PERGAMON Computers and Mathematics with Applications 0 (2005) 1–0
www.elsevier.com/locate/camwa

Numerical Approximation of the Newtonian Film Blowing Problem

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(Received August 2004; revised and accepted January 2005)

Abstract—In this article, we study the numerical approximation of a Newtonian model for film blowing. We prove that the approximations for the bubble radius, and the film thickness, converges to the true solution and establish the convergence rates. Numerical results are given which demonstrate the theoretical results obtained.

Keywords—Film blowing, Finite-element method, *A priori* error estimate.

1. INTRODUCTION

This article examines the simplest mathematical model of the *film-blowing* process, the widely-employed industrial process used in the manufacturing of thin polymer film of thickness of the order of microns. In its physical reality, film-blowing involves complex physical and chemical changes occurring during manufacture; and a complete analysis of the most realistic models of this process would involve complex nonlinear problems, reflecting those changes. On the other hand, the relatively simple model considered here, which ignores a great deal of the detail of more realistic models, avoids many of the analytical difficulties associated with these. Moreover, the solutions obtained from use of this model retain much of the basic structure seen in them.

Before considering the details of the model used here, it is convenient to outline the overall features of the film manufacturing process itself. The elements of this are displayed schematically in Figure 1. A tube of molten polymer film is extruded from an annular die of radius R_0, at velocity $\mathbf{V}_0$, with thickness W_0. An applied internal pressure difference ΔP causes this tube to eventually expand to an increased radius, as shown. In appropriate circumstances, an initial narrowing, or *necking* may occur. As it develops, this tube or *bubble* of polymer is cooled by

*This research was undertaken while a visitor at the RMIT University.
Valuable input into deriving the optimal error estimates was provided by Professor Norbert Heuer, University of Concepcion, Chile.

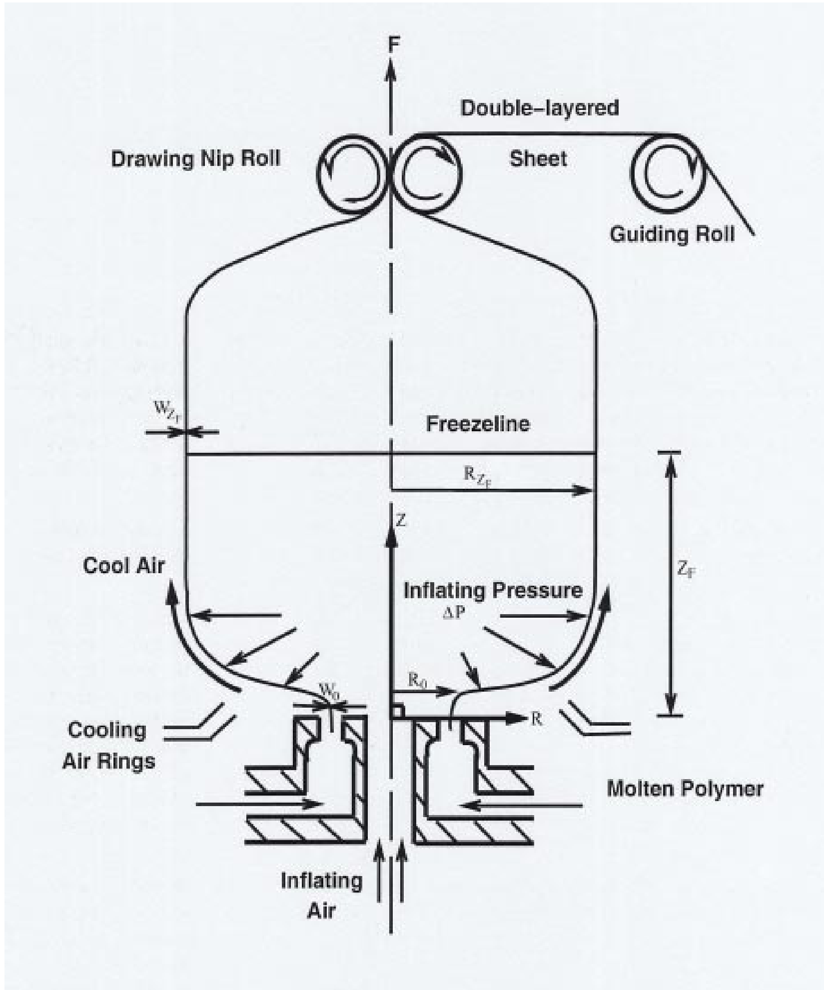

Figure 1. Blown film process.

external air jets from an air ring located above the die. This cooling causes the film to solidify, eventually reaching a constant radius R_{Z_F} with thickness W_{Z_F} at the *freezeline*, $(Z = Z_F)$, where its velocity is $\mathbf{V}_{Z_F}$. After the freezeline is reached the overall bubble shape remains unaltered; with the tube of film eventually being rolled flat as a double layered film and drawn off on to a roller.

The literature relating to the film blowing process is vast, with most attention being directed towards experimental investigations supporting empirical observations. Fundamental work involving the simplest film models is given in the series of papers by Pearson and Petrie [1,2], and Han and Park [3–5]; and a recent survey of research on this topic is given [6]. More general blown film models are discussed in [7]. Where numerical simulations are applied, most effort is directed towards computing a stable approximation to the non-linear system and the reconciliation of the computational results with experimental observations.

Our interest in this article is on the numerical simulation of the steady-state film blowing process. Specifically, we seek to address the question of how the results of the numerical simulation relate to the solution of the modeling equations. Note that this presupposes the existence of such a solution—to our knowledge, this question has not been addressed in any of the relevant literature; and our analysis is the first of its kind.

Since the equations modelling film blowing are highly nonlinear, our study will investigate the simplest film blowing problem—that of the steady isothermal blowing of an incompressible Newtonian film. While this model is of great simplicity, it retains many of the features of more complicated models, and, as noted above, avoids much mathematical complexity, allowing a much higher level of rigor to be applied. For this situation, the bubble structure shown in Figure 1 may be assumed to be axially symmetric, and the resulting film structure may be completely described by two unknown quantities—the (nondimensionalized) bubble radius, r, and (nondimensionalized) bubble thickness, w.

In Section 2, the equations determining r and w, together with boundary conditions are given. The determination of r will be seen to be the result of solving a nonlinear two-point boundary-value problem, separated from the determination of w. In Section 3, we show that under small data assumptions the Galerkin approximation, r_h, converges to the true solution, r, and specify the theoretical convergence rate. The theoretical convergence rate is then confirmed by numerical computations. In Section 4, we analyze the convergence of w_h to w, taking into account the error in the approximation due to using r_h instead of r. Numerical results are given which confirm the theoretically predicted convergence rates. A consequence of the analysis is the observation that the order of the approximating elements used for the film thickness w should be the same as that used for approximating the bubble radius r.

2. MODELING EQUATIONS

Under the assumptions that (see [2,8]):

(i) the forces controlling the flow are viscous forces arising in the steady isothermal flow of a homogeneous Newtonian liquid,
(ii) the film is thin enough for variations in the flow field across it to be ignored,
(iii) the film is thin enough for the velocity gradients to be approximated locally by those of a plane film being extended bi-axially,
(iv) the effects of gravity, surface tensions, air drag and the inertia of the fluid are negligible,

the nondimensionalized equations describing the film blowing process are as follows.

The (dimensionless) bubble radius $r(z)$ satisfies:

$$-2r^2\left(Br^2+F_c\right)r''+6r'+r\left(F_c-3Br^2\right)\left(1+r'^2\right)=0, \qquad 0<z<L, \tag{2.1}$$

subject to the boundary conditions

$$r(0)=1, \qquad r'(L)=0. \tag{2.2}$$

The associated equation for the (dimensionless) film thickness $w(z)$ is

$$w'+\left(\frac{1}{2r}r'+\frac{1}{4}\left(Br^2+F_c\right)\left(1+r'^2\right)\right)w=0, \qquad 0<z<L, \tag{2.3}$$

with the boundary condition

$$w(0)=1. \tag{2.4}$$

In the above, B and F_c are positive dimensionless parameters, with B being a measure of the pressure difference ΔP, and F_c a measure of the pulling force exerted at the freezeline.

In relation to Figure 1, in (2.1)–(2.4) $r(z)=R(Z)/R_0$, $w(z)=W(Z)/W_0$, $z=Z/R_0$, and $L=Z_F/R_0$.

Note that the two-point boundary-value problem (2.1),(2.2) for $r(z)$ is completely independent of the variable $w(z)$. In principle, (2.1),(2.2) can be solved for $r(z)$, and the result incorporated into (2.3),(2.4), an initial-value problem determining $w(z)$.

An alternative to boundary condition (2.2)(b) is to impose

$$r(L)=\text{BUR},$$

where BUR represents the blowup ratio. The following analysis can be modified to handle this boundary condition, resulting in the same convergence rate for the numerical approximations.

We make the following assumptions for $r(z)$.

A1 There exists a constant $r_m > 0$, such that $r(z) \geq r_m$, for $0 \leq z \leq L$.

A2 The exists constants $c_1, c_2 > 0$, such that $c_1 \leq (Br^2 + F_c)(1 + r'^2)/4 + r'/(2r) \leq c_2$.

For notational convenience, we let $c_0 > 0$ denote

$$c_0 := 2r_m^2 \left(Br_m^2 + F_c \right). \tag{2.5}$$

REMARK. Assumption A1 simply states that the *film bubble* does not collapse upon itself. Similar to Assumption A1, the constant $c_2 < \infty$ implies that the film bubble does not collapse on itself ($r = 0$), and additionally, does not *explode* ($r' \to \infty$). Physically we expect the film thickness to be strictly monotonically decreasing as a function of z, for $0 < z < L$. From (2.3), at the freezeline ($z = L$) we have $w'(L) = -1/4(Br(L)^2 + F_c)w(L) < 0$. Hence the existence of $c_1 > 0$ is a physically realistic assumption.

3. NUMERICAL APPROXIMATION OF R(Z)

In this section, we study the numerical approximation of (2.1),(2.2). We begin by reformulating the problem (2.1),(2.2) as a variational equation, suitable for establishing the existence of a numerical approximation scheme and its convergence properties. To this end, we introduce some mathematical notation.

3.1. The Variational Equation

The following notation will be used. Let I denote the interval $(0, L)$. The $L_2(I)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$. Likewise, the $L_p(I)$ norms and the Sobolev $W_p^k(I)$ norms are denoted by $\|\cdot\|_{L_p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the seminorm in $W_p^k(I)$, we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space W_2^k, and $\|\cdot\|_k$, $|\cdot|_k$ denotes the norm and semi-norm in H^k. The following function space is used in the analysis

$$X := \tilde{H}_0^1(I) := \left\{ v \in H^1(I) : v(0) = 0 \right\}.$$

To enable us to approximate the solution of (2.1),(2.2) in a subspace, we introduce the change of variable $\tilde{r} = r - 1$ which transforms (2.1),(2.2) into the following equations for $\tilde{r}$.

$$-2 \left(\tilde{r} + 1 \right)^2 \left(B(\tilde{r} + 1)^2 + F_c \right) \tilde{r}'' + 6\tilde{r}' + (\tilde{r} + 1) \left(F_c - 3B \left(\tilde{r} + 1 \right)^2 \right) \left(1 + \tilde{r}'^2 \right) = 0, \quad 0 < z < L, \tag{3.1}$$

subject to the boundary conditions

$$\tilde{r}(0) = 0, \qquad \tilde{r}'(L) = 0. \tag{3.2}$$

The boundary value problem (3.1),(3.2) for $\tilde{r}$ may be reformulated in a generalized form, suitable for the subsequent analysis. If we let $v \in \tilde{H}_0^1(I)$, we obtain, on multiplying (3.1) by v, integrating by parts, and applying the condition (3.2), the equation

$$\int_0^L 2 \left(\tilde{r} + 1 \right)^2 \left(B \left(\tilde{r} + 1 \right)^2 + F_c \right) \tilde{r}' v' \, dz + \int_0^L \left(2 \left(\tilde{r} + 1 \right)^2 \left(B \left(\tilde{r} + 1 \right)^2 + F_c \right) \right)' \tilde{r}' v \, dz + \int_0^L 6\tilde{r}' v \, dz + \int_0^L (\tilde{r} + 1) \left(F_c - 3B \left(\tilde{r} + 1 \right)^2 \right) \left(1 + \tilde{r}'^2 \right) v \, dz = 0, \qquad \forall v \in X. \tag{3.3}$$

When (3.3) holds for some $\tilde{r}$, for every $v \in X$, we will term $\tilde{r}$ a variational solution of the problem (3.1),(3.2). Clearly, any solution of (3.3) that is sufficiently smooth will also be a solution of (3.1),(3.2). However, there may be nonsmooth functions $\tilde{r}$ that satisfy (3.3).

