Variational Formulation for the Stationary Fractional Advection Dispersion Equation

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Abstract

In this paper a theoretical framework for the Galerkin finite element approximation to the steady state fractional advection dispersion equation is presented. Appropriate fractional derivative spaces are defined and shown to be equivalent to the usual fractional dimension Sobolev spaces H^s. Existence and uniqueness results are proven, and error estimates for the Galerkin approximation derived. Numerical results are included which confirm the theoretical estimates.

Key words. Finite element method, fractional differential operator, fractional diffusion equation, fractional advection dispersion equation.

AMS Mathematics subject classifications. 65N30, 35J99

1 Introduction

In this paper, we investigate the Galerkin approximation to the steady state Fractional Advection Dispersion Equation (FADE)

$$-D\, a\,(p\ {}_0D_x^{-\beta} + q\ {}_xD_1^{-\beta})Du \ + \ b(x)Du \ + c(x)u \ = \ f \ , \qquad (1)$$

where D represents a single spatial derivative, and ${}_0D_x^{-\beta}$, ${}_xD_1^{-\beta}$ represent left and right fractional integral operators, respectively, with $0 \leq \beta < 1$, and $0 \leq p, q \leq 1$, satisfying $p + q = 1$.

Our interest in (1) arises from its application as a model for physical phenomena exhibiting *anomalous* diffusion, i.e. diffusion not accurately modeled by the usual advection dispersion equation. Anomalous diffusion has been used in modeling turbulent flow [4, 12], and chaotic dynamics of classical conservative systems [14]. In viscoelasticity, fractional differential operators have been used to describe materials' constitutive equations [7]. Recently articles involving fractional differential operators have appeared in

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Physics Today [13], and *Nature* [6]. The application of central interest to us is that of contaminant transport in groundwater flow. In [2] the authors state that solutes moving through aquifers do not generally follow a Fickian, second-order, governing equation because of large deviations from the stochastic process of Brownian motion. This give rise to *superdiffusive* motion.

Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to L'Hospital in 1695. Early mathematicians who contributed to the study of fractional differential operators include Liouville, Riemann and Holmgrem. (See [8] for a history of the development of fractional differential operators). A number of definitions for the fractional derivative has emerged over the years: Grüwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, and the Caputo fractional derivative [9]. In this article we restrict our attention to the use the Riemann-Liouville fractional derivative.

To date most solution techniques for equations involving fractional differential operators have exploited the properties of the Fourier and Laplace transforms of the operators to determine a classical solution. Finite difference have also been applied to construct numerical approximation [9]. Aside from [5] we are not aware of any other papers in the literature which investigate the Galerkin approximation and associated error analysis for the FADE.

There are two properties of fractional differential operators which make the analysis of the variational solution to the FADE more complicated than than that for the usual advection dispersion equation. These are

(i) fractional differential operators are <u>not</u> local operators, and

(ii) the adjoint of a fractional differential operator is <u>not</u> the negative of itself.

Because of (i) and (ii) the correct function space setting for the variational solution is not obvious. Related to the left fractional derivative we introduce the J_L^μ space and corresponding to the right fractional derivative the J_R^μ space. We are able to relate these spaces to the fractional Sobolev space H^μ through an intermediate space J_S^μ.

This paper is organized as follows. In Section 2 we develop the appropriate functional setting for the variational solution of FADEs. In Section 3 we then prove existence and uniqueness of the variational solution. The Galerkin approximation is introduced in Section 4, and convergence results for the Galerkin approximation are derived. Numerical results demonstrating the convergence of the Galerkin approximation are presented in Section 5. Contained in the Appendix are the definitions of the Riemann-Liouville fractional derivative and integral operators, together with some other useful properties of these operators.

2 Fractional Derivative Spaces

In this section we develop the abstract setting for the analysis of the approximation to FADEs. We introduce associated left, right, and symmetric fractional derivative spaces. The equivalence of the fractional derivative spaces with fractional order Hilbert spaces is then established.

In order to define the spaces, for $G \subset \mathbb{R}$ an open interval (which may be unbounded) we let $C_0^\infty(G)$ denote the set of all functions $u \in C^\infty(G)$ that vanish outside a compact subset K of G.

Definition 2.1 [Left Fractional Derivative Space] Let $\mu > 0$. Define the semi-norm

$$|u|_{J_L^\mu(\mathbb{R})} := \|\mathbf{D}^\mu u\|_{L^2(\mathbb{R})},$$

and norm

$$\|u\|_{J_L^\mu(\mathbb{R})} := \left(\|u\|_{L^2(\mathbb{R})}^2 + |u|_{J_L^\mu(\mathbb{R})}^2\right)^{1/2}, \tag{2}$$

and let $J_L^\mu(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{J_L^\mu(\mathbb{R})}$.

In the following analysis, we define a semi-norm for functions in $H^\mu(\mathbb{R})$ in terms of the Fourier transform.

Definition 2.2 Let $\mu > 0$. Define the semi-norm

$$|u|_{H^\mu(\mathbb{R})} := \|\,|\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})}, \tag{3}$$

and norm

$$\|u\|_{H^\mu(\mathbb{R})} := \left(\|u\|_{L^2(\mathbb{R})}^2 + |u|_{H^\mu(\mathbb{R})}^2\right)^{1/2},$$

and let $H^\mu(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{H^\mu(\mathbb{R})}$.

Theorem 2.1 The spaces $J_L^\mu(\mathbb{R})$ and $H^\mu(\mathbb{R})$ are equal with equivalent semi-norms and norms.

Proof. The proof follows immediately from the following lemma. ■

Lemma 2.2 Let $\mu > 0$, be given. A function $u \in L^2(\mathbb{R})$ belongs to $J_L^\mu(\mathbb{R})$ if and only if

$$|\omega|^\mu \hat{u} \in L^2(\mathbb{R}). \tag{4}$$

Specifically,

$$|u|_{J_L^\mu(\mathbb{R})} = \|\,|\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})}. \tag{5}$$

Proof. Let $u \in J_L^\mu(\mathbb{R})$ be given. Then $\mathbf{D}^\mu u \in L^2(\mathbb{R})$, and from (49) and (45)

$$\mathcal{F}(\mathbf{D}^\mu u) = (i\omega)^\mu \hat{u}. \tag{6}$$

Using Plancherel's theorem, we have

$$\int_{\mathbb{R}} |\omega|^{2\mu} |\hat{u}|^2 d\omega = \int_{\mathbb{R}} |\mathbf{D}^\mu u|^2\, dx.$$

Hence,

$$\|\,|\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})} = |u|_{J_L^\mu(\mathbb{R})}.$$

■

Analogous to $J_L^\mu(\mathbb{R})$ we introduce $J_R^\mu(\mathbb{R})$, the right fractional derivative space, and establish their equivalence.

Definition 2.3 [Right Fractional Derivative Space] Let $\mu > 0$. Define the semi-norm

$$|u|_{J_R^\mu(\mathbb{R})} := \|\mathbf{D}^{\mu *} u\|_{L^2(\mathbb{R})},$$

and norm

$$\|u\|_{J_R^\mu(\mathbb{R})} := \left(\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{J_R^\mu(\mathbb{R})}\right)^{1/2}, \tag{7}$$

and let $J_R^\mu(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{J_R^\mu(\mathbb{R})}$.

Theorem 2.3 Let $\mu > 0$. The spaces $J_L^\mu(\mathbb{R})$ and $J_R^\mu(\mathbb{R})$ are equal, with equivalent semi-norms and norms.

