Spectral approximation of a variable coefficient fractional diffusion equation in one space dimension

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Abstract

In this article we consider the approximation of a variable coefficient (two-sided) fractional diffusion equation (FDE), having unknown u. By introducing an intermediate unknown, q, the variable coefficient FDE is rewritten as a lower order, constant coefficient FDE. A spectral approximation scheme, using Jacobi polynomials, is presented for the approximation of q, q_N. The approximate solution to u, u_N, is obtained by post processing q_N. An a priori error analysis is given for $(q - q_N)$ and $(u - u_N)$. Two numerical experiments are presented whose results demonstrate the sharpness of the derived error estimates.

Key words. Fractional diffusion equation, Jacobi polynomials, spectral method

AMS Mathematics subject classifications. 65N30, 35B65, 41A10, 33C45

1 Introduction

In recent years the numerical approximation of fractional differential equations has received increased attention as their incorporation into models, to address phenomena not well captured using usual differential equations, has increased. Examples of applications using fractional differential equations include contaminant transport in ground water flow [2], viscoelasticity [23], image processing [3, 13], turbulent flow [23, 31], and chaotic dynamics [42]. Approximation schemes including finite difference methods [8, 21, 27, 33, 34], finite element methods [12, 18, 22, 36], discontinuous Galerkin methods [41], mixed methods [6, 20], spectral methods [7, 11, 19, 26, 24, 26, 25, 40, 43], enriched subspace methods [17] have all been applied to fractional differential equations.

Our interest in this paper is on the numerical approximation of the two-sided variable-coefficient FDE of order $1 < \alpha < 2$

$$\mathcal{K}_r^{\alpha} u(x) \ := \ -D\big(\big(r \, {}_0I_x^{2-\alpha} \ + (1-r) \, {}_xI_1^{2-\alpha}\big) \, K(x) \, Du(x)\big) \ = f(x), \quad x \in (0,1) \,, \tag{1.1}$$

$$u(0) = u(1) = 0 \,, \tag{1.2}$$

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where $K(x)$ is the diffusivity coefficient with $0 < K_{min} \le K(x) \le K_{max}$, $0 \le r \le 1$, $f(x)$ the source or sink term, and D^j denotes the derivative operator d^j/dx^j, for $j \in \mathbb{N}$. The left and right fractional integrals of order $0 < \sigma < 1$ are defined as [29, 30]

$$ {}_0I_x^\sigma w(x) := \frac{1}{\Gamma(\sigma)} \int_0^x \frac{w(s)}{(x-s)^{1-\sigma}}\, ds, \qquad {}_xI_1^\sigma w(x) := \frac{1}{\Gamma(\sigma)} \int_x^1 \frac{w(s)}{(s-x)^{1-\sigma}}\, ds\,, $$

where $\Gamma(\cdot)$ is the Gamma function. Equation (1.1) was derived by incorporating a nonlocal Fick's law with variable diffusivity coefficient $K(x)$ into a conventional local mass conservation law [9, 12, 44].

In [12] the Galerkin weak formulation for (1.1) and (1.2) was presented and studied for $K(x)$ a constant. It was shown in [36] that the bilinear form of the Galerkin weak formulation may lose its coercivity for a variable-coefficient $K(x)$, and so its Galerkin finite element approximation might diverge [39]. A Petrov-Galerkin weak formulation was proved to be wellposed on $H_0^{\alpha-1} \times H_0^1$ for $3/2 < \alpha < 2$ for a one-sided version of (1.1) and (1.2) [36]. A Petrov-Galerkin finite element method was developed and analyzed subsequently for the one-sided version of (1.1) and (1.2) [38]. In [20], with the introduction of an auxiliary variable, a mixed method approximation scheme for problem (1.1) and (1.2) was studied and error estimates derived. In [26], a spectral Galerkin method for the two-sided steady-state FDE with variable coefficient was analyzed, in which the outside and inside fractional derivatives are chosen carefully so that the corresponding Galerkin weak formulation are self-adjoint and coercive. Optimal error estimates were also derived under suitable smoothness assumption on the solution.

It was shown in [37] that for one-dimensional FDEs smoothness of the coefficients and the right-hand side function is not sufficient to guarantee the smoothness of the solution, especially at the endpoints of the interval, which is different from the case of the classical second order diffusion equation. Hence, seeking proper regularity solution spaces for FDEs becomes a key issue in the study of FDEs. Jin et al. [18] conducted a thorough analysis of the regularity issue in the context of a one-sided constant-coefficient FDE by fully utilizing the explicit solution expression. An indirect Legendre spectral Galerkin method [40] and a finite element method [39] were developed for the one-sided FDE with variable coefficient, in which the solution to the FDE is expressed as a fractional derivative of the solution to a second-order differential equations. Consequently, high-order convergence rates of numerical approximations were proved using only regularity assumptions on the coefficients and right-hand side, but not on the true solution (which is not smooth in fact). However, many aforementioned works for one-sided FDEs do not apply for two-sided FDEs.