Let T_h denote a partition of I into subintervals. For $K \in T_h$ let h_K denote the length of the subinterval K. We assume there exists $c_T > 0$, such that

$$c_T \leq \frac{\min_{K \in T_h} h_K}{\max_{K \in T_h} h_K} \leq 1.$$

For $c_T > 0$, T_h is called a quasi-uniform a partition of I. This assumption is necessary for Lemmas 1 and 2 below.

Let $P_k(K)$ denote the space of polynomials on K of degree no greater that k. Introduce the approximation space for $\tilde{r}$, X_h^r as

$$X_h^r := \left\{ v \in X \cap C\left(\bar{I}\right) : v|_K \in P_k(K), \ \forall K \in T_h \right\}. \tag{3.4}$$

Let $\mathcal{R} \in X_h^r$ be a P_k, continuous, interpolant of $\tilde{r}$. For $\tilde{r} \in H^{k+1}(I)$, we have [9]

$$\|\tilde{r}\| - \mathcal{R} + h \|\tilde{r}' - \mathcal{R}'\| \leq C_p h^{k+1} |\tilde{r}|_{k+1}. \tag{3.5}$$

For $v \in X$, from the fundamental theorem of calculus, we have

$$\|v\|_\infty \leq |I|^{1/2} |v|_1 = L^{1/2} |v|_1, \tag{3.6}$$

as $|I| = L$.

The following two lemmas are used in establishing the error estimates for the numerical approximations [10].

LEMMA1. *For $v \in X_h^r$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $0 \leq m \leq l$, we have that there exists $C = C(l,p,q)$, such that*

$$\left[\sum_{K \in T_h} \|v\|^p_{W^{l,p}(K)} \right]^{1/p} \leq C h^{m-l+\min(0, \acute{d}/p - \acute{d}/q)} \left[\sum_{K \in T_h} \|v\|^q_{W^{m,q}(K)} \right]^{1/q}. \tag{3.7}$$ ∎

LEMMA 2. *Let I_h denote the interpolant of v. Then for all $v \in W^{m,p}(\Omega) \cap C^r(\Omega)$ and $0 \leq s \leq \min\{m, r+1\}$,*

$$\|v - I_h\|_{W^{s,\infty}(\Omega)} \leq C h^{m-s-\acute{d}/p} |v|_{W^{m,p}(\Omega)}. \tag{3.8}$$ ∎

3.2. Numerical Approximation

It is not the purpose of this investigation to establish the existence and uniqueness properties of the equation (3.3) (or of (3.1),(3.2)). Rather, we will proceed to show that, under the assumption of the existence of a suitably smooth solution $\tilde{r}$ of (3.3), a well-defined numerical approximation $\tilde{r}_h$ can be specified that converges to $\tilde{r}$ in an appropriate sense.

Thus, we define the task of determining the numerical approximation $\tilde{r}_h$ to $\tilde{r}$ by: *determine $\tilde{r}_h \in X_h^r$ satisfying*

$$\begin{aligned} &\int_0^L 2\left(\tilde{r}_h+1\right)^2 \left(B\left(\tilde{r}_h+1\right)^2 + F_c\right) \tilde{r}_h' v' \, dz + \int_0^L \left(2\left(\tilde{r}_h+1\right)^2 \left(B\left(\tilde{r}_h+1\right)^2 + F_c\right)\right)' \tilde{r}_h' v \, dz \\ &\quad + \int_0^L 6\tilde{r}_h' v \, dz + \int_0^L \left(\tilde{r}_h+1\right)\left(F_c - 3B\left(\tilde{r}_h+1\right)^2\right)\left(1+\tilde{r}_h'^2\right) v \, dz = 0, \qquad \forall v \in X_h^r. \end{aligned} \tag{3.9}$$

We now show that, under suitable conditions, a unique solution to the discretized system (3.9) exists. Fixed-point theory is used to establish the desired result. The proof is established using the following four steps.

1. Define an iterative map in such a way that a fixed point of the map is a solution to (3.9).
2. Show the map is well-defined, and bounded on bounded sets.
3. Show there exists an invariant ball on which the map is a contraction.
4. Apply Banach's fixed-point theorem to establish the existence and uniqueness of the discrete approximation.

THEOREM 3.1. *For $k \in \mathbb{N}$, assume that (3.1),(3.2) has a solution $\tilde{r} \in X \cap H^{k+1}(I)$. Then, for $\|\tilde{r}\|_{k+1}$, B, F_c, L, and h sufficiently small, there exists a unique solution to (3.9) satisfying*

$$\|\tilde{r}' - \tilde{r}_h'\| + \|\tilde{r} - \tilde{r}_h\| \le Ch^k. \tag{3.10}$$

PROOF.

STEP 1. THE ITERATIVE MAP. A mapping $\Phi : X_h^r \to X_h^r$ is defined via: $\tilde{r}_2 = \Phi(\tilde{r}_1)$, where $\tilde{r}_2$ satisfies

$$A_{\tilde{r}_1}(\tilde{r}_2, v) = F_{\tilde{r}_1}(v), \qquad \forall v \in X_h, \tag{3.11}$$

for

$$\begin{aligned} A_{\tilde{r}_1}(\tilde{r}_2, v) := &\int_0^L 2(\tilde{r}_1+1)^2 \left(B(\tilde{r}_1+1)^2 + F_c\right) \tilde{r}_2' v'\, dz \\ &+ \int_0^L 6\tilde{r}_2' v\, dz + \int_0^L F_c\left(1+\tilde{r}_1'^2\right)\tilde{r}_2 v\, dz, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} F_{\tilde{r}_1}(v) := -\int_0^L \left(2(\tilde{r}_1+1)^2(B(\tilde{r}_1+1)^2+F_c)\right)' \tilde{r}_1' v\, dz + \int_0^L 3B(\tilde{r}_1+1)^2(1+\tilde{r}_1'^2)\tilde{r}_1 v\, dz \\ - \int_0^L (F_c - 3B(\tilde{r}_1+1)^2)(1+\tilde{r}_1'^2) v\, dz. \end{aligned} \tag{3.13}$$

STEP 2. SHOW Φ IS WELL-DEFINED AND BOUNDED ON BOUNDED SETS. To see that Φ is well defined, observe that on choosing $v = \tilde{r}_2$ we have

$$\begin{aligned} A_{\tilde{r}_1}(\tilde{r}_2, \tilde{r}_2) := &\int_0^L 2(\tilde{r}_1+1)^2\left(B(\tilde{r}_1+1)^2+F_c\right)\tilde{r}_2'\tilde{r}_2'\, dz + \int_0^L 6\tilde{r}_2'\tilde{r}_2\, dz \\ &+ \int_0^L F_c\left(1+\tilde{r}_1'^2\right)\tilde{r}_2\tilde{r}_2\, dz \\ &\ge 2(\tilde{r}_1+1)^2\left(B(\tilde{r}_1+1)^2+F_c\right)\|\tilde{r}_2'\|^2 + 3\tilde{r}_2(1)^2 + F_c\|\tilde{r}_2\|^2 \\ &\ge c_0\|\tilde{r}_2'\|^2 + 3\tilde{r}_2(1)^2 + F_c\|\tilde{r}_2\|^2, \end{aligned} \tag{3.14}$$

where $c_0 = \min_I 2(\tilde{r}_1+1)^2(B(\tilde{r}_1+1)^2+F_c) > 0$.

Positivity of $A_{\tilde{r}_1}(\cdot,\cdot)$ guarantees invertibility of the linear system (3.11).

Note that $F_{\tilde{r}_1}(\tilde{r}_2)$ satisfies the bound

$$\begin{aligned} |F_{\tilde{r}_1}(\tilde{r}_2)| \le &\left|\int_0^L \left(2(\tilde{r}_1+1)^2\left(B(\tilde{r}_1+1)^2+F_c\right)\right)'\tilde{r}_1'\tilde{r}_2\, dz\right| + \left|\int_0^L 3B(\tilde{r}_1+1)^2\left(1+\tilde{r}_1'^2\right)\tilde{r}_1\tilde{r}_2\, dz\right| \\ &+ \left|\int_0^L \left(F_c - 3B(\tilde{r}_1+1)^2\right)\left(1+\tilde{r}_1'^2\right)\tilde{r}_2\, dz\right| \\ \le &\|\tilde{r}_2\|_\infty \|\tilde{r}_1'\| \left\|\left(2(\tilde{r}_1+1)^2\left(B(\tilde{r}_1+1)^2+F_c\right)\right)'\right\| \\ &+ \|\tilde{r}_2\|_\infty\|\tilde{r}_1\|_\infty\left\|3B(\tilde{r}_1+1)^2\left(1+\tilde{r}_1'^2\right)\right\|_{L_1} \\ &+ \|\tilde{r}_2\|_\infty\left\|\left(F_c - 3B(\tilde{r}_1+1)^2\right)\left(1+\tilde{r}_1'^2\right)\right\|_{L_1}. \end{aligned} \tag{3.15}$$

Using (3.6), Young's inequality, and (A.3)–(A.5), we have for arbitrary ϵ_1, ϵ_2, $\epsilon_3 > 0$,

$$\begin{aligned} |F_{\tilde{r}_1}(\tilde{r}_2)| \le \epsilon_1\|\tilde{r}_2'\|^2 + \frac{1}{4\epsilon_1}D_4^2\|\tilde{r}_1'\|^4 + \epsilon_2\|\tilde{r}_2'\| + \frac{1}{4\epsilon_2}D_2^2\left(D_5 + D_6\|\tilde{r}_1'\|^2\right)^2 \\ +\epsilon_3\|\tilde{r}_2'\|^2 + \frac{1}{4\epsilon_3}\left(D_7 + D_8\|\tilde{r}_1'\|^2\right)^2. \end{aligned} \tag{3.16}$$

Combining (3.14) and (3.16), we conclude that Φ is bounded on bounded sets.

STEP 3. EXISTENCE OF AN INVARIANT BALL FOR Φ. We begin by defining an invariant ball. Let $R = C_B h^k$, and define the ball B_h^r as

$$B_h^r := \{v \in X_h^r : \|\tilde{r}' - v'\| + \|\tilde{r} - v\| \leq R\}. \tag{3.17}$$

The solution $\tilde{r}$ of (2.3),(2.4) satisfies

$$A_{\tilde{r}}(\tilde{r}, v) = F_{\tilde{r}}(v), \qquad \forall v \in X_h. \tag{3.18}$$

Subtracting (3.11) from (3.18) implies that

$$\begin{aligned}
&A_{\tilde{r}}(\tilde{r}, v) - A_{\tilde{r}_1}(\tilde{r}_2, v) \\
&\quad = \int_0^L 2(\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right)(\tilde{r}' - \tilde{r}_2')\, v'\, dz \\
&\qquad + \int_0^L 2\left((\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right) - (\tilde{r}_1+1)^2 \left(B(\tilde{r}_1+1)^2 + F_c\right)\right) \tilde{r}_2' v'\, dz \\
&\qquad + \int_0^L 6(\tilde{r}' - \tilde{r}_2')\, v\, dz + \int_0^L F_c\left(1 + \tilde{r}'^2\right)(\tilde{r} - \tilde{r}_2)\, v\, dz \\
&\qquad + \int_0^L F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right)\tilde{r}_2 v\, dz \\
&\quad = -\int_0^L \left(2(\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right)\right)' (\tilde{r}' - \tilde{r}_1')\, v\, dz \\
&\qquad - \int_0^L 2\left[\left((\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right)\right)' - \left((\tilde{r}_1+1)^2 \left(B(\tilde{r}_1+1)^2 + F_c\right)\right)'\right] \tilde{r}_1' v\, dz \\
&\qquad + \int_0^L 3B(\tilde{r}+1)^2 \left(1 + \tilde{r}'^2\right)(\tilde{r} - \tilde{r}_1)\, v\, dz \\
&\qquad + \int_0^L 3B\left((\tilde{r}+1)^2\left(1 + \tilde{r}'^2\right) - (\tilde{r}_1+1)^2\left(1 + \tilde{r}_1'^2\right)\right)\tilde{r}_1 v\, dz \\
&\qquad - \int_0^L \left(\left(F_c - 3B(\tilde{r}+1)^2\right)\left(1 + \tilde{r}'^2\right) - \left(F_c - 3B(\tilde{r}_1+1)^2\right)\left(1 + \tilde{r}_1'^2\right)\right) v\, dz \\
&\quad = F_{\tilde{r}}(v) - F_{\tilde{r}_1}(v), \qquad \text{for } v \in X_h^r.
\end{aligned} \tag{3.19}$$

Let $\mathcal{R}$ denote the interpolant of $\tilde{r}$ in X_h^r, and introduce

$$\Lambda = \tilde{r} - \mathcal{R}, \qquad E = \mathcal{R} - \tilde{r}_2. \tag{3.20}$$

Then, $e := \tilde{r} - \tilde{r}_2 = \Lambda + E$.