Proof. We need only verify that the $J_L(\mathbb{R})$ and $J_R(\mathbb{R})$ semi-norms are equivalent. This is done using the Fourier transform. Combining (44), the definitions of $\mathbf{D}^\mu$, $\mathbf{D}^{\mu *}$, and Plancherel's theorem yields

$$\begin{aligned} |u|^2_{J_L^\mu(\mathbb{R})} &= \int_{\mathbb{R}} |(i\omega)^\mu \hat{u}(\omega)|^2 d\omega \\ |u|^2_{J_R^\mu(\mathbb{R})} &= \int_{\mathbb{R}} |(-i\omega)^\mu \hat{u}(\omega)|^2 d\omega. \end{aligned} \tag{8}$$

Thus the semi-norms are equivalent, and, in fact, equal as $|(i\omega)^\mu| = |(-i\omega)^\mu|$. ■

In the finite element analysis of (1), we make use of the bilinear functional $(\mathbf{D}^\mu \cdot, \mathbf{D}^{\mu *} \cdot)$. For the case of the entire real line, we can relate this mapping to $|\cdot|_{J_L^\mu(\mathbb{R})}$.

Lemma 2.4 Let $\mu > 0$, n be the smallest integer greater than μ ($n - 1 \leq \mu < n$), and $\sigma = n - \mu$. Then for $u(x)$ a real valued function

$$(\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) = \cos(\pi\mu) \|\mathbf{D}^\mu u\|^2_{L^2(\mathbb{R})}. \tag{9}$$

Proof. Helpful in establishing this result is the Fourier transform property ($\overline{}$ denotes complex conjugate)

$$\int_{\mathbb{R}} u\bar{v}\, dx = \int_{\mathbb{R}} \hat{u}\bar{\hat{v}}\, d\omega, \tag{10}$$

and the observation that

$$\overline{(i\omega)^\mu} = \begin{cases} \exp(-i\pi\mu)\ \overline{(-i\omega)^\mu} & \text{if } \omega \geq 0 \\ \exp(i\pi\mu)\ \overline{(-i\omega)^\mu} & \text{if } \omega < 0 \end{cases}. \tag{11}$$

Thus

$$\begin{aligned} (\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) &= \left(D^n\ {}_{-\infty}D_x^{-\sigma} u,\ \overline{(-D)^n\ {}_xD_\infty^{-\sigma} u}\right) \\ &= \left((i\omega)^\mu \hat{u},\ \overline{(-i\omega)^\mu \hat{u}}\right) \\ &= \int_{-\infty}^0 (i\omega)^\mu \hat{u}\ \overline{(-i\omega)^\mu \hat{u}}\, d\omega \\ &\quad + \int_0^\infty (i\omega)^\mu \hat{u}\ \overline{(-i\omega)^\mu \hat{u}}\, d\omega. \end{aligned}$$

Using (11) this becomes

$$(\mathbf{D}^\mu u,\ \mathbf{D}^{\mu *} u) = \int_{-\infty}^0 (i\omega)^\mu \hat{u}\ \exp(-i\pi\mu)\ \overline{(i\omega)^\mu \hat{u}}\ d\omega$$

$$
\begin{aligned}
&\quad + \int_0^{\infty} (i\omega)^\mu \hat{u} \, \exp(i\pi\mu) \, \overline{(i\omega)^\mu \hat{u}} \, d\omega \\
&= \cos(\pi\mu) \int_{-\infty}^{\infty} (i\omega)^\mu \hat{u} \, \overline{(i\omega)^\mu \hat{u}} \, d\omega \\
&\quad + i \sin(\pi\mu) \left(\int_0^{\infty} (i\omega)^\mu \hat{u} \, \overline{(i\omega)^\mu \hat{u}} \, d\omega \right. \\
&\qquad \left. - \int_{-\infty}^{0} (i\omega)^\mu \hat{u} \, \overline{(i\omega)^\mu \hat{u}} \, d\omega \right). \qquad (12)
\end{aligned}
$$

For $f(x)$ real we have that $\overline{\mathcal{F}(f)(-\omega)} = \mathcal{F}(f)(\omega)$. Thus

$$
\int_{-\infty}^{0} (i\omega)^\mu \hat{u} \, \overline{(i\omega)^\mu \hat{u}} \, d\omega = \int_0^{\infty} (i\omega)^\mu \hat{u} \, \overline{(i\omega)^\mu \hat{u}} \, d\omega. \qquad (13)
$$

Therefore, combining (12) and (13) we obtain

$$
(\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) = \cos(\pi\mu) \left(\mathbf{D}^\mu u, \mathbf{D}^\mu u \right).
$$

■

Remark. Note that for $\mu = n - 1/2$, $n \in \mathbb{N}$, $(\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) = 0$. For example, $(\mathbf{D}^{1/2} u, \mathbf{D}^{1/2*} u) = (Du, u) = 0$, provided that u vanishes on the boundary.

Definition 2.4 [Symmetric Fractional Derivative Space] Let $\mu > 0, \mu \neq n - 1/2, n \in \mathbb{N}$. Define the semi-norm

$$
|u|_{J_S^\mu(\mathbb{R})} := \left| (\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u)_{L^2(\mathbb{R})} \right|^{1/2},
$$

and norm

$$
\|u\|_{J_S^\mu(\mathbb{R})} := (\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{J_S^\mu(\mathbb{R})})^{1/2} \qquad (14)
$$

and let $J_S^\mu(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|\cdot\|_{J_S^\mu(\mathbb{R})}$.

Theorem 2.5 For $\mu > 0, \mu \neq n - 1/2, n \in \mathbb{N}$, the spaces $J_L^\mu(\mathbb{R})$ and $J_S^\mu(\mathbb{R})$ are equal, with equivalent semi-norms and norms.

Proof. We must show that the $J_L^\mu(\mathbb{R})$ and $J_S^\mu(\mathbb{R})$ semi-norms are equivalent. We have from the previous lemma that

$$
|u|^2_{J_S^\mu(\mathbb{R})} = |\cos(\pi\mu)| \; |u|^2_{J_L^\mu(\mathbb{R})}. \qquad (15)
$$

■

Let $\Omega = (l, r)$ be a bounded open subinterval of $\mathbb{R}$. We now restrict the fractional derivative spaces to Ω.

Definition 2.5. Define the spaces $J_{L,0}^\mu(\Omega)$, $J_{R,0}^\mu(\Omega)$, $J_{S,0}^\mu(\Omega)$ as the closures of $C_0^\infty(\Omega)$ under their respective norms.

Following are several useful intermediate results which we use in order to relate the spaces $J_{L,0}^\mu(\Omega), J_{R,0}^\mu(\Omega)$, and $J_{S,0}^\mu(\Omega)$ to the fractional Sobolev space $H_0^\mu(\Omega)$.

Lemma 2.6 Let $\mu > 0$. The following mapping properties hold.

(i) $\mathbf{D}^{-\mu} : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator.

(ii) $\mathbf{D}^{-\mu} : L^2(\Omega) \to J_L^\mu(\Omega)$ is a bounded linear operator.

(iii) $\mathbf{D}^{\mu} : J_L^\mu(\Omega) \to L^2(\Omega)$ is a bounded linear operator.

(iv) $\mathbf{D}^{-\mu*} : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator.

(v) $\mathbf{D}^{-\mu*} : L^2(\Omega) \to J_R^\mu(\Omega)$ is a bounded linear operator.

(vi) $\mathbf{D}^{\mu*} : J_R^\mu(\Omega) \to L^2(\Omega)$ is a bounded linear operator.