Mao et al. [24] analyzed the solution structure to the constant coefficient version of (1.1) and (1.2) with $r = 1/2$ in terms of spectral polynomials and developed corresponding spectral methods. The solution structure to the constant coefficient version of (1.1) and (1.2) with general $0 \le r \le 1$ was completely resolved in [11]. Additionally, in [11] the spectral method utilizing the weighted Jacobi polynomial was studied and a priori error estimates derived. The two-sided FDE with constant coefficient and Riemann-Liouville fractional derivative was investigated in [25], by employing a Petrov-Galerkin projection in a properly weighted Sobolev space using two-sided Jacobi polyfracnomials as test and trial functions. Spectral methods enjoy many excellent mathematical properties that make them particularly suited for FDEs: (i) They present a clean analytical expression of the true solution to FDEs, that has been fully explored in [11, 24] to analyze its structure and regularity; (ii) Fractional differentiation of many spectral polynomials can be carried out analytically [40], in contrast to finite element methods in which they have to be calculated numerically [39]; (iii) As FDEs are nonlocal operators the appealing property of a sparse coefficient matrix, which arises for a finite element, finite difference, or finite volume approximation of a usual differential equation, is lost. The aforementioned discretizations of FDEs ordinarily lead to Toeplitz-like coefficient matrices, the structure of which can be leveraged to solve the associated linear systems with an almost linear cost (precisely, $O(n \log n)$ with n the degrees of freedom), see [10, 28, 35]. In contrast, the stiffness matrices of spectral methods are often diagonal (at least for constant coefficient FDEs). As the convergence analysis of the finite element/difference/volume approximation relies on the estimate of the interpolation, projection or residue of the asymptotic expansion of the solution, which in turn depends on its regularity, the convergence rate of these methods are strongly constrained due to the singularity nature of the solution to FDEs [12]. The spectral method, for which the convergence rate only depends on the regularity of the input data, circumvents that low regularity constraint and thus provides a high-accuracy approximation of the solutions to FDEs [11]. Because of the advantages mentioned above, spectral methods are appealing for the approximation of FDEs.

The goal of this paper is to extend the application of the spectral method in [11] to the two-sided variable-coefficient FDE (1.1) and (1.2) whose solution may have endpoint singularities. By introducing an intermediate variable, we rewrite the variable coefficient model as a constant coefficient FDE. Then, utilizing Jacobi polynomials which incorporate the possible singularity of solution at endpoints, we apply the spectral method to construct an approximating series for the solution.

This paper is organized as follows. In Section 2 we present the formulation to be used, introduce notation used through the paper, and give some key lemmas used in the analysis. The spectral approximation method is formulated and a detailed analysis of its convergence is given in Sections 3, 4 and 5. Two numerical experiments are presented in Section 6 whose results demonstrate the sharpness of the derived error estimates.

2 Problem formulation and preliminaries

Let $\tilde{q}(x) = -K(x)Du(x)$. Using the homogeneous Dirichlet boundary condition at $x = 0$ yields

$$u(x) = -\int_0^x \frac{\tilde{q}(s)}{K(s)} ds. \tag{2.1}$$

Enforcing the homogeneous Dirichlet boundary condition at $x = 1$ we obtain

$$\int_0^1 \frac{\tilde{q}(s)}{K(s)} ds = 0. \tag{2.2}$$

Thus, with (2.1), problem (1.1), (1.2) can be recast as the following system

$$\mathcal{N}_r^\alpha \tilde{q}(x) := D\left(r\ {}_0I_x^{2-\alpha} + (1-r)\ {}_xI_1^{2-\alpha}\right)\tilde{q}(x) = f(x), \quad x \in (0,1), \tag{2.3}$$

$$\text{with} \quad \int_0^1 \frac{\tilde{q}(s)}{K(s)} ds = 0. \tag{2.4}$$

Jacobi polynomials play a key role in the approximation schemes. We briefly review their definition and properties central to the method [1, 32].

Usual Jacobi polynomials, $P_n^{(\alpha,\beta)}(t)$, on $(-1\,,\,1)$.
Definition: $P_n^{(\alpha,\beta)}(t) \;:=\; \sum_{m=0}^{n} p_{n,m}\,(t-1)^{(n-m)}(t+1)^m$, where $\alpha,\beta\;>-1$, and

$$p_{n,m} \;:=\; \frac{1}{2^n}\left(\begin{array}{c} n+\alpha \\ m \end{array}\right)\left(\begin{array}{c} n+\beta \\ n-m \end{array}\right). \tag{2.5}$$

Orthogonality:

$$\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\,P_j^{(\alpha,\beta)}(t)\,P_k^{(\alpha,\beta)}(t)\,dt \;=\; \left\{\begin{array}{ll} 0, & k\neq j \\ |||P_j^{(\alpha,\beta)}|||^2\,, & k=j \end{array}\right. , \quad \text{where}$$

$$|||P_j^{(\alpha,\beta)}||| \;=\; \left(\frac{2^{(\alpha+\beta+1)}}{(2j\,+\,\alpha\,+\,\beta\,+1)}\,\frac{\Gamma(j+\alpha+1)\,\Gamma(j+\beta+1)}{\Gamma(j+1)\,\Gamma(j+\alpha+\beta+1)}\right)^{1/2}. \tag{2.6}$$

