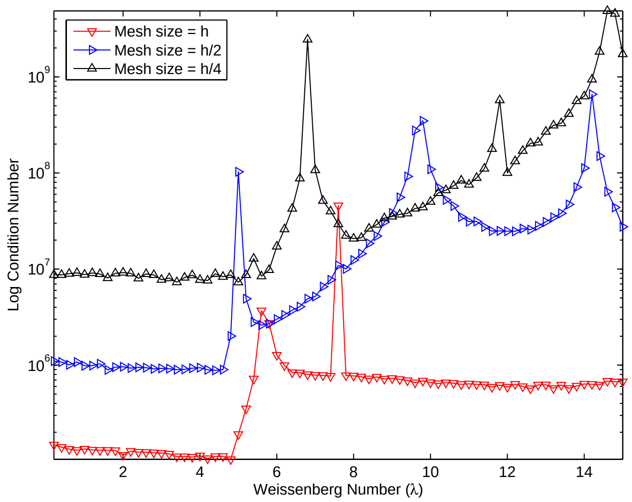

Figure 4.4: Condition number for $M = 1.0$.

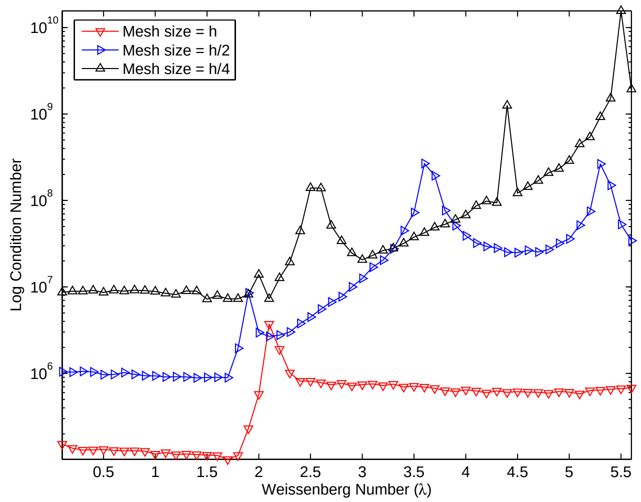

Figure 4.5: Condition number for $M = 2.67$.

flow fields generated. Similar flow fields were also computed for Meshes 1 and 3.

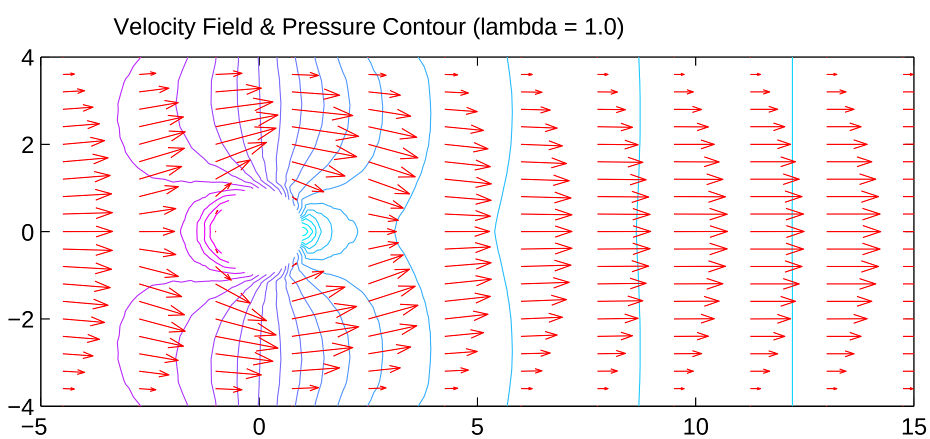

Figure 4.6: Velocity-Pressure plot from Mesh 2 for $\lambda = 1.0$.

From the similar behavior of the condition number on different meshes, it appears that, for the channel flow past a cylinder problem, the Oseen-Viscoelastic equations become singular at a critical value of λ, λ^*.

Illustrated in Figure 4.9 is the difference between the fixed velocity field, $\mathbf{b}$, and the computed velocity field, $\mathbf{u}$, as a function of λ, measured in the L^2 and H^1 norms, on Mesh 2. Recall that for $\mathbf{b} = \mathbf{u}$ the Oseen-viscoelastic equations are the Oldroyd equations. Initially the difference between $\mathbf{b}$ and $\mathbf{u}$ is quite small, indicating that the Oseen-viscoelastic equations are a good approximation to the Oldroyd equations. As λ approaches λ^* the difference between $\mathbf{b}$ and $\mathbf{u}$ becomes very large. (Note that an underlying assumption for the modeling viscoelastic equations (2.1)-(2.4) is a *slow flow* condition for the velocity.)

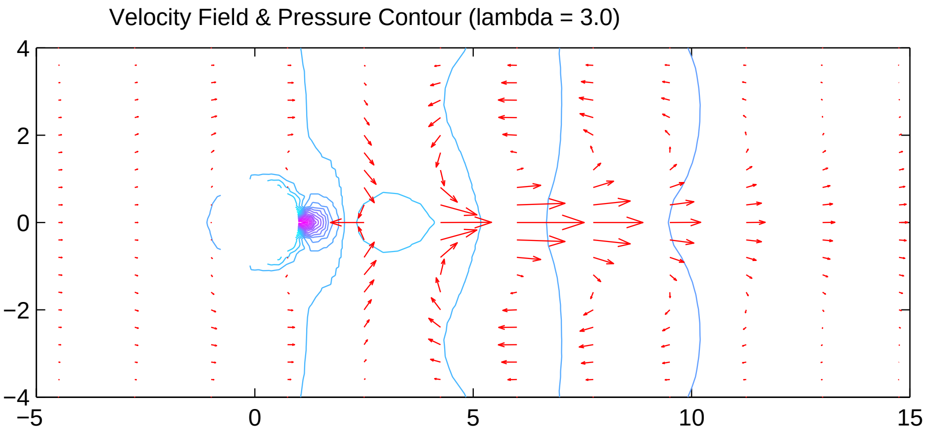

Figure 4.7: Velocity-Pressure plot from Mesh 2 for $\lambda = 3.0$.