Proof. In order to prove (i) we note that $\mathbf{D}^{-\mu}u = \frac{x^{\mu-1}}{\Gamma(\mu)} * u$, where $*$ denotes convolution. Using Young's theorem [1],

$$\|v * w\|_{L^p(\Omega)} \leq \|v\|_{L^1(\Omega)}\|w\|_{L^p(\Omega)},\ 1 \leq p < \infty,$$

we have

$$\begin{aligned}
\|\mathbf{D}^{-\mu}u\|_{L^2(\Omega)} &= \frac{1}{\Gamma(\mu)}\|x^{\mu-1} * u\|_{L^2(\Omega)} \\
&\leq \frac{1}{\Gamma(\mu)}\|x^{\mu-1}\|_{L^1(\Omega)}\|u\|_{L^2(\Omega)} \\
&= \frac{1}{\Gamma(\mu)}\int_\Omega |x|^{\mu-1}\,dx\,\|u\|_{L^2(\Omega)} \\
&\leq \frac{|r|^\mu + |l|^\mu}{\Gamma(\mu+1)}\|u\|_{L^2(\Omega)}.
\end{aligned}$$

From the definition of $J_L^\mu(\Omega)$ and using Property A.4 (in the Appendix), we have

$$\begin{aligned}
\|\mathbf{D}^{-\mu}u\|_{J_L^\mu(\Omega)} &= (\|\mathbf{D}^{-\mu}u\|^2_{L^2(\Omega)} + \|\mathbf{D}^{\mu}\,\mathbf{D}^{-\mu}u\|^2_{L^2(\Omega)})^{1/2}. \\
&= (\|\mathbf{D}^{-\mu}u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})^{1/2}.
\end{aligned}$$

Then, using (i), property (ii) follows.

The result (iii) follows directly from the definition of the $J_L^\mu(\Omega)$ norm as

$$\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \leq (\|u\|^2_{L^2(\Omega)} + \|\mathbf{D}^{\mu}u\|^2_{L^2(\Omega)})^{1/2}.$$

Finally, the proofs of (iv) - (vi) are analogous to the proofs of (i) - (iii). ■

Lemma 2.7 For $u \in J_{L,0}^\mu(\Omega)$, we have $\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u = u$, and for $u \in J_{R,0}^\mu(\Omega)$, we have $\mathbf{D}^{-\mu*}\mathbf{D}^{\mu*}u = u$.

Proof. By the definition of $J_{L,0}^\mu(\Omega)$, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|u - \phi_n\|_{J_L^\mu(\Omega)} = 0.$$

Applying the triangle inequality, we have

$$\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u - u\|_{J_L^\mu(\Omega)} \leq \|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}(u - \phi_n)\|_{J_L^\mu(\Omega)} + \|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}\phi_n - \phi_n\|_{J_L^\mu(\Omega)} + \|\phi_n - u\|_{J_L^\mu(\Omega)}.$$

Since $\phi_n \in C_0^\infty(\Omega)$, Property A.6 (in the Appendix) implies that $\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}\phi_n - \phi_n\|_{J_L^\mu(\Omega)} = 0$. By the mapping properties in Lemma 2.6,

$$\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}(u - \phi_n)\|_{J_L^\mu(\Omega)} \leq C\|u - \phi_n\|_{J_L^\mu(\Omega)}.$$

Thus,

$$\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u - u\|_{J_L^\mu(\Omega)} \leq (C+1)\|u - \phi_n\|_{J_L^\mu(\Omega)}.$$

Taking the limit as $n \to \infty$, we obtain the stated result for $J^{\mu}_{L,0}(\Omega)$. The result for $J^{\mu}_{R,0}(\Omega)$ follows similarly. ■

Corollary 2.8 For $u \in J^{\mu}_{L,0}(\Omega)$, $0 < s < \mu$,

$$\mathbf{D}^{-s}\,\mathbf{D}^{s}\,\mathbf{D}^{\mu-s}u = \mathbf{D}^{\mu-s}u.$$

Proof. Proceeding as in the proof of Lemma 2.7,

$$\begin{aligned}
\|\mathbf{D}^{-s}\,\mathbf{D}^{s}\,\mathbf{D}^{\mu-s}u - \mathbf{D}^{\mu-s}u\|_{L^2(\Omega)} &\leq \|\mathbf{D}^{-s}\,\mathbf{D}^{s}\,\mathbf{D}^{\mu-s}(u-\phi_n)\|_{L^2(\Omega)} \\
&\quad + \|\mathbf{D}^{-s}\mathbf{D}^{s}\mathbf{D}^{\mu-s}\phi_n - \mathbf{D}^{\mu-s}\phi_n\|_{L^2(\Omega)} \\
&\quad + \|\mathbf{D}^{\mu-s}(u-\phi_n)\|_{L^2(\Omega)} \\
&\leq \|\mathbf{D}^{-s}\,\mathbf{D}^{\mu}(u-\phi_n)\|_{L^2(\Omega)} + \|u-\phi_n\|_{J^{\mu-s}_{L,0}(\Omega)} \\
&\quad + \|\mathbf{D}^{-s}\,\mathbf{D}^{s}\,\mathbf{D}^{\mu-s}\phi_n - \mathbf{D}^{\mu-s}\phi_n\|_{L^2(\Omega)}.
\end{aligned}$$

An elementary calculation shows that for $\phi_n \in C_0^{\infty}(\Omega)$,

$$\mathbf{D}^{\mu-s}\phi_n(l) = 0.$$

Hence, by Property A.6,

$$\|\mathbf{D}^{-s}\,\mathbf{D}^{s}\,\mathbf{D}^{\mu-s}\phi_n - \mathbf{D}^{\mu-s}\phi_n\|_{L^2(\Omega)} = 0.$$

The stated result then follows from the convergence of ϕ_n to u and the mapping properties in Lemma 2.6. ■

Analogous to fractional integral operators, fractional differential operators also satisfy a semi-group property.

Lemma 2.9 For $u \in J^{\mu}_{L,0}(\Omega)$, $0 < s < \mu$, we have

$$\mathbf{D}^{\mu}u = \mathbf{D}^{s}\mathbf{D}^{\mu-s}u$$

and, similarly for $u \in J^{\mu}_{R,0}(\Omega)$,

$$\mathbf{D}^{\mu*}u = \mathbf{D}^{s*}\mathbf{D}^{(\mu-s)*}u.$$

Proof. Let $u \in J^{\mu}_{L,0}(\Omega)$. Then by Lemma 2.7 and Property A.1, we have

$$u = \mathbf{D}^{-\mu}\mathbf{D}^{\mu}u = \mathbf{D}^{s-\mu}\mathbf{D}^{-s}\mathbf{D}^{\mu}u.$$

Applying $\mathbf{D}^{s}\mathbf{D}^{\mu-s}$ to both sides and using Property A.4,

$$\mathbf{D}^{s}\mathbf{D}^{\mu-s}u = \mathbf{D}^{s}\mathbf{D}^{\mu-s}\mathbf{D}^{s-\mu}\mathbf{D}^{-s}\mathbf{D}^{\mu}u = \mathbf{D}^{\mu}u.$$

The result for $J^{\mu}_{R,0}(\Omega)$ follows analogously. ■

Theorem 2.10 [Fractional Poincaré-Friedrichs] For $u \in J^{\mu}_{L,0}(\Omega)$, we have

$$\|u\|_{L^2(\Omega)} \leq C|u|_{J^{\mu}_{L}(\Omega)}, \tag{16}$$

and for $u \in J^{\mu}_{R,0}(\Omega)$,

$$\|u\|_{L^2(\Omega)} \le C|u|_{J^{\mu}_{R}(\Omega)} \,. \tag{17}$$

Proof. Combining Lemmas 2.6 and 2.7, for $u \in J^{\mu}_{L,0}(\Omega)$

$$\|u\|_{L^2(\Omega)} = \|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \le C\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \;=\; C|u|_{J^{\mu}_{L}(\Omega)} \,.$$

Inequality (17) follows similarly. ∎

Corollary 2.11 For $u \in J^{\mu}_{L,0}(\Omega)$, $0 < s < \mu$, we have

$$|u|_{J^{s}_{L}(\Omega)} \le C|u|_{J^{\mu}_{L}(\Omega)},$$

and for $u \in J^{\mu}_{R,0}(\Omega)$, $0 < s < \mu$,

$$|u|_{J^{s}_{R}(\Omega)} \le C|u|_{J^{\mu}_{R}(\Omega)}.$$

Proof. Using Lemma 2.9, for $u \in J^{\mu}_{L,0}(\Omega)$

$$\begin{aligned}
\|\mathbf{D}^{s}u\|_{L^2(\Omega)} &= \|\mathbf{D}^{s}\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \\
&= \|\mathbf{D}^{s}\mathbf{D}^{-s}\mathbf{D}^{s-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \\
&= \|\mathbf{D}^{s-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \\
&\le C\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)}.
\end{aligned}$$

The result for $J^{\mu}_{R,0}(\Omega)$ follows analogously. ∎

We next turn to the equivalence of the fractional derivative spaces $J^{\mu}_{L,0}(\Omega)$, $J^{\mu}_{R,0}(\Omega)$, and the fractional order Sobolev spaces $H^{\mu}_{0}(\Omega)$. The analysis of the equivalence of these spaces is complicated by the *non-localness* of the fractional differential operators.