In order to transform the domain of the family of Jacobi polynomials to $[0,1]$, let $t\to 2x-1$ and introduce $G_n^{\alpha,\beta}(x)\;=\;P_n^{\alpha,\beta}(t(x))$. From (2.6),

$$\begin{aligned}
\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\,P_j^{(\alpha,\beta)}(t)\,P_k^{(\alpha,\beta)}(t)\,dt &= \int_0^1 2^{\alpha}\,(1-x)^{\alpha}\,2^{\beta}\,x^{\beta}\,P_j^{(\alpha,\beta)}(2x-1)\,P_k^{(\alpha,\beta)}(2x-1)\,2\,dx \\
&= 2^{\alpha+\beta+1}\int_0^1(1-x)^{\alpha}\,x^{\beta}\,G_j^{(\alpha,\beta)}(x)\,G_k^{(\alpha,\beta)}(x)\,dx \\
&= \left\{\begin{array}{ll} 0, & k\neq j \\ 2^{\alpha+\beta+1}\,|||G_j^{(\alpha,\beta)}|||^2\,, & k=j \end{array}\right. , \text{ where}
\end{aligned}$$

$$|||G_j^{(\alpha,\beta)}||| = \left(\frac{1}{(2j\,+\,\alpha\,+\,\beta\,+1)}\,\frac{\Gamma(j+\alpha+1)\,\Gamma(j+\beta+1)}{\Gamma(j+1)\,\Gamma(j+\alpha+\beta+1)}\right)^{1/2}. \tag{2.7}$$

Note that

$$|||G_j^{(\alpha,\beta)}||| \;=\; |||G_j^{(\beta,\alpha)}|||\,. \tag{2.8}$$

From [24, equation (2.19)] we have that for $k\in\mathbb{N}$

$$\frac{d^k}{dt^k}P_n^{(\alpha,\beta)}(t) \;=\; \frac{\Gamma(n+k+\alpha+\beta+1)}{2^k\,\Gamma(n+\alpha+\beta+1)}P_{n-k}^{(\alpha+k\,,\,\beta+k)}(t)\,. \tag{2.9}$$

Hence,

$$\frac{d^k}{dx^k}G_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}G_{n-k}^{(\alpha+k\,,\,\beta+k)}(x)\,. \tag{2.10}$$

Also, from [24, equation (2.15)],

$$\begin{aligned}
&\frac{d^k}{dt^k}\left\{(1-t)^{\alpha+k}\,(1+t)^{\beta+k}\,P_{n-k}^{(\alpha+k\,,\,\beta+k)}(t)\right\} \\
&\quad= \frac{(-1)^k\,2^k\,n!}{(n-k)!}\,(1-t)^{\alpha}\,(1+t)^{\beta}\,P_n^{(\alpha\,,\,\beta)}(t)\,,\;\; n\geq k\geq 0\,,
\end{aligned} \tag{2.11}$$

from which it follows that

$$\frac{d^k}{dx^k}\left\{\, (1\,-\,x)^{\alpha+k}\, x^{\beta+k}\, G_{n-k}^{(\alpha+k\,,\,\beta+k)}(x)\right\} \;=\; \frac{(-1)^k\, n!}{(n-k)!}\,(1\,-\,t)^{\alpha}\, t^{\beta}\, G_n^{(\alpha\,,\,\beta)}(x)\,. \tag{2.12}$$

For compactness of notation we introduce

$$\mathbb{N}_0 \;:=\; \mathbb{N}\cup\{0\} \;\text{ and }\; \rho^{(\alpha,\beta)} \;:=\; \rho^{(\alpha,\beta)}(x) \;:=\; (1-x)^{\alpha}\, x^{\beta}\,.$$

We use $y_n \sim n^p$ to denote that there exist two constants c and $C \geq 0$ such that, as $n \to \infty$, $c\, n^p \leq |y_n| \leq C\, n^p$.

The weighted $L^2(0,1)$ spaces, $L^2_\omega(0,1)$.
The weighted $L^2(0,1)$ spaces are appropriate for analyzing the convergence of the spectral type methods presented below. For $\omega(x) > 0,\ x \in (0,1)$, let

$$L^2_\omega(0,1) \;:=\; \left\{ f(x) \;:\; \int_0^1 \omega(x)\, f(x)^2\, dx \;<\; \infty \right\}.$$

Associated with $L^2_\omega(0,1)$ is the inner product, $\langle\cdot,\cdot\rangle_\omega$, and norm, $\|\cdot\|_\omega$, defined by

$$\begin{aligned}\langle f\,,\,g\rangle_\omega &:= \int_0^1 \omega(x)\, f(x)\, g(x)\, dx\,, \quad \text{and}\\ \|f\|_\omega &:= \left(\langle f\,,\,f\rangle_\omega\right)^{1/2}\,.\end{aligned}$$

For $1 < \alpha < 2$ and $0 \leq r \leq 1$ given, let β satisfying $\alpha - 1 \leq \beta, \alpha - \beta \leq 1$ be determined by

$$r \;=\; \frac{\sin(\pi\,\beta)}{\sin(\pi(\alpha-\beta)) \,+\, \sin(\pi\,\beta)}\,. \tag{2.13}$$

The operator $\mathcal{N}_r^\alpha$, defined in (2.3), has a nontrivial kernel which is described in the following lemma. The kernel is need in order that functions satisfying (2.3) can be suitably modified so that they then also satisfy (2.4). In the following two lemmas we present mapping properties for $\mathcal{N}_r^\alpha$ which form the basis for the spectral approximation scheme presented in Section 3.