With these definitions, together with the choice $v = E$, the left-hand side of (3.19) becomes

$$\begin{aligned}
&A_{\tilde{r}}(\tilde{r}, E) - A_{\tilde{r}_1}(\tilde{r}_2, E) \\
&\quad = \int_0^L 2(\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right) E'E'\, dz \\
&\qquad + \int_0^L 2(\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right) \Lambda' E'\, dz \\
&\qquad - \int_0^L 2\left((\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right) - (\tilde{r}_1+1)^2 \left(B(\tilde{r}_1+1)^2 + F_c\right)\right) E'E'\, dz \\
&\qquad + \int_0^L 2\left((\tilde{r}+1)^2 \left(B(\tilde{r}+1)^2 + F_c\right) - (\tilde{r}_1+1)^2 \left(B(\tilde{r}_1+1)^2 + F_c\right)\right) \mathcal{R}' E'\, dz
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&+ \int_0^L 6E'E\,dz + \int_0^L 6\Lambda' E\,dz \\
&+ \int_0^L F_c\left(1+\tilde{r}'^2\right) EE\,dz + \int_0^L F_c\left(1+\tilde{r}'^2\right)\Lambda E\,dz \qquad (3.21)(\text{cont.})\\
&- \int_0^L F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right) EE\,dz + \int_0^L F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right)\mathcal{R}E\,dz \\
&:= J_1 + J_2 + \cdots + J_{10}.
\end{aligned}
$$

We need to proceed to bound E in terms of the true solution $\tilde{r}$, the radius of the ball R, and the given data B, F_c, and L.

$$
\begin{aligned}
J_1 &= \int_0^L 2\left(\tilde{r}+1\right)^2\left(B\left(\tilde{r}+1\right)^2 + F_c\right) E'E'\,dz \\
&\geq 2r_m^2\left(Br_m^2 + F_c\right)\|E'\|^2 .
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
J_2 &= \int_0^L 2\left(\tilde{r}+1\right)^2\left(B\left(\tilde{r}+1\right)^2 + F_c\right) \Lambda'E'\,dz \\
&\geq -\epsilon_1\|E'\|^2 - \frac{1}{4\epsilon_1}D_{21}^2\|\Lambda'\|^2 ,
\end{aligned}
\tag{3.23}
$$

where $D_{21} := \|2(\tilde{r}+1)^2(B(\tilde{r}+1)^2+F_c)\|_\infty < \infty$, as $\|\tilde{r}\|_\infty$ is bounded.

$$
\begin{aligned}
J_3 &= -\int_0^L 2\left(\left(\tilde{r}+1\right)^2\left(B\left(\tilde{r}+1\right)^2 + F_c\right) - \left(\tilde{r}_1+1\right)^2\left(B\left(\tilde{r}_1+1\right)^2 + F_c\right)\right) E'E'\,dz \\
&\geq -\|(\tilde{r}-\tilde{r}_1)\|_\infty \left\|2\left(\tilde{r}+\tilde{r}_1+2\right)\left(B\left[\left(\tilde{r}_1+1\right)^2 + \left(\tilde{r}+1\right)^2\right] + F_c\right)\right\|_\infty \|E'\|^2 \\
&\geq -D_{22}\|\tilde{r}'-\tilde{r}_1'\|\,\|E'\|^2 , \qquad \text{using (3.6)},
\end{aligned}
\tag{3.24}
$$

where $D_{22} := L^{1/2}\|2(\tilde{r}+\tilde{r}_1+2)(B[(\tilde{r}_1+1)^2+(\tilde{r}+1)^2]+F_c)\|_\infty < \infty$ (as $\|\tilde{r}_1\|_\infty$ is bounded in terms of $\|\tilde{r}\|_\infty$ and R).

$$
\begin{aligned}
J_4 &= \int_0^L 2\left(\left(\tilde{r}+1\right)^2\left(B\left(\tilde{r}+1\right)^2 + F_c\right) - \left(\tilde{r}_1+1\right)^2\left(B\left(\tilde{r}_1+1\right)^2 + F_c\right)\right) \mathcal{R}'E'\,dz \\
&\geq -\|\tilde{r}-\tilde{r}_1\|_\infty \left\|2\left(\tilde{r}+\tilde{r}_1+2\right)\left(B\left[\left(\tilde{r}_1+1\right)^2 + \left(\tilde{r}+1\right)^2\right] + F_c\right)\right\|_\infty \|\mathcal{R}'\|\,\|E'\| \\
&\geq -\epsilon_2\|E'\|^2 - \frac{1}{4\epsilon_2}D_{22}^2\|\tilde{r}'-\tilde{r}_1'\|^2\|\mathcal{R}'\|^2 .
\end{aligned}
\tag{3.25}
$$

$$
J_5 = \int_0^L 6E'E\,dz = 3E(1)^2. \tag{3.26}
$$

$$
J_6 = \int_0^L 6\Lambda'E\,dz \geq -\epsilon_3\|E\|^2 - \frac{9}{\epsilon_3}\|\Lambda'\|^2 . \tag{3.27}
$$

$$
J_7 = \int_0^L F_c\left(1+\tilde{r}'^2\right) EE\,dz \geq F_c\|E\|^2. \tag{3.28}
$$

$$
\begin{aligned}
J_8 &= \int_0^L F_c\left(1+\tilde{r}'^2\right)\Lambda E\,dz \\
&\geq -\left\|F_c\left(1+\tilde{r}'^2\right)\right\|_{L_1}\|\Lambda\|_\infty\|E\|_\infty \\
&\geq -\epsilon_4\|E'\|^2 - \frac{1}{4\epsilon_4}D_{23}^2\|\Lambda'\|^2 .
\end{aligned}
\tag{3.29}
$$

where $D_{23} := \|F_c(1+\tilde{r}'^2)\|_{L_1} < \infty$, as $\tilde{r}' \in L_2(I)$.

$$\begin{aligned} J_9 &= -\int_0^L F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right) E E\, dz \\ &\leq \left\|F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right)\right\|_{L_1} \|E\|_\infty \|E\|_\infty \\ &\leq D_{24} \|\tilde{r}' - \tilde{r}_1'\| \left\|E'\right\|^2, \end{aligned} \tag{3.30}$$

where $D_{24} := F_c\|\tilde{r}' + \tilde{r}_1'\| < \infty$ (as $\|\tilde{r}_1'\|$ is bounded in terms of $\|\tilde{r}'\|$ and R).

$$\begin{aligned} J_{10} &= \int_0^L F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right) \mathcal{R} E\, dz \\ &\leq \left\|F_c\left(\tilde{r}'^2 - \tilde{r}_1'^2\right)\right\|_{L_1} \|\mathcal{R}\|_\infty \|E\|_\infty \\ &\leq \epsilon_5 \left\|E'\right\|^2 + \frac{1}{4\epsilon_5} D_{24}^2 \left\|\mathcal{R}'\right\|^2 \left\|\tilde{r}' - \tilde{r}_1'\right\|^2. \end{aligned} \tag{3.31}$$

Next the terms on the right-hand side of (3.19) must be similarly bounded for the choice $v = E$.

$$\begin{aligned} J_{11} &= -\int_0^L \left(2\left(\tilde{r}+1\right)^2 \left(B\left(\tilde{r}+1\right)^2 + F_c\right)\right)' \left(\tilde{r}' - \tilde{r}_1'\right) E\, dz \\ &\leq \left\|\left(2\left(\tilde{r}+1\right)^2 \left(B\left(\tilde{r}+1\right)^2 + F_c\right)\right)'\right\| \|\tilde{r}' - \tilde{r}_1'\| \|E\|_\infty \\ &\leq \epsilon_6 \left\|E'\right\|^2 + \frac{1}{4\epsilon_6} D_{25}^2 \left\|\tilde{r}'\right\|^2 \left\|\tilde{r}' - \tilde{r}_1'\right\|^2, \end{aligned} \tag{3.32}$$

where $\| \left(2(\tilde{r}+1)^2(B(\tilde{r}+1)^2 + F_c)\right)' \| \leq D_{25}\|\tilde{r}'\|$. (See (A.6) for existence of $D_{25} < \infty$.)

$$\begin{aligned} J_{12} &= -\int_0^L 2\left[\left(\left(\tilde{r}+1\right)^2 \left(B\left(\tilde{r}+1\right)^2 + F_c\right)\right)' - \left(\left(\tilde{r}_1+1\right)^2 \left(B\left(\tilde{r}_1+1\right)^2 + F_c\right)\right)'\right] \tilde{r}_1' E\, dz \\ &\leq \left\|\left[2\left(\tilde{r}-\tilde{r}_1\right)\left(\tilde{r}+\tilde{r}_1+2\right)\left(B\left[\left(\tilde{r}_1+1\right)^2 + \left(\tilde{r}+1\right)^2\right] + F_c\right)\right]'\right\| \|\tilde{r}_1'\| \|E\|_\infty \\ &\leq \epsilon_7 \left\|E'\right\|^2 + \frac{1}{4\epsilon_7} D_{26}^2 \left\|\tilde{r}' - \tilde{r}_1'\right\|^2. \end{aligned} \tag{3.33}$$

The existence of D_{26} is given in (A.7).

$$\begin{aligned} J_{13} &= \int_0^L 3B\left(\tilde{r}+1\right)^2 \left(1+\tilde{r}'^2\right) \left(\tilde{r}-\tilde{r}_1\right) E\, dz \\ &\leq \left\|3B\left(\tilde{r}+1\right)^2\left(1+\tilde{r}'^2\right)\right\|_{L_1} \|\tilde{r}-\tilde{r}_1\|_\infty \|E\|_\infty \\ &\leq \epsilon_8 \left\|E'\right\|^2 + \frac{1}{4\epsilon_8} D_{27}^2 \left\|\tilde{r}' - \tilde{r}_1'\right\|^2. \end{aligned} \tag{3.34}$$

where $D_{27} := \|3B(\tilde{r}+1)^2(1+\tilde{r}'^2)\|_{L_1} < \infty$, as $\|\tilde{r}\|_\infty$ is bounded and $\tilde{r}' \in L_2(I)$.

$$\begin{aligned} J_{14} &= \int_0^L 3B\left(\left(\tilde{r}+1\right)^2\right)\left(1+\tilde{r}'^2\right) - \left(\tilde{r}_1+1\right)^2\left(1+\tilde{r}_1'^2\right) \tilde{r}_1 E\, dz \\ &\leq \|\tilde{r}-\tilde{r}_1\|_\infty \left\|3B\left[\left(\tilde{r}_1+1\right)\left(\tilde{r}'+\tilde{r}_1'\right) + \left(\tilde{r}+\tilde{r}_1+2\right)\left(1+\tilde{r}'^2\right)\right]\right\|_{L_1} \|\tilde{r}_1\|_\infty \|E\|_\infty \\ &\leq \epsilon_9 \left\|E'\right\|^2 + \frac{1}{4\epsilon_9} D_{28}^2 \left\|\tilde{r}' - \tilde{r}_1'\right\|^2. \end{aligned} \tag{3.35}$$

The existence of D_{28} is given in (A.8).