Theorem 2.12 Let $\mu > 0, \mu \neq n - 1/2, n \in \mathbb{N}$. Then the $J^{\mu}_{S,0}(\Omega)$ and $H^{\mu}_{0}(\Omega)$ spaces are equal, with equivalent semi-norms and norms.

Proof. Let $u \in C^{\infty}_{0}(\Omega)$ and $\tilde{u}$ be the extension of u by zero outside of Ω. Then $\text{supp}(\tilde{u}) \subset (l, r)$. However, as the left and right fractional derivatives are non-local,

$$\text{supp}(\mathbf{D}^{\mu}\tilde{u}) \subset (l, \infty)$$

and

$$\text{supp}(\mathbf{D}^{\mu *}\tilde{u}) \subset (-\infty, r).$$

Nonetheless, the product $\mathbf{D}^{\mu}u\,\mathbf{D}^{\mu *}u$ has support in $\Omega = (l, r)$. Hence,

$$|u|_{J^{\mu}_{S,0}(\Omega)} = |\tilde{u}|_{J^{\mu}_{S}(\mathbb{R})},$$

and

$$|u|_{H^{\mu}_{0}(\Omega)} = |\tilde{u}|_{H^{\mu}(\mathbb{R})}.$$

From Theorem 2.1 and Theorem 2.5 we have that $J^{\mu}_{S}(\mathbb{R})$ and $H^{\mu}(\mathbb{R})$ are equal with equivalent norms. Thus the norms $\|\cdot\|_{J^{\mu}_{S,0}(\mathbb{R})}$ and $\|\cdot\|_{H^{\mu}_{0}(\mathbb{R})}$ are also equivalent. Finally, as $J^{\mu}_{S,0}(\Omega)$ and $H^{\mu}_{0}(\Omega)$ are the closures of $C^{\infty}_{0}(\Omega)$ with respect to equivalent norms, the spaces must also be equal. ∎

The preceding theorem followed from the fact that, for functions with support restricted to the interior of Ω, $|u|_{J^{\mu}_{S,0}(\Omega)} = |\tilde{u}|_{J^{\mu}_{S}(\mathbb{R})}$. This is not the case for the spaces $J^{\mu}_{L,0}(\Omega)$ where we only have $|u|_{J^{\mu}_{L,0}(\Omega)} \le |\tilde{u}|_{J^{\mu}_{L}(\mathbb{R})}$. However, following Theorem 2.13 we show that $|\tilde{u}|_{J^{\mu}_{L}(\mathbb{R})} \le C|\tilde{u}|_{J^{\mu}_{L}(\Omega)}$, which implies that the contribution to $|\tilde{u}|_{J^{\mu}_{L}(\mathbb{R})}$ from outside of Ω is bounded by a constant times that from the interior of Ω.

Theorem 2.13 Let $\mu > 0$. Then the $J^{\mu}_{L,0}(\Omega), J^{\mu}_{R,0}(\Omega)$, and $H^{\mu}_{0}(\Omega)$ spaces are equal. Also, if $\mu \ne n - 1/2, n \in \mathbb{N}$, the spaces $J^{\mu}_{L,0}(\Omega), J^{\mu}_{R,0}(\Omega)$, and $H^{\mu}_{0}(\Omega)$ have equivalent semi-norms and norms.

Proof. Again, let $u \in C^{\infty}_{0}(\Omega)$ and $\tilde{u}$ be the extension of u by zero outside of Ω. From (3) and (8) we have

$$\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \;=\; |u|_{J^{\mu}_{L}(\Omega)} \le |\tilde{u}|_{J^{\mu}_{L}(\mathbb{R})} \;=\; |\tilde{u}|_{H^{\mu}(\mathbb{R})} \;=\; |u|_{H^{\mu}(\Omega)}, \tag{18}$$

and thus $H^{\mu}_{0}(\Omega) \subseteq J^{\mu}_{L,0}(\Omega)$.

In order to show the reverse inequality, we use the equivalence of the $J^{\mu}_{S,0}(\Omega)$ and $H^{\mu}_{0}(\Omega)$ norms for $\mu \ne n - 1/2, n \in \mathbb{N}$ (Theorem 2.12). Using Young's inequality, we obtain

$$\begin{aligned}
|u|^2_{H^{\mu}_{0}(\Omega)} &\le C|u|^2_{J^{\mu}_{S,0}(\Omega)} \\
&= C\left|\int_{\Omega} \mathbf{D}^{\mu}u \; \mathbf{D}^{\mu *}u dx\right| \\
&\le \frac{C}{4\epsilon}\|\mathbf{D}^{\mu}u\|^2_{L^2(\Omega)} + C\epsilon\|\mathbf{D}^{\mu *}u\|^2_{L^2(\Omega)} \\
&= \frac{C}{4\epsilon}|u|^2_{J^{\mu}_{L,0}(\Omega)} + C\epsilon|u|^2_{J^{\mu}_{R,0}(\Omega)} \\
&\le \frac{C}{4\epsilon}|u|^2_{J^{\mu}_{L,0}(\Omega)} + C\epsilon|\tilde{u}|^2_{J^{\mu}_{R}(\mathbb{R})} \\
&= \frac{C}{4\epsilon}|u|^2_{J^{\mu}_{L,0}(\Omega)} + C\epsilon|\tilde{u}|^2_{H^{\mu}(\mathbb{R})} \\
&= \frac{C}{4\epsilon}|u|^2_{J^{\mu}_{L,0}(\Omega)} + C\epsilon|u|^2_{H^{\mu}_{0}(\Omega)}.
\end{aligned}$$

Therefore, taking $\epsilon = 1/(2C)$, we have

$$|u|^2_{H^{\mu}_{0}(\Omega)} \le C^2|u|^2_{J^{\mu}_{L,0}(\Omega)},$$

from which we obtain $J^{\mu}_{L,0}(\Omega) \subseteq H^{\mu}_{0}(\Omega)$.

The equivalence of norms follows then from the equivalence of semi-norms and the definition of the $J^{\mu}_{L,0}(\Omega)$ and $H^{\mu}_{0}(\Omega)$ norms. The result for $J^{\mu}_{R,0}(\Omega)$ follows analogously. ■

Corollary 2.14 Let $\mu \ne n - 1/2, n \in \mathbb{N}$. Then for $u \in J^{\mu}_{L,0}(\Omega)$ with $\tilde{u}$ the extension of u by zero outside of Ω there exists a constant C (independent of u) such that

$$|\tilde{u}|_{J^{\mu}_{L}(\mathbb{R})} \le C|u|_{J^{\mu}_{L,0}(\Omega)}.$$

Proof. Using Theorems 2.1 and 2.13, we have

$$|\tilde{u}|_{J^{\mu}_{L}(\mathbb{R})} = |\tilde{u}|_{H^{\mu}(\mathbb{R})} = |u|_{H^{\mu}_{0}(\Omega)} \le C|u|_{J^{\mu}_{L,0}(\Omega)}.$$

■

For the semi-norm on $H_0^\mu(\Omega)$ defined by (3), we have a fractional Poincaré-Friedrichs inequality for $H_0^\mu(\Omega)$.