Lemma 2.1 *[11] For β determined by* (2.13)*, we have that*

$$ker(\mathcal{N}_r^\alpha) \;=\; span\big\{z(x) \;:=\; (1-x)^{\alpha-\beta-1}\, x^{\beta-1}\big\}\,.$$

where $\mathcal{N}_r^\alpha$ is defined in (2.3). Additionally, [16] (as $G_0^{(\delta,\gamma)}(x) \,=\, 1$)

$$\begin{aligned}\mathcal{N}_r^\alpha(x\, z(x)) &= -(1-r)\,\Gamma(\alpha)\frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta))} \;=\; \lambda_{-1}\, G_0^{(\delta,\gamma)}(x)\,,\\ \text{and}\quad \mathcal{N}_r^\alpha((1-x)\, z(x)) &= -\lambda_{-1}\, G_0^{(\delta,\gamma)}(x)\,,\\ \text{where } \lambda_{-1} &:= -(1-r)\,\Gamma(\alpha)\frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta))}\,.\end{aligned}$$

■

Lemma 2.2 *[11] Let β be determined by* (2.13)*. Then, for* $n = 0, 1, 2, \ldots$

$$\mathcal{N}_r^\alpha \left((1-x)^{\alpha-\beta} x^\beta \, G_n^{(\alpha-\beta\,,\,\beta)}(x) \right) = \lambda_n \, G_{n+1}^{(\beta-1\,,\,\alpha-\beta-1)}(x) \, , \quad \textit{where}$$

$$\lambda_n = \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta)) \; + \; \sin(\pi\beta)} \, \frac{\Gamma(n+\alpha)}{n!} \; > \; 0 \tag{2.14}$$

$$\sim (n+1)^{\alpha-1} \, . \tag{2.15}$$

■

Remark: Equation (2.15) follows from Stirling's formula, specifically,

$$\lim_{n\to\infty} \frac{\Gamma(n+\mu)}{\Gamma(n)\, n^\mu} \; = \; 1 \, , \text{ for } \mu \in \mathbb{R} \, .$$

3 Spectral type approximation to $\mathcal{N}_r^\alpha$.

In this section, using the mapping properties of $\mathcal{N}_r^\alpha$ given in Lemmas 2.1 and 2.2, we present a spectral approximation scheme for $q(x)$ satisfying $\mathcal{N}_r^\alpha q(x) = f(x)$. (The solution $\tilde{q}(x)$ satisfying (2.3),(2.4), is given by $\tilde{q}(x) = c_{-2} \, z(x) + q(x)$ where c_{-2} is chosen in order that $\tilde{q}$ satisfies (2.4).) Also presented in this section is a corresponding a priori error analysis.

In this section we fix the values of α and r as defined by the operator $\mathcal{K}_r^\alpha$ in (1.1), and correspondingly, β determined by (2.13).

Useful in the analysis below is the following result.

Lemma 3.1 *For* $j = 0, 1, 2, \ldots$

$$\frac{1}{2} \; \leq \; \frac{|||G_j^{(\alpha-\beta\,,\,\beta)}|||^2}{|||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} \; = \; \frac{j+1}{j+\alpha} \; \leq \; 1 \, . \tag{3.1}$$

Proof: From (2.7),

$$\frac{|||G_j^{(\alpha-\beta\,,\,\beta)}|||^2}{|||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} = \frac{1}{2j \, + \, \alpha \, + \, 1} \, \frac{\Gamma(j+\alpha-\beta+1)\,\Gamma(j+\beta+1)}{\Gamma(j+1)\,\Gamma(j+\alpha+1)}$$
$$\times \, \frac{2j \, + \, \alpha \, + \, 1}{1} \, \frac{\Gamma(j+2)\,\Gamma(j+\alpha)}{\Gamma(j+\beta+1)\,\Gamma(j+\alpha-\beta+1)} \; = \; \frac{j+1}{j+\alpha} \leq 1 \, . \tag{3.2}$$

■

The solution $u(x)$ to (1.1), (1.2) is computed directly using (2.1) once $\tilde{q}(x)$ satisfying (2.3), (2.4) is determined. Note that as $ker(\mathcal{N}_r^\alpha) = span\{z(x)\}$, then $\tilde{q}$ satisfying (2.3) is only determined up to an additive constant multiple of $z(x)$. Hence, we rewrite $\tilde{q}(x) \ = \ c_{-2}\, z(x) \ + \ q(x)$, where $q(x)$ satisfies

$$\mathcal{N}_r^{\alpha} q(x) \ = \ f(x), \quad x \in (0,1)\,, \tag{3.3}$$

and c_{-2} is determined by (2.4).