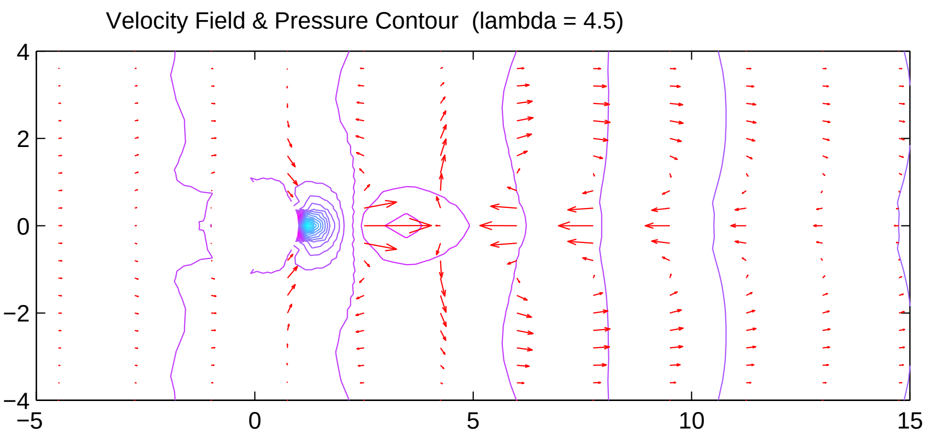

Figure 4.8: Velocity-Pressure plot from Mesh 2 for $\lambda = 4.5$.

Computations were also performed with the function **b** chosen as the velocity approximation to the Oldroyd-B model with $\lambda = 0.5$. The results were similar to those presented above. The corresponding condition number plots are given in Figures 4.10 and 4.11.

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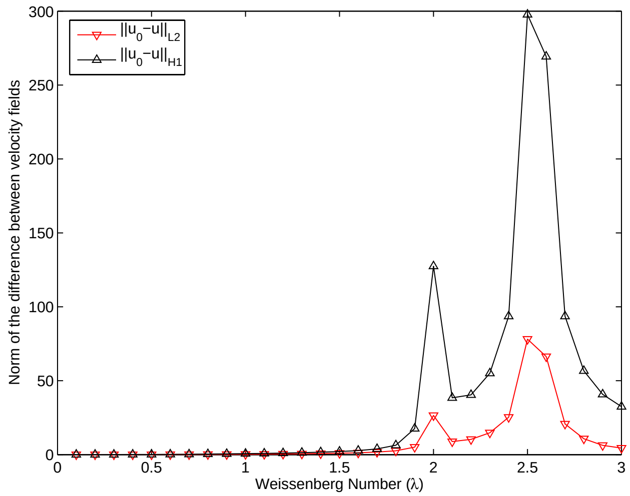

Figure 4.9: $\|\mathbf{u} - \mathbf{b}\|$, from Mesh 2, as a function of λ.

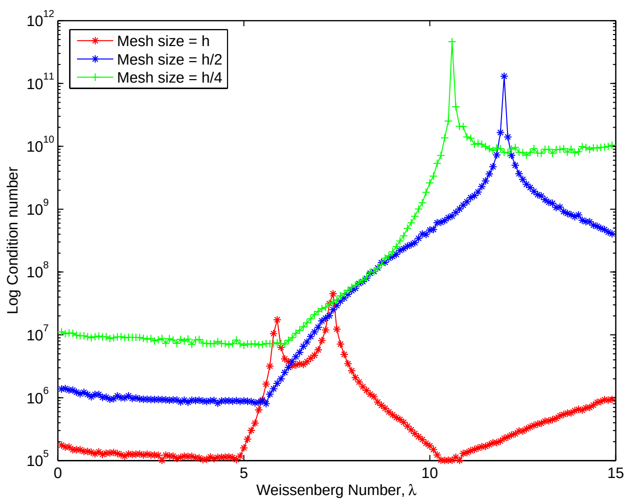

Figure 4.10: Condition number for $M = 1.0$ ($\mathbf{b} \equiv$ Oldroyd-B, $\lambda = 0.5$).

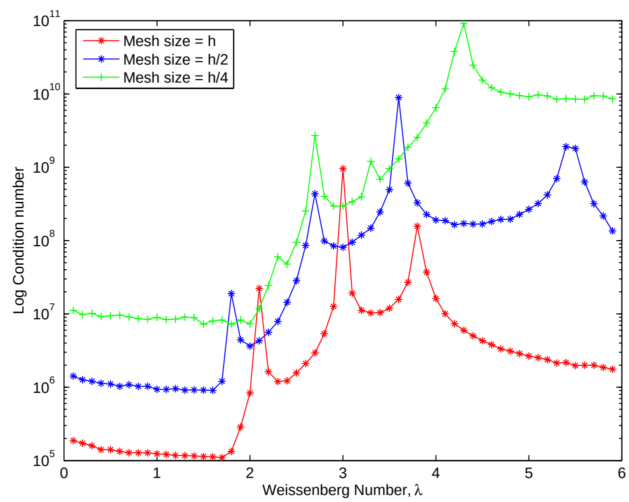

Figure 4.11: Condition number for $M = 2.67$ ($\mathbf{b} \equiv$ Oldroyd-B, $\lambda = 0.5$).

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