Corollary 2.15 [Fractional Poincaré-Friedrichs] For $u \in H_0^\mu(\Omega)$, we have

$$\|u\|_{L^2(\Omega)} \leq C|u|_{H_0^\mu(\Omega)}, \tag{19}$$

and for $0 < s < \mu$, $s \neq n - 1/2, n \in \mathbb{N}$

$$|u|_{H_0^s(\Omega)} \leq C|u|_{H_0^\mu(\Omega)}.$$

Proof. The result follows from Theorem 2.10, Corollary 2.11, and the equivalence of norms, Theorem 2.13. ■

In order to provide regularity estimates for functions $u \in H^\mu(\Omega)$ solving (1), we state an additional estimate for $\mu = n - 1/2, n \in \mathbb{N}$.

Corollary 2.16 For $u \in H_0^\mu(\Omega), \mu = n - 1/2, n \in \mathbb{N}$ and $0 < \epsilon < 1/2$, there exists a constant C depending only upon ϵ, u such that

$$\begin{aligned} |u|_{H^{\mu-\epsilon}(\Omega)} &\leq C|u|_{J_{L,0}^\mu(\Omega)}, \\ |u|_{H^{\mu-\epsilon}(\Omega)} &\leq C|u|_{J_{R,0}^\mu(\Omega)}. \end{aligned}$$

Proof. As $\mu - \epsilon \neq n - 1/2, n \in \mathbb{N}$, Theorem 2.13 implies

$$|u|_{H^{\mu-\epsilon}(\Omega)} \leq C_1|u|_{J_{L,0}^{\mu-\epsilon}(\Omega)}.$$

Using Corollary 2.11, we have

$$|u|_{J_{L,0}^{\mu-\epsilon}(\Omega)} \leq C_2|u|_{J_{L,0}^\mu(\Omega)}.$$

Therefore, the stated result follows. The result for $J_{R,0}^\mu(\Omega)$ follows analogously. ■

3 Variational Formulation

Let $\Omega = (0, 1)$ and $0 \leq \beta < 1$. Define $\alpha := \frac{2-\beta}{2}$, so that $1/2 < \alpha \leq 1$. In this section, we will show that there exists a unique variational solution of (1) in the space $H_0^\alpha(\Omega)$.

Problem 1 [Steady-State Fractional Advection Dispersion Equation]
Given $\Omega = (0, 1)$, $f : \bar{\Omega} \to \mathbb{R}$, *find* $u : \bar{\Omega} \to \mathbb{R}$ *such that*

$$\begin{aligned} Lu &= f, \quad \text{in } \Omega \qquad (20) \\ u &= 0, \quad \text{on } \partial\Omega \qquad (21) \end{aligned}$$

where

$$Lu := -D\, a\, (p\, {}_0D_x^{-\beta} + q\, {}_xD_1^{-\beta})Du + b(x)Du + c(x)u,$$

$0 \leq \beta < 1,\ a > 0,\ b(x) \in C^1(\bar{\Omega}), c(x) \in C(\bar{\Omega})$ *with* $c - 1/2Db \geq 0$, *and* $p + q = 1,\ 0 \leq p, q \leq 1.$

In order to derive a variational form of Problem 1, we assume that u is a sufficiently smooth solution of (20)-(21), and multiply by an arbitrary $v \in C_0^\infty(\Omega)$ to obtain

$$\int_\Omega -Da(p\,{}_0D_x^{-\beta} + q\,{}_xD_1^{-\beta})Du\,v + bDu\,v + cuv\,dx = \int_\Omega fvdx.$$

Integrating by parts in the first integral, and noting that $v = 0$ on $\partial\Omega$ gives

$$\int_\Omega a(p\,{}_0D_x^{-\beta} + q\,{}_xD_1^{-\beta})Du\,Dv + bDu\,v + cuv\,dx = \int_\Omega fvdx.$$

Thus, we define the associated bilinear form $B : H_0^\alpha(\Omega) \times H_0^\alpha(\Omega) \to \mathbb{R}$ as

$$B(u,v) := a\,p\left\langle {}_0D_x^{-\beta}Du, Dv\right\rangle \,+\, a\,q\left\langle {}_xD_1^{-\beta}Du, Dv\right\rangle + \langle b\,Du, v\rangle + (c\,u, v)\,, \tag{22}$$

where $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$, and $< \cdot,\cdot >$ the duality pairing of $H^{-\mu}(\Omega)$ and $H_0^\mu(\Omega)$, $\mu \ge 0$.

For a given $f \in H^{-\alpha}(\Omega)$, we define the associated linear functional $F : H_0^\alpha(\Omega) \to \mathbb{R}$ as

$$F(v) := \langle f, v\rangle\,. \tag{23}$$

Note that the duality pairings in (22) are well defined for $u, v \in H_0^\alpha(\Omega)$.

Thus, the Galerkin variational solution of (20)-(21) may be defined as follows.

Definition 3.1 [Variational Solution] A function $u \in H_0^\alpha(\Omega)$ is a *variational solution* of Problem 1 provided that

$$B(u,v) = F(v), \quad \forall\ v \in H_0^\alpha(\Omega). \tag{24}$$

Using the results of Section 2, we show that there exists a unique solution to (24). To do this we begin by establishing coercivity and continuity of $B(\cdot,\cdot)$.

Lemma 3.1 The bilinear form $B(\cdot,\cdot)$ is coercive over $H_0^\alpha(\Omega)$, i.e. there exists a constant $C_0 > 0$ such that

$$B(u,u) \ge C_0\|u\|^2_{H^\alpha(\Omega)}, \quad \forall\ u \in H_0^\alpha(\Omega). \tag{25}$$

Proof. We have that

$$\begin{aligned} B(u,u) &= a\,p\left\langle {}_0D_x^{-\beta}Du,\,Du\right\rangle + a\,q\left\langle {}_xD_1^{-\beta}Du,\,Du\right\rangle \,+\, \langle(b\,Du\,+\,cu), u\rangle \\ &= a\,p\left\langle {}_0D_x^{-\beta}Du,\,Du\right\rangle + a\,q\left\langle {}_xD_1^{-\beta}Du,\,Du\right\rangle + \left\langle\left(c\,-\,\frac{1}{2}Db\right)u,\,u\right\rangle \\ &\ge a\,p\left\langle {}_0D_x^{-\beta}Du,\,Du\right\rangle + a\,q\left\langle {}_xD_1^{-\beta}Du,\,Du\right\rangle \end{aligned}$$

as $c\,-\,\frac{1}{2}Db \ge 0$.

Applying the semi-group and adjoint properties of the Riemann-Liouville fractional integral operators, we have

$$B(u,u) \ge a\left\langle {}_0D_x^{-\beta/2}Du,\ \ {}_xD_1^{-\beta/2}Du\right\rangle.$$

As $u = 0$ on $\partial\Omega$,

$$\begin{aligned} {}_0D_x^{-\beta/2}Du &= \mathbf{D}^\alpha u, \\ {}_xD_1^{-\beta/2}Du &= -\mathbf{D}^{\alpha*}u. \end{aligned}$$

Thus

$$B(u,u) \geq -a\,(\mathbf{D}^{\alpha}u,\,\mathbf{D}^{\alpha *}u) = a|u|^2_{J^{\alpha}_{S,0}(\Omega)}.$$

Note that as $1/2 < \alpha \leq 1$, Lemma 2.4 implies $(\mathbf{D}^{\alpha}u,\,\mathbf{D}^{\alpha *}u) < 0$.