Finally,

$$\begin{aligned}
J_{15} &= -\int_0^L \left(\left(F_c - 3B\left(\tilde{r}+1\right)^2\right)\left(1+\tilde{r}'^2\right) - \left(F_c - 3B\left(\tilde{r}_1+1\right)^2\right)\left(1+\tilde{r}_1'^2\right)\right) E\,dz \\
&= -\int_0^L F_c\left(\tilde{r}'^2 - \tilde{r}_1^2\right) E\,dz \\
&\quad + \int_0^L 3B\left(\left(\tilde{r}+1\right)^2\left(1+\tilde{r}'^2\right) - \left(\tilde{r}_1+1\right)^2\left(1+\tilde{r}_1'^2\right)\right) E\,dz \\
&\le \left\|F_c\left(\tilde{r}'+\tilde{r}_1'\right)\left(\tilde{r}'-\tilde{r}_1'\right)\right\|_{L_1}\|E\|_\infty \\
&\quad + \|\tilde{r}-\tilde{r}_1\|_\infty \left\|3B\left[\left(\tilde{r}_1+1\right)\left(\tilde{r}'+\tilde{r}_1'\right) + \left(\tilde{r}+\tilde{r}_1+2\right)\left(1+\tilde{r}'^2\right)\right]\right\|_{L_1}\|E\|_\infty \\
&\le F_c\|\tilde{r}'+\tilde{r}_1'\|\,\|\tilde{r}'-\tilde{r}_1'\|\,\|E\|_\infty \\
&\quad + \|\tilde{r}-\tilde{r}_1\|_\infty \left\|3B\left[\left(\tilde{r}_1+1\right)\left(\tilde{r}'+\tilde{r}_1'\right) + \left(\tilde{r}+\tilde{r}_1+2\right)\left(1+\tilde{r}'^2\right)\right]\right\|_{L_1}\|E\|_\infty \\
&\le \epsilon_{10}\|E'\|^2 + \frac{1}{4\epsilon_{10}}D_{29}^2\|\tilde{r}'-\tilde{r}_1'\|^2 + \epsilon_{11}\|E'\|^2 + \frac{1}{4\epsilon_{11}}D_{28}^2\|\tilde{r}'-\tilde{r}_1'\|^2,
\end{aligned} \tag{3.36}$$

where $D_{29} := F_c(\|\tilde{r}'\| + \|\tilde{r}_1'\|)$.

Combining (3.19) with the estimates (3.22)–(3.36), we have that

$$\begin{aligned}
&\left(2r_m^2\left(Br_m^2+F_c\right) - \left(\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_9+\epsilon_{10}+\epsilon_{11}\right)\right. \\
&\qquad \left. - \left(D_{22}+D_{24}\right)\|\tilde{r}'-\tilde{r}_1'\|\right)\|E'\| + 3E(1)^2 + \left(F_c-\epsilon_3\right)\|E\| \\
&\quad \le \left(\frac{1}{4\epsilon_1}D_{21}^2 + \frac{9}{\epsilon_3} + \frac{1}{4\epsilon_4}D_{23}^2\right)\|\Lambda'\|^2 \\
&\qquad + \left(\frac{1}{4\epsilon_2}D_{22}^2\|\mathcal{R}'\|^2 + \frac{1}{4\epsilon_5}D_{24}^2\|\mathcal{R}'\|^2 + \frac{1}{4\epsilon_6}D_{25}^2\|\tilde{r}'\|^2 + \frac{1}{4\epsilon_7}D_{26}^2\right. \\
&\qquad \left. + \frac{1}{4\epsilon_8}D_{27}^2 + \frac{1}{4\epsilon_9}D_{28}^2 + \frac{1}{4\epsilon_{10}}D_{29}^2 + \frac{1}{4\epsilon_{11}}D_{28}^2\right)\|\tilde{r}'-\tilde{r}_1'\|^2
\end{aligned} \tag{3.37}$$

Note that the constants $D_{\cdot\cdot}$ only depend upon $\|\tilde{r}\|_1$, and the data B, F_c, and L. Additionally, from (3.5), $\|\mathcal{R}'\| \le \|\tilde{r}'\| + C_p h^k \|\tilde{r}\|_{k+1}$. For h sufficiently small, and the ϵ_i, $i = 1, \ldots 11$, appropriately chosen, we have that the coefficients of $\|E'\|$ and $\|E\|$ in (3.37) are greater that $r_m^2(Br_m^2 + F_c)$, and $F_c/2$, respectively. Hence, we have that for positive constants $\tilde{D}_1$ and $\tilde{D}_2$, determined by $\|\tilde{r}\|_1$ and the data,

$$\begin{aligned}
\|E'\|^2 + \|E\|^2 &\le \tilde{D}_1\|\Lambda'\|^2 + \tilde{D}_2\|\tilde{r}'-\tilde{r}_1'\|^2 \\
&\le \tilde{D}_1^2 C_p^2\|\tilde{r}\|_{k+1}^2 h^{2k} + \tilde{D}_2^2 C_B^2 h^{2k}.
\end{aligned} \tag{3.38}$$

Finally, using (3.5) and (3.38), we have

$$\begin{aligned}
\|\tilde{r}'-\tilde{r}_2'\| + \|\tilde{r}-\tilde{r}_2\| &\le \|\Lambda'\| + \|E'\| + \|\Lambda\| + \|E\| \\
&\le C_p\|\tilde{r}\|_{k+1}h^k + 2\tilde{D}_1 C_p\|\tilde{r}\|_{k+1}h^k + 2\tilde{D}_2 C_B h^k + C_p\|\tilde{r}\|_{k+1}h^{k+1}.
\end{aligned} \tag{3.39}$$

Hence, for h, $\|\tilde{r}\|_{k+1}$, and the data sufficiently small, and C_B appropriately chosen, from (3.39) we have that $\|\tilde{r}'-\tilde{r}_2'\| + \|\tilde{r}-\tilde{r}_2\| < C_B h^k$. Thus, Φ is a strict contraction on the *ball*, B_h^r, defined in (3.17).

Step 4. A direct application of Banach's fixed-point theorem now establishes the uniqueness of the approximation and the stated error estimates. ∎

Helpful in establishing the error estimate for the width of the film w, presented in the next section, is the following estimate.

Corollary 3.1. *For $\tilde{r} \in X \cap H^2(I)$, there a constant $C < \infty$, such that for h sufficiently small, $\tilde{r}_h' < C$, i.e., $\tilde{r}_h'$ remains bounded as $h \to \infty$.*

Proof. We have that for $\Lambda = \tilde{r} - \mathcal{R}$, and $E = \mathcal{R} - \tilde{r}_h$

$$\begin{aligned} \|\tilde{r}_h'\|_\infty &\leq \|\tilde{r}' - \tilde{r}_h'\|_\infty + \|\tilde{r}'\|_\infty \\ &\leq \|\tilde{r}' - \tilde{r}_h'\|_\infty + \|\tilde{r}'\|_\infty \\ &\leq \|E'\|_\infty + \|\Lambda'\|_\infty + \|\tilde{r}'\|_\infty \\ &\leq C_1 h^{-1/2} \|E'\| + C_2 h^{1/2} \|\tilde{r}\|_2 + \|\tilde{r}'\|_\infty , \qquad \text{using (3.7) and (3.8).} \end{aligned}$$

As $\tilde{r} \in H^2(I)$, $\|\tilde{r}\|_2$, and $\|\tilde{r}'\|_\infty$ are bounded. From (3.38) it follows that $h^{-1/2}\|E'\|$ is also bounded. ∎

3.3. Numerical results for $r_h(z)$

In this section, we present numerical results for the approximation of the (dimensionless) radius of the bubble, $r(z) = 1+\tilde{r}(z)$. The numerical results are compared with the predicted theoretical results given in Theorem 3.1.

Table 1. Experimental rates of convergence for $\|r_h' - r'\|$.

	P/W Linear Approx. ($k = 1$)		P/W Quad. Approx. ($k = 2$)		P/W Cubic Approx. ($k = 3$)	
	$\|\tilde{r}_h' - \tilde{r}_{2h}'\|$	Cvge. Rate	$\|\tilde{r}_h' - \tilde{r}_{2h}'\|$	Cvge. Rate	$\|\tilde{r}_h' - \tilde{r}_{2h}'\|$	Cvge. Rate
$h = L/40$	1.81E − 01		1.98E − 02		2.14E − 03	
$h = L/80$	7.62E − 02	1.25	4.71E − 03	2.07	3.08E − 04	2.80
$h = L/160$	3.68E − 02	1.05	1.17E − 03	2.01	3.86E − 05	3.00
$h = L/320$	1.83E − 02	1.01	2.93E − 04	2.00	4.83E − 06	3.00
$h = L/640$	9.11E − 03	1.00	7.32E − 05	2.00	6.04E − 07	3.00
Predicted		1.0		2.0		3.0

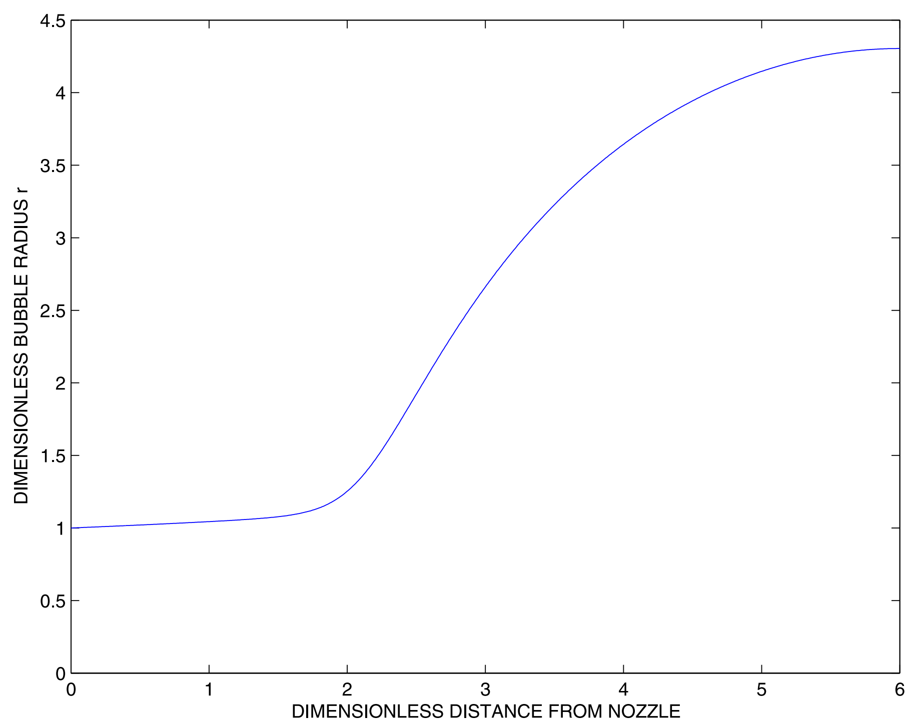

Figure 2. Plot of the dimensionless bubble radius.

For the numerical experiments, we used the following parameter values (taken from [11]): $B = 0.15$, $F_c = 0.2207$, $L = 6.0$. Computations were performed on a sequence of uniform partitions of $[0, L]$, using 40, 80, 160, 320, and 640 subintervals. On successive partitions the difference between the derivatives of the approximations, $\|\tilde{r}'_h - \tilde{r}'_{2h}\|$, was calculated. From (3.10), we have that

$$\begin{aligned} \|\tilde{r}'_h - \tilde{r}'_{2h}\| &\leq \|\tilde{r}'_h - \tilde{r}'\| + \|\tilde{r}' - \tilde{r}'_{2h}\| \\ &\leq Ch^k + C(2h)^k = \tilde{C}h^k, \end{aligned} \tag{3.40}$$

where k denotes the degree of the approximating, piecewise polynomial.

Presented in Table 1 are the results for $\|\tilde{r}'_h - \tilde{r}'_{2h}\|$ for linear ($k = 1$), quadratic ($k = 2$), and cubic ($k = 3$) piecewise polynomial approximations. A plot of the approximation of the (dimensionless) radius of the bubble is shown in Figure 2, generated using a piecewise quadratic approximation with 320 subintervals. The bubble profile is consistent with that physically observed. The numerical rates of convergence $\tilde{\alpha}$, defined by

$$\tilde{\alpha} := \frac{\log\left(\|\tilde{r}'_{2h} - \tilde{r}'_{4h}\| \,/\, \|\tilde{r}'_h - \tilde{r}'_{2h}\|\right)}{\log(2)}, \tag{3.41}$$

agree with those predicted theoretically by Theorem 3.1, namely, k.