For $\gamma,\, \delta > -1$, $\left\{ G_i^{(\gamma\,,\,\delta)}(x) \right\}_{i=0}^{\infty}$ is a basis for $L^2_{\rho^{(\gamma\,,\,\delta)}}(0,1)$. In view of Lemmas 2.1 and 2.2, in order that the mapping $\mathcal{N}_r^{\alpha} \,:\, S \rightarrow L^2_{\rho^{(\beta\,,\,\alpha-\beta)}}(0,1)$ is onto we can choose

$$S \ := \ \left\{ h \,:\, h(x) \ = \ c_{-1} x\, z(x) \ + \ \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{i=0}^{\infty} c_i\, G_i^{(\alpha-\beta\,,\,\beta)}(x), \quad c_i \in \mathbb{R}, \ i \ = \ -1,0,1,2,\ldots \right\}, \tag{3.4}$$

or

$$S \ := \ \left\{ h \,:\, h(x) \ = \ c_{-1}(1-x)\, z(x) \ + \ \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{i=0}^{\infty} c_i\, G_i^{(\alpha-\beta\,,\,\beta)}(x), \quad c_i \in \mathbb{R}, \ i \ = \ -1,0,1,2,\ldots \right\}. \tag{3.5}$$

Observe that if $\beta - 1 \ < \ \alpha - \beta$, i.e., $r > 1/2$, then $x\, z(x)$ is a more regular function on $(0,1)$ than $(1-x)\, z(x)$. However, if $\beta - 1 \ > \ \alpha - \beta$, i.e., $r < 1/2$, then $(1-x)\, z(x)$ is a more regular function on $(0,1)$ than $x\, z(x)$. Below we will assume that $r \geq 1/2$ and use (3.4) as the domain for $\mathcal{N}_r^{\alpha}$. This choice is reflected in the representation for $q_N(x)$, the approximation of $q(x)$, given in (3.7).

Remark: Note that $f(x) \in L^2_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$ may be expressed as $f(x) \ = \ \sum_{i=0}^{\infty} \frac{f_i}{|||G_i^{(\beta-1\,,\,\alpha-\beta-1)}|||^2}\, G_i^{(\beta-1\,,\,\alpha-\beta-1)}(x)$, where f_i is given by

$$f_i \ := \ \langle f\,,\, G_i^{(\beta-1\,,\,\alpha-\beta-1)} \rangle_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \ = \ \int_0^1 \rho^{(\beta-1\,,\,\alpha-\beta-1)}(x)\, f(x)\, G_i^{(\beta-1\,,\,\alpha-\beta-1)}(x)\, dx\,. \tag{3.6}$$

With f_i defined in (3.6), let

$$q_N(x) = \ c_{-1} x\, z(x) \ + \ \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{i=0}^{N-1} c_i\, G_i^{(\alpha-\beta\,,\,\beta)}(x)\,, \tag{3.7}$$

where,

$$c_i \ = \ \frac{1}{\lambda_i\, |||G_{i+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} f_{i+1}\,, \text{for } i = -1,0,1,\ldots N-1. \tag{3.8}$$

With the above framework we can establish the convergence of $q_N(x)$ to $q(x)$ satisfying (3.3).

Theorem 3.1 *Let* $f(x) \in L^2_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$ *and* $q_N(x)$ *be as defined in* (3.7)*. Then,* $(q(x) - c_{-1} x\, z(x)) \ := \ \lim_{N\rightarrow\infty} (q_N(x) - c_{-1} x\, z(x)) \in L^2_{\rho^{(-(\alpha-\beta)\,,\,-\beta)}}(0,1)$*. In addition,* $\mathcal{N}_r^{\alpha} q(x) \ = \ f(x)$*.*

Proof: For $f_N(x) \ := \ \sum_{i=0}^{N} \frac{f_i}{|||G_i^{(\beta-1\,,\alpha-\beta-1)}|||^2} \, G_i^{(\beta-1\,,\alpha-\beta-1)}(x)$, we have that $f(x) \ = \ \lim_{N\to\infty} f_N(x)$, and $\{f_N(x)\}_{N=0}^{\infty}$ is a Cauchy sequence in $L^2_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$. A straightforward calculation shows that $(q_N(x) - c_{-1}x\, z(x)) \in L^2_{\rho^{(-(\alpha-\beta)\,,-\beta)}}(0,1)$. Then, (without loss of generality, assume $M > N$)

$$
\begin{aligned}
&\| \, (q_M(x) - c_{-1}x\, z(x)) - \ (q_N(x) - c_{-1}x\, z(x)) \, \|^2_{\rho^{(-(\alpha-\beta)\,,-\beta)}} \ = \ \|q_M(x) \ - \ q_N(x)\|^2_{\rho^{(-(\alpha-\beta)\,,-\beta)}} \\
&= \ \left\langle \rho^{(-(\alpha-\beta)\,,-\beta)}(x) \ \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{j=N}^{M-1} c_j \, G_j^{(\alpha-\beta\,,\,\beta)}(x) \ , \ \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{j=N}^{M-1} c_j \, G_j^{(\alpha-\beta\,,\,\beta)}(x) \right\rangle \\
&= \ \left\langle \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{j=N}^{M-1} \frac{G_j^{(\alpha-\beta\,,\,\beta)}(x)}{\lambda_j \, |||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} \, f_{j+1} \ , \ \sum_{j=N}^{M-1} \frac{G_j^{(\alpha-\beta\,,\,\beta)}(x)}{\lambda_j \, |||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} \, f_{j+1} \right\rangle \\
&= \ \sum_{j=N}^{M-1} \frac{f_{j+1}^2}{\lambda_j^2 \, |||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^4} |||G_j^{(\alpha-\beta\,,\,\beta)}|||^2 \ \le \ \sum_{j=N}^{M-1} \frac{f_{j+1}^2}{\lambda_j^2 \, |||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} \qquad \text{(using (3.1))} \\
&\le \ C \left\langle \rho^{(\beta-1\,,\,\alpha-\beta-1)}(x) \sum_{j=N+1}^{M} \frac{G_j^{(\beta-1\,,\,\alpha-\beta-1)}(x)}{|||G_j^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} f_j \ , \ \sum_{j=N+1}^{M} \frac{G_j^{(\beta-1\,,\,\alpha-\beta-1)}(x)}{|||G_j^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} f_j \right\rangle \\
&\qquad \text{(using that } \lambda_j \text{ are bounded away from zero, see (2.14))} \\
&= \ C \, \|f_N(x) \ - \ f_M(x)\|^2_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \, .
\end{aligned}
$$