The semi-norm equivalence of $J^{\alpha}_{S,0}(\Omega)$ and $H^{\alpha}_0(\Omega)$, Theorem 2.12, implies that

$$B(u,u) \geq C|u|^2_{H^{\alpha}(\Omega)}.$$

By the fractional Poincaré-Friedrichs inequality, Corollary 2.15, we have

$$|u|_{H^{\alpha}(\Omega)} \geq \tilde{C}\|u\|_{L^2(\Omega)}.$$

Therefore,
$$B(u,u) \geq C\|u\|^2_{H^{\alpha}(\Omega)}.$$
■

To establish the continuity of $B(\cdot,\cdot)$,we make use of the following lemma.

Lemma 3.2 Let $b \in C^1(\bar{\Omega})$. Then for all $v \in H^{\alpha}_0(\Omega)$, there exists a constant C depending only upon b and α such that

$$\|bv\|_{H^{\alpha}(\Omega)} \leq C\|v\|_{H^{\alpha}(\Omega)}.$$

Proof. As $b \in C^1(\bar{\Omega})$, we can show that the linear mappings T_0, T_1 defined by

$$T_0(v) = T_1(v) = bv$$

are bounded. First, we have that $T_0 : L^2(\Omega) \to L^2(\Omega)$ is bounded as

$$\|T_0(v)\|^2_{L^2(\Omega)} = \|bv\|^2_{L^2(\Omega)} \leq \|b\|^2_{\infty}\|v\|^2_{L^2(\Omega)}.$$

Next, we have that $T_1 : H^1(\Omega) \to H^1(\Omega)$ is bounded as

$$\begin{aligned}
\|T_1(v)\|^2_{H^1(\Omega)} &= \|bv\|^2_{H^1(\Omega)} \\
&= \|bv\|^2_{L^2(\Omega)} + \|bDv\|^2_{L^2(\Omega)} + \|vDb\|^2_{L^2(\Omega)} \\
&\leq \|b\|^2_{\infty}\|v\|^2_{H^1(\Omega)} + \|Db\|_{\infty}\|v\|^2_{L^2(\Omega)} \\
&\leq (\|b\|^2_{\infty} + \|Db\|^2_{\infty})\|v\|^2_{H^1(\Omega)}.
\end{aligned}$$

Therefore, using operator interpolation (see [11], p. 358),

$$\begin{aligned}
\|bv\|_{H^{\alpha}(\Omega)} &\leq \|T_1\|^{\alpha}\|T_0\|^{1-\alpha}\|v\|_{H^{\alpha}(\Omega)} \\
&\leq C\|v\|_{H^{\alpha}(\Omega)},
\end{aligned}$$

where C depends only upon b and α. ■

Lemma 3.3 The bilinear form $B(\cdot,\cdot)$ is continuous on $H^{\alpha}_0(\Omega) \times H^{\alpha}_0(\Omega)$, i.e. there exists a constant C_1 such that

$$|B(u,v)| \leq C_1\|u\|_{H^{\alpha}(\Omega)}\ \|v\|_{H^{\alpha}(\Omega)}, \quad \forall\ u,v \in H^{\alpha}_0(\Omega). \tag{26}$$

Proof. From the definition of B we have

$$|B(u,v)| \leq a\,p\,|(\mathbf{D}^{\alpha}u,\,\mathbf{D}^{\alpha *}v)| + a\,q\,|(\mathbf{D}^{\alpha *}u,\,\mathbf{D}^{\alpha}v)| + |\langle b\,Du,\,v\rangle| + |(c\,u,\,v)|\,.$$

Using the equivalence of norms, Theorem 2.13, and Lemma 3.2,

$$\begin{aligned} |\langle b\, Du,\, v\rangle| &= \left|\left(\mathbf{D}^{\alpha}u,\, \mathbf{D}^{\beta/2*}(b\, v)\right)\right| \\ &\leq |u|_{J_L^{\alpha}(\Omega)} |b\, v|_{J_R^{\alpha}(\Omega)} \\ &\leq C\|u\|_{H^{\alpha}(\Omega)}\|bv\|_{H^{\alpha}(\Omega)}, \qquad (27) \\ &\leq C\|u\|_{H^{\alpha}(\Omega)}\|v\|_{H^{\alpha}(\Omega)}. \qquad (28) \end{aligned}$$

Thus,

$$\begin{aligned} |B(u,v)| &\leq ap\|u\|_{J_L^{\alpha}(\Omega)}\,\|v\|_{J_R^{\alpha}(\Omega)} \;+\; aq\|u\|_{J_R^{\alpha}(\Omega)}\,\|v\|_{J_L^{\alpha}(\Omega)} \\ &\quad +C\|u\|_{H^{\alpha}(\Omega)}\|v\|_{H^{\alpha}(\Omega)} \;+\; \|c\|_{\infty}\|u\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \\ &\leq C\|u\|_{H^{\alpha}(\Omega)}\|v\|_{H^{\alpha}(\Omega)}\,. \end{aligned}$$

■

Lemma 3.4 The linear functional $F(\cdot)$ is continuous over $H_0^{\alpha}(\Omega)$.

Proof. The result follows from the fact that

$$F(v) = \langle f, v\rangle \leq \|f\|_{H^{-\alpha}(\Omega)}\|v\|_{H^{\alpha}(\Omega)}. \qquad (29)$$

■

Theorem 3.5 There exists a unique solution $u \in H_0^{\alpha}(\Omega)$ to (24) satisfying

$$\|u\|_{H^{\alpha}(\Omega)} \leq C\|f\|_{H^{-\alpha}(\Omega)}\,. \qquad (30)$$

Proof. By Lemmas 3.1, 3.3, and 3.4, B, F satisfy the hypotheses of the Lax-Milgram theorem, from which existence and uniqueness of a solution to (24) immediately follow. The estimate (30) is obtained from (25), (26), and (29). ■

4 Finite Element Convergence Estimates

Let S_h denote a partition of Ω such that $\bar{\Omega} = \{\cup K : K \in S_h\}$. Assume that there exist positive constants c_1, c_2 such that $c_1 h \leq h_K \leq c_2 h$, where h_K is the width of the subinterval K, and $h = \max_{K\in S_h} h_K$.

Let $P_k(K)$ denote the space of polynomials of degree less than or equal to k on $K \in S_h$. Associated with S_h, define the finite-dimensional subspace $X_h \subset H_0^{\alpha}(\Omega)$ as

$$X_h := \{v \in H_0^{\alpha}(\Omega) \cap C^0(\bar{\Omega}) :\, v|_K \in P_{m-1}(K),\, \forall K \in S_h\}\,.$$

Denote by $\mathcal{I}^h u$ the piecewise polynomial interpolant of u in S_h.

Let u_h be the solution to the finite-dimensional variational problem

$$B(u_h, v_h) = F(v_h), \quad \forall\, v_h \in X_h. \qquad (31)$$

We define the energy norm associated with (24) as

$$\|u\|_E := B(u,u)^{1/2}. \tag{32}$$

Note that from (25) and (26) we have norm equivalence of $\|\cdot\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_E$.

Theorem 4.1 Let u denote the solution to (24). There exists a unique solution to (31) which satisfies the estimate

$$\begin{aligned} \|u-u_h\|_E &\leq C_I \inf_{v\in X_h} \|u-v\|_E \\ &\leq C_I \|u-\mathcal{I}^h u\|_E. \end{aligned} \tag{33}$$

Proof. Existence and uniqueness follow from the fact that X_h is a subset of the space $H_0^\alpha(\Omega)$, and thus (31) satisfies the hypotheses of the Lax-Milgram lemma over the finite-dimensional subspace X_h. The estimate (33) is a result of Ceá's lemma. ■

The finite-dimensional subspace X_h and the interpolant $\mathcal{I}^h u$ are chosen specifically so that they satisfy an approximation property over subspaces of $H^m(\Omega)$. That is to say that $\mathcal{I}^h u$ satisfies the following theorem [3].