4. NUMERICAL APPROXIMATION OF $w(z)$

In this section, we study the numerical approximation of (2.3),(2.4).

Helpful in establishing the error in the approximation of w is the following lemma.

LEMMA 3. *Let $\|\delta(z)\| \leq 1$ and $g(z)$ satisfy*

$$g' + \delta(z)g = -\delta(z), \qquad 0 < z < L, \tag{4.1}$$

$$g(0) = 0. \tag{4.2}$$

There exists a constant C, such that

$$\|g'(z)\| \leq C\|\delta(z)\|. \tag{4.3}$$

PROOF. Observe that (4.1),(4.2) is a linear system of equations whose unique solution is given by

$$g(z) = 1 - e^{-\int_0^z \delta(t)\,dt}. \tag{4.4}$$

Hence,

$$\begin{aligned} g'(z) &= \delta(z)e^{-\int_0^z \delta(t)\,dt}, \qquad \text{and} \\ \|g'(z)\| &\leq \|\delta(z)\| \left\| e^{-\int_0^z \delta(t)\,dt} \right\|_\infty \\ &\leq \|\delta(z)\| e^{\int_0^L |\delta(t)|\,dt} \leq \|\delta(z)\| e^{L^{1/2}\|\delta(z)\|} \\ &\leq C\|\delta(z)\|. \end{aligned}$$

∎

In order to approximate w over a subspace, we introduce the change of variable

$$\tilde{w} := w - 1 \quad \Leftrightarrow \quad w = \tilde{w} + 1.$$

Rearranging (2.3), we obtain the following differential equation and homogeneous boundary condition for $\tilde{w}$.

$$\begin{aligned}
&\tilde{w}' + \left(\frac{\tilde{r}'}{2(\tilde{r}+1)} + \frac{1}{4}(B(\tilde{r}+1)^2 + F_c)\left(1+\tilde{r}'^2\right)\right)\tilde{w} \\
&= -\left(\frac{\tilde{r}'}{2(\tilde{r}+1)} + \frac{1}{4}\left(B(\tilde{r}+1)^2 + F_c\right)\left(1+\tilde{r}'^2\right)\right),
\end{aligned} \tag{4.5}$$

$$\tilde{w}(0) = 0. \tag{4.6}$$

We introduce the approximation space for $\tilde{w}$, X_h^w as

$$X_h^w := \left\{v \in C\left(\bar{I}\right) : v|_K \in P_m(K),\ \forall K \in T_h\right\}.$$

For $\tilde{r}_h$ defined by (3.9), we define the numerical approximation of (4.5),(4.6) as: *determine* $\tilde{w}_h \in X_h^w$ *satisfying*

$$\begin{aligned}
&\int_0^L \tilde{w}_h' \hat{v}\, dz + \int_0^L \left(\frac{\tilde{r}_h'}{2(\tilde{r}_h+1)} + \frac{1}{4}\left(B(\tilde{r}_h+1)^2 + F_c\right)\left(1+\tilde{r}_h'^2\right)\right)\tilde{w}_h\hat{v}\, dz \\
&= -\int_0^L \left(\frac{\tilde{r}_h'}{2(\tilde{r}_h+1)} + \frac{1}{4}\left(B(\tilde{r}_h+1)^2 + F_c\right)\left(1+\tilde{r}_h'^2\right)\right)\hat{v}\, dz,
\end{aligned} \tag{4.7}$$

where $\hat{v} := v + \nu h v'$, $v \in X_h^w$, and ν is a small positive constant.

We now proceed to establish the existence of $\tilde{w}_h$, and its convergence properties.

For notational convenience, we make the following definitions.

$$\mathsf{H}(r) := \left(B(r+1)^2 + F_c\right)\left(1+r'^2\right)/4 + r'/\left(2(r+1)\right), \tag{4.8}$$

$$A(r; w, v) := \int_0^L w'\hat{v}\, dz + \int_0^L \mathsf{H}(r) w \hat{v}\, dz \tag{4.9}$$

$$F(r; v) := -\int_0^L \mathsf{H}(r)\hat{v}\, dz. \tag{4.10}$$

Note that (4.7) is equivalent to: $A(\tilde{r}_h; \tilde{w}_h, v) = F(\tilde{r}_h; v)$, for all $v \in X_h^w$.

Before discussing the error in the approximation, we prove the following estimate for $\mathsf{H}(\tilde{r}) - \mathsf{H}(\tilde{r}_h)$, which is used in the subsequent analysis.

Lemma 4. *For h sufficiently small, and $\tilde{r} \in X \cap H^{k+1}(I)$ there exists a constant C_0, such that*

$$\|\mathsf{H}(\tilde{r}) - \mathsf{H}(\tilde{r}_h)\| \le C_0 h^k. \tag{4.11}$$

Proof. We establish (4.11) by considering two separate pieces. First, we have that

$$\begin{aligned}
\left(B(\tilde{r}+1)^2 + F_c\right)\left(1+\tilde{r}'^2\right) &= \left[B(\tilde{r}_h + 1 + \tilde{r} - \tilde{r}_h)^2 + F_c\right]\left(1 + \left(\tilde{r}_h'^2 + \tilde{r}^2 - \tilde{r}_h'^2\right)\right) \\
&= \left(B(\tilde{r}_h+1)^2 + F_c\right)\left(1+\tilde{r}_h'^2\right) + \left(B(\tilde{r}_h+1)^2 + F_c\right)\left(\tilde{r}' - \tilde{r}_h'\right)\left(\tilde{r}' + \tilde{r}_h'\right) \\
&\quad + B(\tilde{r} - \tilde{r}_h)(\tilde{r} + \tilde{r}_h + 2)\left(1+\tilde{r}'^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|\left(B(\tilde{r}+1)^2 + F_c\right)\left(1+\tilde{r}'^2\right) - \left(B(\tilde{r}_h+1)^2 + F_c\right)\left(1+\tilde{r}_h'^2\right)\right\| \\
&\le \left\|B(\tilde{r}_h+1)^2 + F_c\right\|_\infty \|\tilde{r}' - \tilde{r}_h'\| \left(\|\tilde{r}'\|_\infty + \|\tilde{r}_h'\|_\infty\right) + B\|\tilde{r} + \tilde{r}_h + 2\|_\infty \left\|1+\tilde{r}'^2\right\|_\infty \|\tilde{r} - \tilde{r}_h\| \\
&\le D_1 C_r h^k \left(\|\tilde{r}'\|_\infty + \|\tilde{r}_h'\|_\infty\right) + D_2 C_r h^k \\
&\le D_3 h^k,
\end{aligned} \tag{4.12}$$

as, $\|\tilde{r}'\|_\infty$ and $\|\tilde{r}_h'\|_\infty$ are bounded.

Secondly,

$$\frac{\tilde{r}'}{\tilde{r}+1} - \frac{\tilde{r}_h'}{\tilde{r}_h+1} = \frac{\tilde{r}'\left(\tilde{r}_h - \tilde{r}\right) + \tilde{r}\left(\tilde{r}' - \tilde{r}_h'\right) + \left(\tilde{r}' - \tilde{r}_h'\right)}{\left(\tilde{r}+1\right)\left(\tilde{r}_h+1\right)}.$$

Using A1 and (3.10), we obtain

$$\begin{aligned} \left\| \frac{\tilde{r}'}{\tilde{r}+1} - \frac{\tilde{r}_h'}{\tilde{r}_h+1} \right\| &\le \frac{1}{r_m r_m/2}\left(\|\tilde{r}'\|_\infty \|\tilde{r}_h - \tilde{r}\| + (1+\|\tilde{r}\|_\infty)\|\tilde{r}' - \tilde{r}_h'\|\right) \\ &\le D_4 h^k. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we obtain (4.11). ∎

LEMMA 5. *For h sufficiently small (4.7) determines a unique $\tilde{w}_h \in X_h^w$.*

PROOF. As (4.7) represents a square linear system of equations, existence and uniqueness of $\tilde{w}_h$ is equivalent to the invertible of the coefficient matrix.

Choosing $v = \tilde{w}_h$ in (4.7), we have

$$\begin{aligned} A\left(\tilde{r}_h; \tilde{w}_h, \tilde{w}_h\right) &= \int_0^1 \tilde{w}_h'\left(\tilde{w}_h + \nu h \tilde{w}_h'\right) dz + \int_0^1 \mathsf{H}\left(\tilde{r}_h\right) \tilde{w}_h \left(\tilde{w}_h + \nu h \tilde{w}_h'\right) dz \\ &= \nu h \|\tilde{w}_h'\|^2 + \int_0^1 \tilde{w}_h' \tilde{w}_h \, dz + \int_0^1 \mathsf{H}\left(\tilde{r}_h\right) \tilde{w}_h \tilde{w}_h \, dz + \nu h \int_0^1 \mathsf{H}\left(\tilde{r}_h\right) \tilde{w}_h \tilde{w}_h' \, dz \\ &= \nu h \|\tilde{w}_h'\|^2 + \frac{1}{3}\tilde{w}_h(1)^2 + \int_0^1 \mathsf{H}\left(\tilde{r}\right) \tilde{w}_h \tilde{w}_h \, dz + \nu h \int_0^1 \mathsf{H}\left(\tilde{r}\right) \tilde{w}_h \tilde{w}_h' \, dz \\ &\quad + \int_0^1 \left(\mathsf{H}\left(\tilde{r}_h\right) - \mathsf{H}\left(\tilde{r}\right)\right) \tilde{w}_h \tilde{w}_h \, dz + \nu h \int_0^1 \left(\mathsf{H}\left(\tilde{r}_h\right) - \mathsf{H}\left(\tilde{r}\right)\right) \tilde{w}_h \tilde{w}_h' \, dz. \end{aligned}$$

Now, using Assumption A2, (4.11), (3.7), we have that

$$\begin{aligned} A\left(\tilde{r}_h; \tilde{w}_h, \tilde{w}_h\right) &\ge \nu h \|\tilde{w}_h'\|^2 + \frac{1}{2}\tilde{w}_h(1)^2 + c_1 \|\tilde{w}_h\|^2 - \nu h c_2 \|\tilde{w}_h\| \, \|\tilde{w}_h'\| \\ &\quad - \|\mathsf{H}\left(\tilde{r}_h\right) - \mathsf{H}\left(\tilde{r}\right)\| \, \|\tilde{w}_h\|_{L_4}^2 - \nu h \|\mathsf{H}\left(\tilde{r}_h\right) - \mathsf{H}\left(\tilde{r}\right)\| \, \|\tilde{w}_h\|_{L_4} \|\tilde{w}_h'\|_{L_4} \\ &\ge \nu h \left(1 - \frac{\nu h c_2^2}{4\epsilon_1}\right) \|\tilde{w}_h'\|^2 + \frac{1}{2}\tilde{w}_h(1)^2 + \left(c_1 - \epsilon_1\right) \|\tilde{w}_h\|^2 \\ &\quad - C_0 h^k h^{-1/2} \|\tilde{w}_h\|^2 - \nu h C_0 h^k h^{-1/2} \|\tilde{w}_h\| \|\tilde{w}_h'\| \\ &\ge \nu h \left(1 - \frac{\nu h c_2^2}{4\epsilon_1} - \frac{\nu h^{2k} C_0^2}{4\epsilon_2}\right) \|\tilde{w}_h'\|^2 \\ &\quad + \frac{1}{2}\tilde{w}_h(1)^2 + \left(c_1 - \epsilon_1 - C_0 h^{k-1/2} - \epsilon_2\right) \|\tilde{w}_h\|^2. \end{aligned} \tag{4.14}$$

Hence, for h sufficiently small (4.14) establishes the positivity of $A(\tilde{r}_h; \tilde{w}_h, v)$ which guarantees the invertibility of the approximating linear system. ∎

Next, consider the function $q(z)$ denoting the solution of

$$\begin{aligned} & q' + \left(\frac{\tilde{r}_h'}{2\left(\tilde{r}_h+1\right)} + \frac{1}{4}\left(B\left(\tilde{r}_h+1\right)^2 + F_c\right)\left(1+\tilde{r}_h'^2\right)\right) q \\ & \quad = -\left(\frac{\tilde{r}_h'}{2\left(\tilde{r}_h+1\right)} + \frac{1}{4}\left(B\left(\tilde{r}_h+1\right)^2 + F_c\right)\left(1+\tilde{r}_h'^2\right)\right), \end{aligned} \tag{4.15}$$

$$q(0) = 0. \tag{4.16}$$

Note that (4.15),(4.16) is a linear differential differential equation for $q(z)$. Existence and uniqueness of $q(z)$ follows from Lemma 3. Important in the subsequent error analysis for $q - \tilde{w}_h$ is the order of approximation of q by its interpolant in X_h^w. We address this issue in the following lemma.