Hence $\{(q_N(x) - c_{-1}x\, z(x))\}_{N=0}^{\infty}$ is a Cauchy sequence in $L^2_{\rho^{(-(\alpha-\beta)\,,-\beta)}}(0,1)$. As $L^2_{\rho^{(-(\alpha-\beta)\,,-\beta)}}(0,1)$ is complete [15], $q(x) - c_{-1}x\, z(x) \ := \ \lim_{N\to\infty} q_N(x) - c_{-1}x\, z(x) \ \in \ L^2_{\rho^{(-(\alpha-\beta)\,,-\beta)}}(0,1)$.

Next, as $f_N(x) \to f(x)$ in $L^2_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$, given $\epsilon > 0$ there exists $\tilde{N}$ such that for $N > \tilde{N}$, $\|f(x) \ - \ f_N(x)\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \ < \epsilon$. Then, for $N > \tilde{N}$, using Lemmas 2.1 and 2.2

$$
\begin{aligned}
&\|f(x) \ - \ \mathcal{N}_r^{\alpha} q_N(x)\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \\
&= \ \left\| f(x) \ - \ \mathcal{N}_r^{\alpha} \left(c_{-1}x\, z(x) \ + \ \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{j=0}^{N-1} \frac{G_j^{(\alpha-\beta\,,\,\beta)}(x)}{\lambda_j \, |||G_{j+1}^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} f_{j+1} \right) \right\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \\
&= \ \left\| f(x) \ - \ \sum_{j=0}^{N} \frac{G_j^{(\beta-1\,,\,\alpha-\beta-1)}(x)}{|||G_j^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} \, f_j \right\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \\
&= \ \|f(x) \ - \ f_N(x)\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \ < \ \epsilon \, .
\end{aligned}
$$

Hence, $f(x) \ = \ \mathcal{N}_r^{\alpha} q(x)$.

■

For $q \ - \ q_N$ we have the following a priori error estimate.

Theorem 3.2 *For $f(x) \in L^2_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$ and $q_N(x)$ given by (3.7), there exists $C > 0$ such that*

$$
\|q \ - \ q_N\|_{\rho^{(-(\alpha-\beta)\,,-\beta)}} \ \le \ \frac{1}{\lambda_N} \, \|f\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \ \le \ C \, (N+1)^{-\alpha+1} \, \|f\|_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}} . \tag{3.9}
$$

Proof: Using the definition of the $\| \cdot \|_{\rho^{(-(\alpha-\beta),-\beta)}}$ norm,

$$
\begin{aligned}
\|q - q_N\|^2_{\rho^{(-(\alpha-\beta),-\beta)}} &= \int_0^1 \rho^{(-(\alpha-\beta),-\beta)}(x) \Big(\rho^{(\alpha-\beta,\beta)}(x) \sum_{i=N}^{\infty} \frac{G_i^{(\alpha-\beta,\beta)}(x)}{(\lambda_i \, |||G_{i+1}^{(\beta-1,\alpha-\beta-1)}|||^2)} f_{i+1} \Big)^2 dx \\
&\leq \max_{k \geq N} \left(\frac{1}{\lambda_k^2} \right) \sum_{i=N}^{\infty} \frac{f_{i+1}^2}{|||G_{i+1}^{(\beta-1,\alpha-\beta-1)}|||^4} |||G_i^{(\alpha-\beta,\beta)}|||^2 \\
&\leq \frac{1}{\lambda_N^2} \sum_{i=N}^{\infty} \frac{f_{i+1}^2}{|||G_{i+1}^{(\beta-1,\alpha-\beta-1)}|||^4} |||G_{i+1}^{(\beta-1,\alpha-\beta-1)}|||^2 \quad \text{(using (3.1))} \\
&\leq \frac{1}{\lambda_N^2} \int_0^1 \rho^{(\beta-1,\alpha-\beta-1)}(x) \left(\sum_{i=0}^{\infty} \frac{G_i^{(\beta-1,\alpha-\beta-1)}(x)}{|||G_i^{(\beta-1,\alpha-\beta-1)}|||^2} f_i \right)^2 dx \\
&= \frac{1}{\lambda_N^2} \int_0^1 \rho^{(\beta-1,\alpha-\beta-1)}(x) \, f(x)^2 \, dx \leq \frac{1}{\lambda_N^2} \|f\|^2_{\rho^{(\beta-1,\alpha-\beta-1)}} \\
&\leq C \, (N+1)^{-2(\alpha-1)} \|f\|^2_{\rho^{(\beta-1,\alpha-\beta-1)}} , \quad \text{(using (2.15))}.
\end{aligned}
$$

■

Immediately following from (3.9) we obtain an L^2 error estimate for $u(x) - u_N(x)$.