Theorem 4.2 [Approximation Property] Let $u \in H^r(\Omega), 0 < r \leq m$, and $0 \leq s \leq r$. Then there exists a constant C_A depending only on Ω such that

$$\|u-\mathcal{I}^h u\|_{H^s(\Omega)} \leq C_A h^{r-s} \|u\|_{H^r(\Omega)}. \tag{34}$$

■

We can combine the previous results into an estimate for $e := u - u_h$ in the energy norm.

Corollary 4.3 Let $u \in H_0^\alpha(\Omega) \cap H^r(\Omega)$ $(\alpha \leq r \leq m)$ solve (24), and u_h solve (31). Then there exists a constant C such that the error $e = u - u_h$ satisfies

$$\|e\|_{H^\alpha(\Omega)} \leq C h^{r-\alpha} \|u\|_{H^r(\Omega)}. \tag{35}$$

Proof. From Theorem 4.1, we have that the error satisfies

$$\|e\|_E \leq C_I \|u-\mathcal{I}^h u\|_E.$$

Applying the approximation property and continuity yields

$$\|e\|_E \leq \sqrt{C_1} C_I C_A h^{r-\alpha} \|u\|_{H^r(\Omega)}. \tag{36}$$

Finally, we obtain (35) via the norm equivalence of $\|\cdot\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_E$. ■

We now apply the Aubin-Nitsche trick to obtain a convergence estimate in the L^2 norm. First, we must make an assumption concerning the regularity of the solution to the adjoint problem

$$\begin{aligned} -D(q\, {}_0D_x^{-\beta} + p\, {}_xD_1^{-\beta})\, a\, Dw - D(b(x)w) + c(x)w &= g \quad \text{in } \Omega \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{37}$$

Assumption 4.1 For w solving (37) with $g \in L^2(\Omega)$, we have

$$\|w\|_{H^{2\alpha}(\Omega)} \leq C_E \|g\|_{L^2(\Omega)}.$$

Theorem 4.4 Let $u \in H_0^\alpha(\Omega) \cap H^r(\Omega)$ $(\alpha \leq r \leq m)$ solve (24), and u_h solve (31). Then, under Assumption 4.1, there exists a constant C such that the error $e = u - u_h$ satisfies

$$\begin{aligned} \|e\|_{L^2(\Omega)} &\leq Ch^r \|u\|_{H^r(\Omega)}, \quad \alpha \neq 3/4, \\ \|e\|_{L^2(\Omega)} &\leq Ch^{r-\epsilon} \|u\|_{H^r(\Omega)}, \quad \alpha = 3/4, \ 0 < \epsilon < 1/2. \end{aligned} \tag{38}$$

Proof. Introduce w as the solution to (37) with $g = e = u - u_h \in L^2(\Omega)$. Then w satisfies the variational form

$$B(v, w) = (e, v), \quad \forall v \in H_0^\alpha(\Omega), \tag{39}$$

and the regularity estimate

$$\|w\|_{H^{2\alpha}(\Omega)} \leq C_E \|e\|_{L^2(\Omega)}.$$

Substituting $v = e$ in (39), and applying Galerkin orthogonality, we have

$$\begin{aligned} \|e\|^2_{L^2(\Omega)} &= B(e, w) \\ &= B(e, w - \mathcal{I}^h w) \\ &\leq C_1 \|e\|_{H^\alpha(\Omega)} \|w - \mathcal{I}^h w\|_{H^\alpha(\Omega)} \\ &\leq C_1 C_A h^\alpha \|e\|_{H^\alpha(\Omega)} \|w\|_{H^{2\alpha}(\Omega)} \\ &\leq C_1 C_A C_E h^\alpha \|e\|_{H^\alpha(\Omega)} \|e\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, dividing through by $\|e\|_{L^2(\Omega)}$ yields the estimate

$$\|e\|_{L^2(\Omega)} \leq C_1 C_A C_E h^\alpha \|e\|_E,$$

and applying (35) we obtain (38). The stated result for $\alpha = 3/4$ follows analogously. ∎

Note that for $\alpha = 3/4$, the L^2 convergence rate is effectively r for $u \in H^r(\Omega)$.

5 Numerical Calculations for Piecewise Linear Polynomials

Let $\Omega = (0, 1)$ and let X_h denote the space of continuous piecewise linear polynomials over S_h, i.e. $m = 2$. In this section, we present numerical calculations which support the error estimates in the Corollary 4.3 and Theorem 4.4.

Example 1. Let $u = x^2$, $p = 1$, $q = 0$, $\beta = 1/2$, $\alpha = 3/4$, $a = 1$, and $b = c = 0$. Then, u is the exact solution to the boundary value problem

$$\begin{aligned} -D\, {}_0D_x^{-1/2} Du &= \frac{-2\sqrt{x}}{\Gamma(3/2)} \\ u(0) &= 0 \\ u(1) &= 1. \end{aligned}$$

As $u \in H^2(\Omega)$, the results of Corollary 4.3 and Theorem 4.4 predict a convergence rate of 2 in the L^2 norm and 5/4 in the energy norm. Table 1 includes numerical calculations over a regular partition of $[0,1]$ which support the predicted rates of convergence.

h	$\|u-u_h\|_{L^2(\Omega)}$	cvge. rate	$\|u-u_h\|_{J_L^{3/4}(\Omega)}$	cvge. rate
1/4	$9.319448 \cdot 10^{-3}$		$6.078802 \cdot 10^{-2}$	
1/8	$2.287113 \cdot 10^{-3}$	2.03	$2.506011 \cdot 10^{-2}$	1.28
1/16	$5.626150 \cdot 10^{-4}$	2.02	$1.049400 \cdot 10^{-2}$	1.27
1/32	$1.391495 \cdot 10^{-4}$	2.02	$4.354633 \cdot 10^{-3}$	1.26
1/64	$3.439341 \cdot 10^{-5}$	2.02	$1.825171 \cdot 10^{-3}$	1.25

Table 1: Experimental error results for Example 1.

Example 2. Let $u = x^2 - x^3$, $\beta = 1/2$, $\alpha = 3/4$, $a = 2$, and $b = c = 1$. Then, u is the exact solution to the boundary value problem

$$\begin{aligned} -2D(p\,{}_0D_x^{-1/2} + q\,{}_xD_1^{-1/2})Du + Du + u &= f \\ u(0) &= 0 \\ u(1) &= 0. \end{aligned}$$

As $u \in H^2(\Omega)$, Theorem 4.4 predicts a rate of convergence of 2 in the L^2 norm. Table 2 includes numerical calculations over a regular partition of $[0,1]$ which support the predicted rates of convergence for different values of p and q.

h	$\|u-u_h\|_{L^2(\Omega)}$ $p=1, q=0$	cvge. rate	$\|u-u_h\|_{L^2(\Omega)}$ $p=0, q=1$	cvge. rate	$\|u-u_h\|_{L^2(\Omega)}$ $p=0.5, q=0.5$	cvge. rate
1/4	$1.291090 \cdot 10^{-2}$		$1.036369 \cdot 10^{-2}$		$8.307486 \cdot 10^{-3}$	
1/8	$3.230796 \cdot 10^{-3}$	2.00	$2.338583 \cdot 10^{-3}$	2.15	$1.868104 \cdot 10^{-3}$	2.15
1/16	$8.123848 \cdot 10^{-4}$	1.99	$5.623415 \cdot 10^{-4}$	2.06	$4.274829 \cdot 10^{-4}$	2.12
1/32	$2.049679 \cdot 10^{-4}$	1.99	$1.387338 \cdot 10^{-4}$	2.02	$1.003665 \cdot 10^{-4}$	2.09
1/64	$5.189776 \cdot 10^{-5}$	1.98	$3.476001 \cdot 10^{-5}$	2.00	$2.434186 \cdot 10^{-5}$	2.04

Table 2: Experimental error results for Example 2.