LEMMA 6. *For q satisfying (4.15),(4.16), we have that $q \in H^{3/2-\epsilon}(I)$, for any $\epsilon > 0$. In addition, for $r \in H^{k+1}(I)$, the interpolant of $q \in X_h^w$, $\mathcal{Q}$, satisfies*

$$h\,\|q' - \mathcal{Q}'\| + \|q - \mathcal{Q}\| \leq C h^{\min\{k+1, m+1\}}. \tag{4.17}$$

PROOF. From (4.4) $q'(z)$ is given by

$$q'(z) = -\mathsf{H}\left(\tilde{r}_h\right)(z) e^{-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt}.$$

Using Corollary 3.1, we have that $\|\tilde{r}_h'\|_\infty$ and $\|\tilde{r}_h\|_\infty$ are bounded, and for h sufficiently small $(\tilde{r}_h + 1)$ is bounded away from zero. Thus, $\exp(-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt)$ is bounded for $0 \leq z \leq L$, as is $\|q'\|_\infty$. As q' is only a piecewise continuous function it follows that $q \in H^{3/2-\epsilon}(I)$, for any $\epsilon > 0$.

Because of this lack of regularity for $q(z)$ on I, we cannot use the standard interpolation result to conclude (4.17). However, on each subinterval $K \in T_h$ $q(z)$ has much higher regularity, and as the interpolants are constructed on each subinterval, it is the regularity within K which is important.

Without loss of generality assume that $n = \min\{k, m\}$. Then, on each subinterval K, we have

$$h\,\|q' - \mathcal{Q}'\|_{L^2(K)} + \|q - \mathcal{Q}\|_{L^2(K)} \leq C h^{n+1} \left\|q^{(n+1)}\right\|_{L^2(K)}. \tag{4.18}$$

We need to establish that $\|q^{n+1}\|_{L^2(K)}$ is bounded independent of h.

Note that $q^{(n+1)}$ is of the form

$$\begin{aligned} q^{(n+1)}(z) = -\mathsf{H}\left(\tilde{r}_h\right)^{(n)}(z) e^{-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt} + C\mathsf{H}\left(\tilde{r}_h\right)^{(n-1)}(z) \mathsf{H}\left(\tilde{r}_h\right)(z) e^{-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt} \\ + \cdots + \left(\mathsf{H}\left(\tilde{r}_h\right)(z)\right)^{n+1} e^{-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt}. \end{aligned} \tag{4.16}$$

We first show that $\|\mathsf{H}(\tilde{r}_h)^{(n)}(z)\exp(-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt)\|_{L^2(K)}$ is bounded independent of h. As $\exp(-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt)\|_\infty$ is bounded all we need show is that $\|\mathsf{H}(\tilde{r}_h)^{(n)}(z)\|_{L^2(K)}$ is bounded independent of h.

Note that the highest derivative term in $\mathsf{H}(\tilde{r}_h)^{(n)}(z)$ is $\tilde{r}_h^{(n+1)}$. For $\mathcal{R}$ the interpolant of $\tilde{r}$, and $E = \mathcal{R} - \tilde{r}_h$ (as defined above)

$$\begin{aligned} \left\|\tilde{r}_h^{(n+1)}\right\|_{L^2(K)} &\leq \left\|\mathcal{R}^{(n+1)}\right\|_{L^2(K)} + \left\|\mathcal{R}^{(n+1)} - \tilde{r}_h^{(n+1)}\right\|_{L^2(K)} \\ &\leq C \left\|\tilde{r}^{(n+1)}\right\|_{L^2(K)} + \left\|E^{(n+1)}\right\|_{L^2(K)} \\ &\leq C \left\|\tilde{r}^{(n+1)}\right\|_{L^2(K)} + h^{-n}\,\|E'\|_{L^2(K)}, \qquad \text{using (3.7).} \end{aligned}$$

Summing across all the subintervals K, we have using (3.38)

$$\begin{aligned} \sum_{K \in T_h} \left\|\tilde{r}_h^{(n+1)}\right\|_{L^2(K)}^2 &\leq \tilde{C}_1 \left\|\tilde{r}^{(n+1)}\right\|_{L^2(I)}^2 + \tilde{C}_2 h^{-2n}\,\|E'\|_{L^2(I)}^2 \\ &\leq \tilde{C}_1 \left\|\tilde{r}^{(n+1)}\right\|_{L^2(I)}^2 + \tilde{C}_2 h^{-2n} h^{2k} \left(\tilde{C}_3 \left\|\tilde{r}^{(k+1)}\right\|_{L^2(I)}^2 + \tilde{C}_4\right). \end{aligned}$$

As $\|\tilde{r}_h^{(n+1)}\|_{L^2(K)}$ is uniformly bounded across all the subintervals, independent of h, it follows that $\|\tilde{r}_h^{(j)}\|_{L^\infty(K)}$ is also uniformly bounded across all the subintervals, independent of h, for $j = 0, 1, \ldots, n$. Thus, all the terms on the right-hand side of (4.19), except the first term, are uniformly bounded independent of h and n. Moreover,

$$\begin{aligned} &\sum_{K \in T_h} \left\|\mathsf{H}\left(\tilde{r}_h\right)^{(n)}(z) e^{-\int_0^z \mathsf{H}(\tilde{r}_h)(t)\,dt}\right\|_{L^2(K)}^2 \\ &\qquad\qquad \leq C_1 \left\|\tilde{r}^{(n+1)}\right\|_{L^2(I)}^2 + h^{2(k-n)} \left(C_2 \left\|\tilde{r}^{(k+1)}\right\|_{L^2(I)}^2 + C_3\right) + C_4. \end{aligned}$$

Therefore, we have that

$$\left(\sum_{K\in T_h}\left\|q^{(n+1)}\right\|^2_{L^2(K)}\right)^{1/2} \le C_6\left(\left\|\tilde{r}^{(n+1)}\right\|_{L^2(I)} + h^{k-n}\left\|\tilde{r}^{(k+1)}\right\|_{L^2(I)}\right) + C_7,$$

which in view of (4.18) the stated result follows. ∎

We have the following estimate for $(q - \tilde{w}_h)$.

LEMMA 7. *For q satisfying (4.15),(4.16), h sufficiently small, and $\tilde{w}_h$ determined by (4.7), we have that*

$$\sqrt{h}\,\|q' - \tilde{w}_h'\| + \|q - \tilde{w}_h\| \le C_w h^{\min\{m+1/2, k+1/2\}}. \tag{4.20}$$

PROOF. Note that q satisfies $A(\tilde{r}_h; q, v) = F(\tilde{r}_h; v)$ for all $v \in X_h$. Therefore, we have that

$$A\left(\tilde{r}_h; q, v\right) - A\left(\tilde{r}_h; \tilde{w}_h, v\right) = \int_0^L \left(q' - \tilde{w}_h'\right)\hat{v}\,dz + \int_0^L \mathsf{H}\left(\tilde{r}_h\right)\left(q - \tilde{w}_h\right)\hat{v}\,dz = 0.$$

Let $\mathcal{Q}$ denote the interpolant of q in X_h^w, and introduce

$$\Lambda = q - \mathcal{Q}, \qquad E = \mathcal{Q} - \tilde{w}_h. \tag{4.21}$$

Then, $e := q - \tilde{w}_h = \Lambda + E$.

With these definitions, and the choice $v = E$, we have

$$\begin{aligned} A\left(\tilde{r}; \tilde{w}, E\right) - A\left(\tilde{r}_h; \tilde{w}_h, E\right) &= \int_0^L E'\left(E + \nu h E'\right)\,dz + \int_0^L \Lambda'\left(E + \nu h E'\right)\,dz \\ &\quad + \int_0^L \mathsf{H}\left(\tilde{r}_h\right) E\left(E + \nu h E'\right)\,dz + \int_0^L \mathsf{H}\left(\tilde{r}_h\right)\Lambda\left(E + \nu h E'\right)\,dz \\ &:= J_1 + J_2 + J_3 + J_4 = 0. \end{aligned} \tag{4.22}$$

We now proceed to estimate the terms J_1 through J_4.

$$\begin{aligned} J_1 &:= \int_0^L E'\left(E + \nu h E'\right)\,dz \\ &= \nu h\left\|E'\right\|^2 + \frac{1}{2}E(L)^2. \end{aligned} \tag{4.23}$$

$$\begin{aligned} J_2 &:= \int_0^L \Lambda'\left(E + \nu h E'\right)\,dz = \int_0^L \Lambda'\nu h E'\,dz + \int_0^L \Lambda' E\,dz \\ &\ge -\epsilon_{11}\nu h\left\|E'\right\|^2 - \frac{\nu h}{4\epsilon_{11}}\left\|\Lambda'\right\|^2 + \Lambda(L)E(L) - \int_0^L \Lambda E'\,dz \\ &\ge -\epsilon_{11}\nu h\left\|E'\right\|^2 - \frac{\nu h}{4\epsilon_{11}}\left\|\Lambda'\right\|^2 - \frac{1}{4}E(L)^2 - \Lambda(L)^2 - \epsilon_1 h\left\|E'\right\|^2 - \frac{1}{4\epsilon_1}h^{-1}\left\|\Lambda\right\|^2 \\ &\ge -\left(\epsilon_1 h + \epsilon_{11}\nu h\right)\left\|E'\right\|^2 - \frac{1}{4\epsilon_1}h^{-1}\left\|\Lambda\right\|^2 - \frac{\nu h}{4\epsilon_{11}}\left\|\Lambda'\right\|^2 - \frac{1}{4}E(L)^2 - \left\|\Lambda\right\|^2_\infty \end{aligned} \tag{4.24}$$

$$\begin{aligned} J_3 &:= \int_0^L \mathsf{H}\left(\tilde{r}_h\right) E\left(E + \nu h E'\right)\,dz \\ &= \int_0^L \left(\mathsf{H}\left(\tilde{r}_h\right) - \mathsf{H}\left(\tilde{r}\right)\right) E\left(E + \nu h E'\right)\,dz + \int_0^L \mathsf{H}\left(\tilde{r}\right) E\left(E + \nu h E'\right)\,dz \\ &\ge -\left\|\mathsf{H}\left(\tilde{r}\right) - \mathsf{H}\left(\tilde{r}_h\right)\right\|\left\|E\right\|^2_{L_4} - \left\|\mathsf{H}\left(\tilde{r}\right) - \mathsf{H}\left(\tilde{r}_h\right)\right\|\left\|E\right\|_\infty \nu h\left\|E'\right\| \\ &\quad + c_1\|E\|^2 - \left\|\mathsf{H}\left(\tilde{r}\right)\right\|_\infty\left\|E\right\|\nu h\left\|E'\right\| \\ &\ge -C_0 h^k\left(h^{-1/4}\|E\|\right)^2 - C_0 h^k h^{-1/2}\|E\|\nu h\left\|E'\right\| + c_1\|E\|^2 - c_2\|E\|\nu h\left\|E'\right\| \\ &\ge \left(c_1 - C_0 h^{k-1/2} - \frac{\nu}{4\epsilon_2}C_0^2 h^{2k} - \frac{\nu h}{4\epsilon_{12}}c_2^2\right)\|E\|^2 - \nu h\left(\epsilon_2 + \epsilon_{12}\right)\left\|E'\right\|^2 \end{aligned} \tag{4.25}$$