Corollary 3.1 *For $f(x) \in L^2_{\rho^{(\beta-1,\alpha-\beta-1)}}(0,1)$ and $q_N(x)$ given by (3.7), there exists $C > 0$ such that*

$$
\|q - q_N\| \leq \frac{1}{\lambda_N} \|f\|_{\rho^{(\beta-1,\alpha-\beta-1)}} \leq C \, (N+1)^{-\alpha+1} \|f\|_{\rho^{(\beta-1,\alpha-\beta-1)}} . \tag{3.10}
$$

Proof: As $\rho^{(-(\alpha-\beta),-\beta)}(x) = (1-x)^{-(\alpha-\beta)} x^{-\beta} > 1$, for $0 < x < 1$, then $\|u - u_N\| \leq \|u - u_N\|_{\rho^{(-(\alpha-\beta),-\beta)}}$. Hence the bound (3.10) follows immediately from (3.9) .

■

4 Regularity of $D^j\big((q(x) - c_{-1}xz(x))/\rho^{(\alpha-\beta,\beta)}(x)\big)$

The endpoint behavior of $q(x)$ is determined by the functions $x\,z(x)$ and $\rho^{(\alpha-\beta,\beta)}(x)$. In order to gain insight into the regularity of $q(x)$ "away from the endpoints," in this section we investigate the regularity of $(q - c_{-1}x\,z(x))/\rho^{(\alpha-\beta,\beta)}(x)$. We do this by establishing that $\{D^j\big((q_N - c_{-1}x\,z(x))/\rho^{(\alpha-\beta,\beta)}(x)\big)\}$ is a Cauchy sequence in an appropriately weighted L^2 function space.

Let

$$
f_N(x) = \sum_{i=0}^{N} \frac{f_i}{|||G_i^{(\beta-1,\alpha-\beta-1)}(x)|||^2} G_i^{(\beta-1,\alpha-\beta-1)}(x) .
$$

Hence, using (2.10) and reindexing

$$
D^j f_N(x) = \sum_{i=-1}^{N-1} \frac{f_{i+1}}{|||G_{i+1}^{(\beta-1,\alpha-\beta-1)}|||^2} \frac{\Gamma(i+j+\alpha)}{\Gamma(i+\alpha)} G_{i-j+1}^{(\beta+j-1,\alpha-\beta+j-1)}(x) . \tag{4.1}
$$

Helpful in establishing the regularity of $(q(x) \ - \ c_{-1}x\, z(x))/\rho^{(\alpha-\beta,\beta)}(x)$ is the following lemma.

Lemma 4.1 *For $j \in \mathbb{N}$, there exists $C > 0$ such that*

$$\frac{1}{\lambda_i^2} \left(\frac{i+j+\alpha}{i+\alpha} \right)^2 \frac{|||G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}|||^2}{|||G_{i-j+1}^{(\beta+j-1\,,\,\alpha-\beta+j-1)}|||^2} \ \leq \ C\, i^{-2(\alpha-1)}. \tag{4.2}$$

Proof: From (2.8) and (2.7) ,

$$\begin{aligned} \frac{|||G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}|||^2}{|||G_{i-j+1}^{(\beta+j-1\,,\,\alpha-\beta+j-1)}|||^2} \ &= \ \frac{|||G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}|||^2}{|||G_{i-j+1}^{(\alpha-\beta+j-1\,,\,\beta+j-1)}|||^2} \\ &= \ \frac{1}{(2i\ +\ \alpha\ +\ 1)} \, \frac{\Gamma(i+\alpha-\beta+1)\,\Gamma(i+\beta+1)}{\Gamma(i-j+1)\,\Gamma(i+j+\alpha+1)} \\ &\qquad \cdot\ (2i\ +\ \alpha\ +\ 1)\, \frac{\Gamma(i-j+2)\,\Gamma(i+j+\alpha)}{\Gamma(i+\alpha-\beta+1)\,\Gamma(i+\beta+1)} \\ &= \ \frac{(i-j+1)}{(i+j+\alpha)}. \end{aligned} \tag{4.3}$$

Using Stirling's formula,

$$\frac{1}{|\lambda_i|} \ = \ C\, \frac{\Gamma(i+1)}{\Gamma(i+\alpha)} \sim\ (i+1)^{-(\alpha-1)} \ \sim\ i^{-(\alpha-1)}\,. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain

$$\frac{1}{\lambda_i^2} \left(\frac{i+j+\alpha}{i+\alpha} \right)^2 \frac{|||G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}|||^2}{|||G_{i-j+1}^{(\beta+j-1\,,\,\alpha-\beta+j-1)}|||^2} \sim \left(i^{-(\alpha-1)} \right)^2 \left(\frac{i+j+\alpha}{i+\alpha} \right)^2 \frac{(i-j+1)}{(i+j+\alpha)} \ \sim\ i^{-2(\alpha-1)}\,,$$

from which (4.2) follows.