Example 3. Let $u = x^\lambda - x$, $p = 1$, $q = 0$, $a = 1$, $b = c = 0$ and $0 \le \beta < 1$. Then, u is the exact solution to the boundary value problem

$$-D\,{}_0D_x^{-\beta} Du = \frac{-\Gamma(\lambda+1)x^{\lambda+\beta-2}}{\Gamma(\lambda+\beta-1)} + \frac{x^{\beta-1}}{\Gamma(\beta)}$$

$$\begin{aligned} u(0) &= 0 \\ u(1) &= 0. \end{aligned}$$

Suppose that $\lambda < 3/2$. Then $u \in H^{r-\epsilon}(\Omega)$ where $r = \lambda + 1/2$, and so the result of Theorem 4.4 predicts a rate of convergence of r in the L^2 norm. For this example, we provide two sets of calculations,

$$\text{(A) } \lambda = 1.1,\ \beta = 0.8,\ \alpha = 0.6,\ r = 1.6$$
$$\text{(B) } \lambda = 1.4,\ \beta = 0.4,\ \alpha = 0.8,\ r = 1.9$$

Table 3 contains numerical results which support the predicted rate convergence.

h	$\|u - u_h\|_{L^2(\Omega)}$ (A)	cvge. rate	$\|u - u_h\|_{L^2(\Omega)}$ (B)	cvge. rate
1/4	$3.815961 \cdot 10^{-3}$		$5.508758 \cdot 10^{-3}$	
1/8	$1.064472 \cdot 10^{-3}$	1.84	$1.496356 \cdot 10^{-3}$	1.88
1/16	$3.458776 \cdot 10^{-4}$	1.62	$4.052072 \cdot 10^{-4}$	1.88
1/32	$1.143203 \cdot 10^{-4}$	1.60	$1.098377 \cdot 10^{-4}$	1.88
1/64	$3.772030 \cdot 10^{-5}$	1.60	$2.986717 \cdot 10^{-5}$	1.88

Table 3: Experimental error results for Example 3.

A Riemann-Liouville Fractional Integral Operators

We define the fractional integral operators in terms of the Riemann-Liouville definition given in [9, 10].

Definition A.1 [Left Riemann-Liouville Fractional Integral] Let u be a function defined on (a, b), and $\sigma > 0$. Then the *left Riemann-Liouville fractional integral of order* σ is defined to be

$$_aD_x^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)} \int_a^x (x-s)^{\sigma-1} u(s)ds. \tag{40}$$

Definition A.2 [Right Riemann-Liouville Fractional Integral] Let u be a function defined on (a, b), and $\sigma > 0$. Then the *right Riemann-Liouville fractional integral of order* σ is defined to be

$$_xD_b^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)} \int_x^b (s-x)^{\sigma-1} u(s)ds. \tag{41}$$

With these definitions, we note some of the properties of the Riemann-Liouville fractional integral operators, as outlined in [9, 10].

Property A.1 [Semigroup Property] The left and right Riemann-Liouville fractional integral operators follow the properties of a semigroup, i.e. for $u \in L^p[a, b]$ for $p \geq 1$,

$$\begin{aligned} {_aD_x^{-\mu}}\, {_aD_x^{-\sigma}} u(x) &= {_aD_x^{-\mu-\sigma}} u(x),\ \forall\, x \in [a, b],\ \forall\, \mu, \sigma > 0, \\ {_xD_b^{-\mu}}\, {_xD_b^{-\sigma}} u(x) &= {_xD_b^{-\mu-\sigma}} u(x),\ \forall\, x \in [a, b],\ \forall\, \mu, \sigma > 0. \end{aligned} \tag{42}$$

Property A.2 [Adjoint Property] The left and right Riemann-Liouville fractional integral operators are adjoints in the L^2 sense, i.e. for all $\sigma > 0$,

$$\left({}_aD_x^{-\sigma} u, v \right)_{L^2(a,b)} = \left(u, {}_xD_b^{-\sigma} v \right)_{L^2(a,b)}, \ \forall\, u, v \in L^2(a,b). \tag{43}$$

Property A.3 [Fourier Transform Property] Let $\sigma > 0$, $u \in L^p(\mathbb{R}), \mathrm{p} \geq 1$. The Fourier transform of the left and right Riemann-Liouville fractional integrals satisfy the following,

$$\begin{aligned} \mathcal{F}({}_{-\infty}D_x^{-\sigma} u(x)) &= (i\omega)^{-\sigma} \hat{u}(\omega), \\ \mathcal{F}({}_xD_\infty^{-\sigma} u(x)) &= (-i\omega)^{-\sigma} \hat{u}(\omega), \end{aligned} \tag{44}$$

where $\hat{u}(\omega)$ denotes the Fourier transform of u,

$$\hat{u}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} u(x) dx.$$

Definition A.3 [Left Riemann-Liouville Fractional Derivative] Let u be a function defined on $\mathbb{R}$, $\mu > 0$, n be the smallest integer greater than μ ($n - 1 \leq \mu < n$), and $\sigma = n - \mu$. Then the *left fractional derivative of order* μ is defined to be

$$\mathbf{D}^\mu u := {}_{-\infty}D_x^\mu u = D^n \, {}_{-\infty}D_x^{-\sigma} u(x) = \frac{1}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x - \xi)^{\sigma - 1} u(\xi) d\xi. \tag{45}$$

Definition A.4 [Right Riemann-Liouville Fractional Derivative] Let u be a function defined on $\mathbb{R}$, $\mu > 0$, n be the smallest integer greater than μ ($n - 1 \leq \mu < n$), and $\sigma = n - \mu$. Then the *right fractional derivative of order* μ is defined to be

$$\mathbf{D}^{\mu*} u := {}_xD_\infty^\mu u = (-D)^n \, {}_xD_\infty^{-\sigma} u(x) = \frac{(-1)^n}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_{x}^{\infty} (\xi - x)^{\sigma - 1} u(\xi) d\xi. \tag{46}$$

Note. If $\overline{supp(u)} \subset (a,b)$, then $\mathbf{D}^\mu u = {}_aD_x^\mu u$ and $\mathbf{D}^{\mu*} u = {}_xD_b^\mu u$, where ${}_aD_x^\mu$ and ${}_xD_b^\mu u$ are the left and right Riemann-Liouville fractional derivatives of order μ [9].

Property A.4 The left (right) Riemann-Liouville fractional derivative of order μ acts as a left inverse of the left (right) Riemann-Liouville fractional integral of order μ, i.e.

$$ {}_aD_x^\mu \, {}_aD_x^{-\mu} u(x) = u(x), \tag{47}$$

$$ {}_xD_b^\mu \, {}_xD_b^{-\mu} u(x) = u(x), \ \forall\, \mu > 0. \tag{48}$$

Property A.5 [Fourier Transform Property] Let $\mu > 0$, $u \in C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}$. The Fourier transform of the left and right Riemann-Liouville fractional derivatives satisfy the following,

$$\begin{aligned} \mathcal{F}({}_{-\infty}D_x^{\mu} u(x)) &= (i\omega)^{\mu} \hat{u}(\omega), \\ \mathcal{F}({}_xD_\infty^{\mu} u(x)) &= (-i\omega)^{\mu} \hat{u}(\omega). \end{aligned} \tag{49}$$

Property A.6 Let $\mu > 0$, $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}$. The following composition rule holds for the left (right) Riemann-Liouville fractional integral and fractional derivative operators.

$$ {}_aD_x^{-\mu} \, {}_aD_x^\mu u(x) = u(x), \quad \forall\, u(x) \text{ such that } \overline{supp(u)} \subset (a, \infty), \tag{50}$$

$$ {}_xD_b^{-\mu} \, {}_xD_b^\mu u(x) = u(x), \quad \forall\, u(x) \text{ such that } \overline{supp(u)} \subset (-\infty, b). \tag{51}$$

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