Similarly,

$$
\begin{aligned}
J_4 &:= \int_0^L \mathsf{H}(\tilde{r}_h)\,\Lambda\,(E+\nu h E')\;dz \\
&= \int_0^L (\mathsf{H}(\tilde{r}_h)-\mathsf{H}(\tilde{r}))\,\Lambda\,(E+\nu h E')\;dz + \int_0^L \mathsf{H}(\tilde{r})\,\Lambda\,(E+\nu h E')\;dz \\
&\geq -\|\mathsf{H}(\tilde{r})-\mathsf{H}(\tilde{r}_h)\|\,\|E\|\|\Lambda\|_\infty - \|\mathsf{H}(\tilde{r})-\mathsf{H}(\tilde{r}_h)\|\,\|\Lambda\|_\infty \nu h\,\|E'\| \\
&\quad - \|\mathsf{H}(\tilde{r})\|_\infty\,\|E\|\|\Lambda\| - \|\mathsf{H}(\tilde{r})\|_\infty\,\|\Lambda\|\nu h\,\|E'\| \\
&\geq -C_0 h^k\|E\|\|\Lambda\|_\infty - C_0 h^k\|\Lambda\|_\infty \nu h\,\|E'\| - c_2\|E\|\|\Lambda\| - c_2\|\Lambda\|\nu h\,\|E'\| \\
&\geq -(\epsilon_3+\epsilon_{14})\,\|E\|^2 - \nu h\,(\epsilon_{13}+\epsilon_{15})\,\|E'\|^2 \\
&\quad - \left(\frac{1}{4\epsilon_3}C_0^2 h^{2k} + \frac{\nu}{4\epsilon_{13}}C_0^2 h^{2k+1}\right)\|\Lambda\|_\infty^2 - \left(\frac{1}{4\epsilon_{14}}c_2^2 + \frac{\nu h}{4\epsilon_{15}}c_2^2\right)\|\Lambda\|^2.
\end{aligned}
\tag{4.26}
$$

Combining estimates (4.23)–(4.26) with (4.22), we have

$$
\begin{aligned}
&(\nu h\,(1-(\epsilon_2+\epsilon_{11}+\epsilon_{12}+\epsilon_{13}+\epsilon_{15})) - \epsilon_1 h)\,\|E'\|^2 + \frac{1}{4}E(L)^2 \\
&+ \left(c_1 - (\epsilon_3+\epsilon_{14}) - C_0 h^{k-1/2} - \frac{\nu}{4\epsilon_2}C_0^2 h^{2k} - \frac{\nu h}{4\epsilon_{12}}c_2^2\right)\|E\|^2 \\
&\qquad = \left(\frac{1}{4\epsilon_1}h^{-1} + \frac{1}{4\epsilon_{14}}c_2^2 + \frac{\nu h}{4\epsilon_{15}}c_2^2\right)\|\Lambda\|^2 \\
&+ \left(1 + \frac{1}{4\epsilon_3}C_0^2 h^{2k} + \frac{\nu}{4\epsilon_{13}}C_0^2 h^{2k+1}\right)\|\Lambda\|_\infty^2 + \left(\frac{\nu h}{4\epsilon_{11}}\right)\|\Lambda'\|^2.
\end{aligned}
\tag{4.27}
$$

Note that as we have a quasi-uniform mesh partition, applying (3.6) across each of the subintervals, we have that

$$\|\Lambda\|_\infty \leq C h^{1/2}\,\|\Lambda'\|\,.$$

Thus, for h sufficiently small,

$$h\,\|E'\|^2 + \|E\|^2 \leq C\left(h^{-1}\|\Lambda\|^2 + h\|\Lambda'\|^2\right). \tag{4.28}$$

Finally, as

$$\sqrt{h}\,\|q' - \tilde{w}_h'\| + \|q - \tilde{w}_h\| \leq \sqrt{h}\,(\|\Lambda'\| + \|E'\|) + (\|\Lambda\| + \|E\|)\,,$$

(4.20) follows from (4.28) and the interpolation properties of $\mathcal{Q}$. ∎

We now combine the above results to establish the convergence estimate.

Theorem 4.2. *For $m \in \mathbb{N}$, and assuming the conditions of Theorem 3.1, and A1 and A2 are satisfied, then there exists unique solutions $\tilde{w}$ and $\tilde{w}_h$ to (4.5), (4.6), and (4.7), respectively. In addition,*

$$\|\tilde{w}' - \tilde{w}_h'\| \leq C h^{\min\{m,k\}}. \tag{4.29}$$

Proof. From Lemma 3 and (4.19), $\tilde{w}$ is given by

$$
\begin{aligned}
\tilde{w}(z) &= -1 + e^{-\int_0^z \mathsf{H}(\tilde{r})(t)\,dt}, && (4.30)\\
\tilde{w}'(z) &= -\mathsf{H}(\tilde{r})\,(z) e^{-\int_0^z \mathsf{H}(\tilde{r})(t)\,dt} && (4.31)\\
\tilde{w}^{(j)}(z) &= -\mathsf{H}(\tilde{r})^{(j-1)}\,(z) e^{-\int_0^z \mathsf{H}(\tilde{r})(t)\,dt} + c\mathsf{H}(\tilde{r})^{(j-2)}\,(z)\mathsf{H}(\tilde{r})\,(z) e^{-\int_0^z \mathsf{H}(\tilde{r})(t)\,dt} \\
&\quad + \cdots + (\mathsf{H}(\tilde{r})\,(z))^j\, e^{-\int_0^z \mathsf{H}(\tilde{r})(t)\,dt}. && (4.32)
\end{aligned}
$$

If $\tilde{r} \in H^1(I)$ then from (4.31), $\tilde{w} \in W_1^1(I)$. For $\tilde{r} \in H^n(I)$, $n > 1$, then from (4.32) and the discussion in the proof of Lemma 6, $\tilde{w} \in H^n(I)$ also.

We have that

$$\|\tilde{w}' - \tilde{w}_h'\| \leq \|\tilde{w}' - q'\| + \|q' - \tilde{w}_h'\|\,. \tag{4.33}$$

From (4.5),(4.6) and (4.15),(4.16), we have that $g(z) := \tilde{w}(z) - q(z)$ satisfies (4.1),(4.2) for $\delta(z) := \mathsf{H}(\tilde{r}) - \mathsf{H}(\tilde{r}_h)$. Thus, (4.29) follows from (4.33), (4.11), and (4.20). ∎

4.1. Numerical Results for $w_h(z)$

In this section, we present numerical results for the approximation of the (dimensionless) film thickness, $w(z) = 1 + \tilde{w}(z)$. The numerical results are compared with the predicted theoretical results given in Theorem 4.2.

As described in Section 3.3, computations were performed on a sequence of uniform partitions of $[0, L]$. The approximation $\tilde{w}_h$ was computed as follows. First, on the given partition, $\tilde{r}_h$ was computed by solving (3.9). Then, (4.7) was solved for $\tilde{w}_h$. The values used for B, F_c, and L were the same as in Section 3.3. For ν, the value $\nu = 1$ was used. Various combinations of polynomial degrees were used for the approximation of $\tilde{r}$ and $\tilde{w}$. Analogous to (3.40), and using Theorem 4.2, we have that

$$\begin{aligned} \|\tilde{w}_h' - \tilde{w}_{2h}'\| &\leq \|\tilde{w}_h' - \tilde{w}'\| + \|\tilde{w}' - \tilde{w}_{2h}'\| \\ &\leq C h^{\min\{m,k\}} + C(2h)^{\min\{m,k\}} = \tilde{C} h^{\min\{m,k\}}, \end{aligned} \tag{4.34}$$

where k and m denotes the degree of the approximating, piecewise polynomials used for $\tilde{r}_h$ and $\tilde{w}_h$, respectively.

Table 2. Experimental rates of convergence for $\|w_h' - w'\|$, using a quadratic approximation for $\tilde{r}$.

	P/W Linear Approx. $(m=1)$		P/W Quad. Approx. $(m=2)$		P/W Cubic Approx. $(m=3)$	
$k=2$	$\|\tilde{w}_h' - \tilde{w}_{2h}'\|$	Cvge. Rate	$\|\tilde{w}_h' - \tilde{w}_{2h}'\|$	Cvge. Rate	$\|\tilde{w}_h' - \tilde{w}_{2h}'\|$	Cvge. Rate
$h = L/40$	5.01E − 02		7.86E − 03		8.49E − 03	
$h = L/80$	2.52E − 02	0.99	1.88E − 03	2.07	2.01E − 03	2.08
$h = L/160$	1.26E − 02	1.00	4.68E − 04	2.00	4.99E − 04	2.01
$h = L/320$	6.28E − 03	1.00	1.17E − 04	2.00	1.25E − 04	2.00
$h = L/640$	3.14E − 03	1.00	2.92E − 05	2.00	3.12E − 05	2.00
Predicted		1.0		2.0		2.0

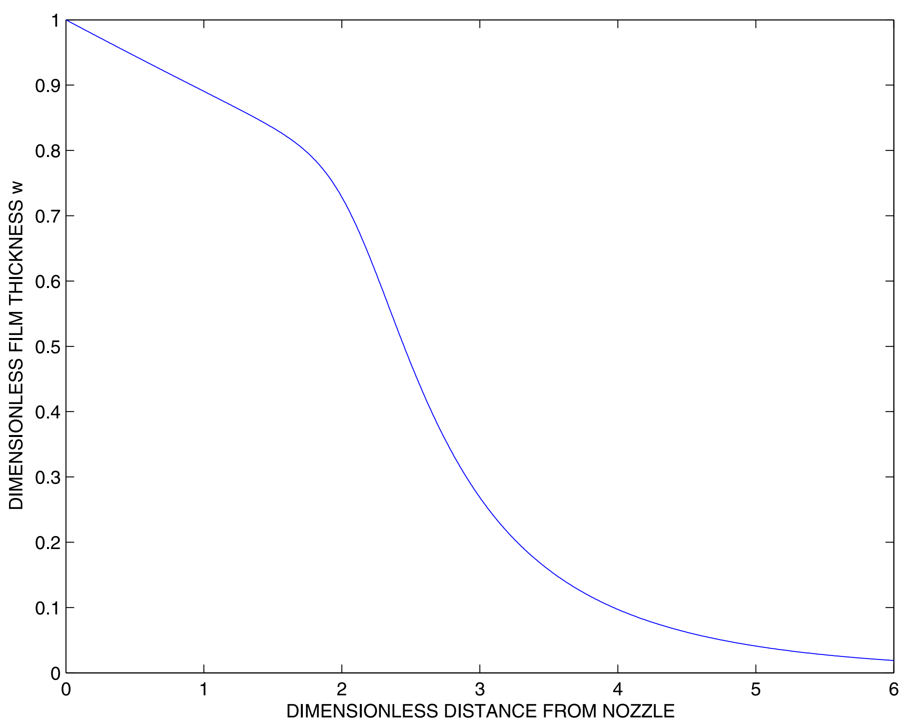

Figure 3. Plot of the dimensionless film thickness.

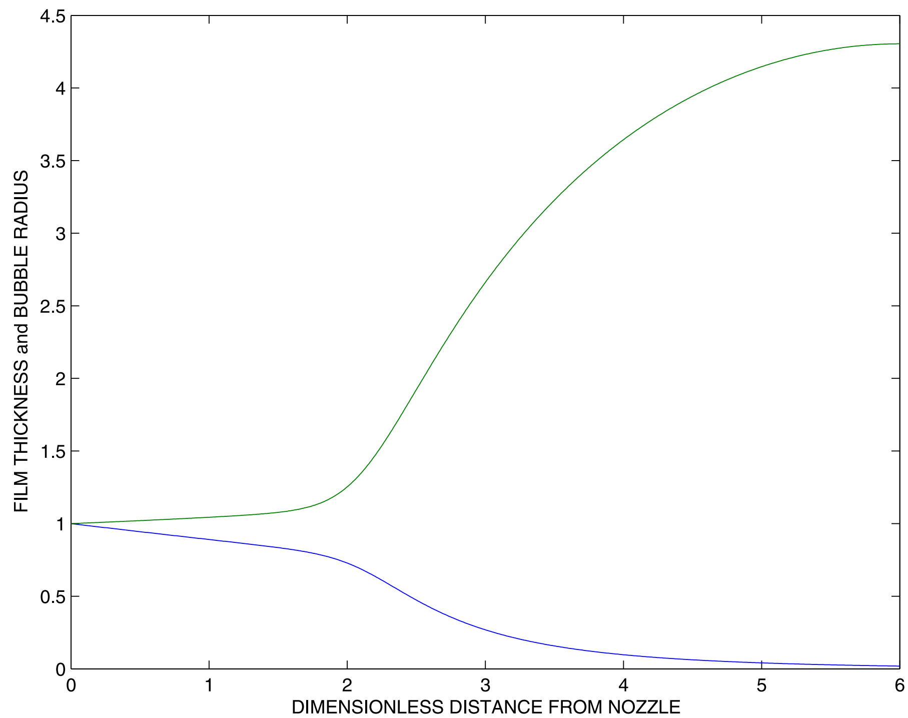

Figure 4. Plot of the (dimensionless) film thickness and bubble radius.

Presented in Table 2 are computations for $\|\tilde{w}'_h - \tilde{w}'_{2h}\|$, obtained using a piecewise quadratic approximation for $\tilde{r}$, i.e., $k = 2$, and piecewise linear, quadratic and cubic approximations for $\tilde{w}$. The numerical convergence rates agree with those predicted (see (4.34)). Computations were also performed using piecewise linear and piecewise quadratic approximations for $\tilde{r}$. The numerical converge rates for these cases also agree with those predicted by Theorem 4.2.

Displayed in Figure 3 is a plot of the (dimensionless) film thickness, computed using 320 subdivisions and quadratic approximations for $\tilde{r}$ and $\tilde{w}$. The profile of the film thickness is consistent with physically expectations. In Figure 4, both the quadratic approximations for $\tilde{r}$ and $\tilde{w}$, computed using 320 subdivisions, are displayed. Of interest to note is the consistency of the rate of change of $\tilde{r}_h$ and $\tilde{w}_h$, what is expected to occur.

APPENDIX

DETAILED BOUND DERIVATIONS

First, note that

$$\|\tilde{r}'_1\| \le \|\tilde{r}'\| + \|\tilde{r}' - \tilde{r}'_1\| \le \|\tilde{r}'\| + R := D_2, \tag{A.1}$$

$$\|\tilde{r}_1\|_\infty \le L^{1/2}\, \|\tilde{r}'_1\| \le D_2. \tag{A.2}$$

Lemma 8. *There exists a constant $D_4 < \infty$, such that*

$$\left\| \left(2\,(\tilde{r}_1 + 1)^2 \left(B\,(\tilde{r}_1 + 1)^2 + F_c \right) \right)' \right\| \le D_4\, \|\tilde{r}'_1\|\,. \tag{A.3}$$

PROOF. Expanding the derivative,

$$\begin{aligned}\left\|\left(2\left(\tilde{r}_1+1\right)^2\left(B\left(\tilde{r}_1+1\right)^2+F_c\right)\right)'\right\| &= 2\left\|2\tilde{r}_1'\left(\tilde{r}_1+1\right)\left(B\left(\tilde{r}_1+1\right)^2+F_c\right)\right.\\ &\quad \left.+\left(\tilde{r}_1+1\right)^2 2B\tilde{r}_1'\left(\tilde{r}_1+1\right)\right\|\\ &\le 4\left(\|\tilde{r}_1\|_\infty+1\right)\left(B\left(\|\tilde{r}_1\|_\infty+1\right)^2+F_c\right)\|\tilde{r}_1'\|\\ &\quad +4B\left(\|\tilde{r}_1\|_\infty+1\right)^2\left(\|\tilde{r}_1\|_\infty+1\right)\|\tilde{r}_1'\|\\ &= D_4\|\tilde{r}_1'\|.\end{aligned}$$

∎

LEMMA 9. *There exists constants $D_5, D_6 < \infty$, such that*

$$\left\|3B\left(\tilde{r}_1+1\right)^2\left(1+\tilde{r}_1'^2\right)\right\|_{L_1} \le D_5 + D_6\|\tilde{r}_1'\|^2. \tag{A.4}$$

PROOF. We have that

$$\begin{aligned}\left\|3B\left(\tilde{r}_1+1\right)^2\left(1+\tilde{r}_1'^2\right)\right\|_{L_1} &\le \left\|3B\left(\tilde{r}_1+1\right)^2\right\|_{L_1}+\left\|3B\left(\tilde{r}_1+1\right)^2\tilde{r}_1'^2\right\|_{L_1}\\ &\le L3B\left(\|\tilde{r}_1\|_\infty+1\right)^2+3B\left(\|\tilde{r}_1\|_\infty+1\right)^2\|\tilde{r}_1'\|^2\\ &= D_5+D_6\|\tilde{r}_1'\|^2\end{aligned}$$

∎

LEMMA 10. *There exists constants $D_7, D_8 < \infty$, such that*

$$\left\|\left(F_c-3B\left(\tilde{r}_1+1\right)^2\right)\left(1+\tilde{r}_1'^2\right)\right\|_{L_1} \le D_7 + D_8\|\tilde{r}_1'\|^2. \tag{A.5}$$

PROOF. The proof of (A.5) follows as that of (A.4). ∎

LEMMA 11. *There exists a constant $D_{25} < \infty$, such that*

$$\left\|\left(2G^2(w+1)^2\left(f_0+B\left((w+1)^2-1\right)\right)\right)'\right\| \le D_{25}\|\tilde{r}'\|. \tag{A.6}$$

PROOF. The proof of equation (A.6) follows similarly to that of (A.3). ∎

LEMMA 12. *There exists a constant $D_{26} < \infty$, such that*

$$\left\|\left[2\left(\tilde{r}-\tilde{r}_1\right)\left(\tilde{r}+\tilde{r}_1+2\right)\left(B\left[\left(\tilde{r}_1+1\right)^2+\left(\tilde{r}+1\right)^2\right]+F_c\right)\right]'\right\|\|\tilde{r}_1'\| \le D_{26}\|\tilde{r}'-\tilde{r}_1'\|. \tag{A.7}$$

PROOF. Expanding the derivative, we have

$$\begin{aligned}&\left\|\left[2\left(\tilde{r}-\tilde{r}_1\right)\left(\tilde{r}+\tilde{r}_1+2\right)\left(B\left[\left(\tilde{r}_1+1\right)^2+\left(\tilde{r}+1\right)^2\right]+F_c\right)\right]'\right\|\\ &= \left\|2\left(\tilde{r}'-\tilde{r}_1'\right)\left(\tilde{r}+\tilde{r}_1+2\right)\left(B\left[\left(\tilde{r}_1+1\right)^2+\left(\tilde{r}+1\right)^2\right]+F_c\right)\right.\\ &\quad +2\left(\tilde{r}-\tilde{r}_1\right)\left(\tilde{r}'+\tilde{r}_1'\right)\left(B\left[\left(\tilde{r}_1+1\right)^2+\left(\tilde{r}+1\right)^2\right]+F_c\right)\\ &\quad \left.+2\left(\tilde{r}-\tilde{r}_1\right)\left(\tilde{r}+\tilde{r}_1+2\right)2B\left(\left(\tilde{r}_1+1\right)\tilde{r}_1'+\left(\tilde{r}+1\right)\tilde{r}'\right)\right\|\\ &\le 2\|\tilde{r}'-\tilde{r}_1'\|\left(\|\tilde{r}\|_\infty+\|\tilde{r}_1\|_\infty+2\right)\left(B\left[\left(\|\tilde{r}_1\|_\infty+1\right)^2+\left(\|\tilde{r}\|_\infty+1\right)^2\right]+F_c\right)\\ &\quad +2\|\tilde{r}-\tilde{r}_1\|_\infty\left(\|\tilde{r}'\|+\|\tilde{r}_1'\|\right)\left(B\left[\left(\|\tilde{r}_1\|_\infty+1\right)^2+\left(\|\tilde{r}\|_\infty+1\right)^2\right]+F_c\right)\\ &\quad +4B\|\tilde{r}-\tilde{r}_1\|_\infty\left(\|\tilde{r}\|_\infty+\|\tilde{r}_1\|_\infty+2\right)\left(\left(\|\tilde{r}_1\|_\infty+1\right)\|\tilde{r}_1'\|+\left(\|\tilde{r}\|_\infty+1\right)\|\tilde{r}'\|\right).\end{aligned}$$

Using (3.6), we therefore obtain (A.7), where

$$\begin{aligned}D_{26} &= 2\left(\|\tilde{r}\|_\infty+\|\tilde{r}_1\|_\infty+2\right)\left(B\left[\left(\|\tilde{r}_1\|_\infty+1\right)^2+\left(\|\tilde{r}\|_\infty+1\right)^2\right]+F_c\right)\|\tilde{r}_1'\|\\ &\quad +2L^{1/2}\left(\|\tilde{r}'\|+\|\tilde{r}_1'\|\right)\left(B\left[\left(\|\tilde{r}_1\|_\infty+1\right)^2+\left(\|\tilde{r}\|_\infty+1\right)^2\right]+F_c\right)\|\tilde{r}_1'\|\\ &\quad +4BL^{1/2}\left(\|\tilde{r}\|_\infty+\|\tilde{r}_1\|_\infty+2\right)\left(\left(\|\tilde{r}_1\|_\infty+1\right)\|\tilde{r}_1'\|+\left(\|\tilde{r}\|_\infty+1\right)\|\tilde{r}'\|\right)\|\tilde{r}_1'\|.\end{aligned}$$

∎

LEMMA 13. *There exists a constant $D_{28} < \infty$, such that*

$$\|\tilde{r}' - \tilde{r}_1'\|_\infty \left\|3B\left[(\tilde{r}_1+1)(\tilde{r}'+\tilde{r}_1') + (\tilde{r}+\tilde{r}_1+2)\left(1+\tilde{r}'^2\right)\right]\right\|_{L_1} \|\tilde{r}_1\|_\infty \le D_{28}\|\tilde{r}'-\tilde{r}_1'\| \qquad \text{(A.8)}$$

PROOF. We have that

$$\begin{aligned}&\left\|3B\left[(\tilde{r}_1+1)(\tilde{r}'+\tilde{r}_1') + (\tilde{r}+\tilde{r}_1+2)\left(1+\tilde{r}'^2\right)\right]\right\|_{L_1}\\ &\qquad\le 3B\left[(\|\tilde{r}_1\|_\infty+1)L^{1/2}(\|\tilde{r}'\|+\|\tilde{r}_1'\|) + (\|\tilde{r}\|_\infty+\|\tilde{r}_1\|_\infty+2)\left(1+\|\tilde{r}'\|^2\right)\right]\end{aligned}$$

as $\|\cdot\|_{L_1} \le L^{1/2}\|\cdot\|$. Using (3.6), we therefore obtain (A.8), where

$$D_{28} = L^{1/2}3B\left[(\|\tilde{r}_1\|_\infty+1)L^{1/2}(\|\tilde{r}'\|+\|\tilde{r}_1'\|) + (\|\tilde{r}\|_\infty+\|\tilde{r}_1\|_\infty+2)\left(1+\|\tilde{r}'\|^2\right)\right]\|\tilde{r}_1\|_\infty . \qquad \blacksquare$$

REFERENCES

1. J.R.A. Pearson and C.J.S. Petrie, The flow of a tubular film. Part 1: Formal mathematical representation, *J. Fluid Mech.* **40**, 1–19, (1970).
2. J.R.A. Pearson and C.J.S. Petrie, The flow of a tubular film. Part 2: Interpretation of the model and discussion of solutions, *J. Fluid Mech.* **42**, 609-625, (1970).
3. C.D. Han and J.Y. Park, Studies on blown film extrusion I. Experimental determination of elongational viscosity, *J. Appl. Polym. Sci.* **19**, 3257–3276, (1975).
4. C.D. Han and J.Y. Park, Studies on blown film extrusion II. Analysis of deformation and heat transfer processes, *J. Appl. Polym. Sci.* **19**, 3277–3290, (1975).
5. C.D. Han and J.Y. Park, Studies on blown film extrusion III. Bubble instability, *J. Appl. Polym. Sci.* **19**, 3291–3297, (1975).
6. D.C.H. Tam, Mathematical analysis of a blown Newtonian film, M. App. Sc. Thesis, RMIT, (2003).
7. A.K. Doufas and A.J. McHugh, Simulation of film blowing including flow-induced crystallization, *J. Rheol.* **45**, 1085–1104, (2001).
8. V. Sidiropoulos, J.J. Tian and J. Vlachopoulos, Computer simulation of film blowing, *TAPPI Journal* **79** (8), 113–118, (1996).
9. C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge, (1987).
10. S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag Publishing, (1994).
11. X.L. Luo and R.I. Tanner, A computer study of film blowing, *Polym. Eng. Sci.* **25** (10), 620–629, (1985).