■

The following theorem describes the regularity of $q(x)$ "away from the endpoints."

Theorem 4.1 *For $j \in \mathbb{N}$, if $D^j f \in L^2_{\rho^{(\beta+j-1\,,\,\alpha-\beta+j-1)}}(0,1)$, then $D^j \big((q(x) - c_{-1}x\, z(x))/\rho^{(\alpha-\beta,\beta)}(x) \big) \in L^2_{\rho^{(\alpha-\beta+j\,,\,\beta+j)}}(0,1)$.*

Proof: From (2.12) and (3.7),

$$D^j \left(\frac{q_N \ - \ c_{-1}x\, z(x)}{\rho^{(\alpha-\beta,\beta)}(x)} \right) \ = \ D^j \left(\sum_{i=0}^{N-1} c_i\, G_i^{(\alpha-\beta\,,\,\beta)}(x) \right) \ = \ \sum_{i=0}^{N-1} c_i\, \frac{\Gamma(i+j+\alpha+1)}{\Gamma(i+\alpha+1)}\, G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}(x),$$

where $G_k^{(a,b)}(x) = 0$ for $k < 0$.

Then,

$$
\begin{aligned}
&\left\| D^j \left(\frac{q_M \ - \ c_{-1} x z(x)}{\rho^{(\alpha-\beta,\beta)}(x)} \right) \ - \ D^j \left(\frac{q_N \ - \ c_{-1} x z(x)}{\rho^{(\alpha-\beta,\beta)}(x)} \right) \right\|^2_{\rho^{(\alpha-\beta+j\,,\,\beta+j)}} \\
&= \left(\rho^{(\alpha-\beta+j\,,\,\beta+j)} \sum_{i=N}^{M-1} c_i \, \frac{\Gamma(i+j+\alpha+1)}{\Gamma(i+\alpha+1)} \, G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)} \ , \ \sum_{i=N}^{M-1} c_i \, \frac{\Gamma(i+j+\alpha+1)}{\Gamma(i+\alpha+1)} \, G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)} \right) \\
&= \sum_{i=N}^{M-1} c_i^2 \left(\frac{\Gamma(i+j+\alpha+1)}{\Gamma(i+\alpha+1)} \right)^2 |||G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}(x)|||^2 \\
&= \sum_{i=N}^{M-1} \frac{f_{i+1}^2}{\lambda_i^2 |||G_{i+1}^{(\beta-1,\alpha-\beta-1)}(x)|||^4} \left(\frac{\Gamma(i+j+\alpha+1)}{\Gamma(i+\alpha+1)} \right)^2 |||G_{i-j}^{(\alpha-\beta+j\,,\,\beta+j)}(x)|||^2 \\
&\le C \sum_{i=N}^{M-1} \frac{f_{i+1}^2}{|||G_{i+1}^{(\beta-1,\alpha-\beta-1)}(x)|||^4} \, i^{-2(\alpha-1)} \left(\frac{\Gamma(i+j+\alpha)}{\Gamma(i+\alpha)} \right)^2 |||G_{i+1-j}^{(\beta+j-1\,,\,\alpha-\beta+j-1)}(x)|||^2 \quad \text{(using (4.2))} \\
&\le C \, N^{-2(\alpha-1)} \, \|D^j f_M(x) \ - \ D^j f_N(x)\|^2_{\rho^{(\beta+j-1\,,\,\alpha-\beta+j-1)}} \qquad \text{(using (4.1))}\,, \qquad (4.5) \\
&= C \, \|D^j f_M(x) \ - \ D^j f_N(x)\|^2_{\rho^{(\beta+j-1\,,\,\alpha-\beta+j-1)}} \,.
\end{aligned}
$$

Assuming that $D^j f \in L^2_{\rho^{(\beta+j-1\,,\,\alpha-\beta+j-1)}}(0,1)$, then $\{D^j f_n\}$ is a Cauchy sequence in $L^2_{\rho^{(\beta+j-1\,,\,\alpha-\beta+j-1)}}(0,1)$. Thus we can conclude that $D^j\big((q \ - \ c_{-1} x \, z(x))/\rho^{(\alpha-\beta,\beta)}(x)\big) \in L^2_{\rho^{(\alpha-\beta+j\,,\,\beta+j)}}(0,1)$.

$\blacksquare$

4.1 Additional error estimate for $q(x) \ - \ c_{-1} x z(x)$

From Theorems 3.1 and 4.1 we have that $q(x) \ - \ c_{-1} x \, z(x) \in L^2_{\rho^{(-(\alpha-\beta)\,,\,-\beta)}}(0,1)$, and $D^j\big((q(x) \ - \ c_{-1} x \, z(x))/\rho^{(\alpha-\beta,\beta)}(x)\big) \in L^2_{\rho^{(\alpha-\beta+j\,,\,\beta+j)}}(0,1)$, $j \in \mathbb{N}$, for a sufficiently smooth rhs function, $f(x)$. Thus the power on the weight function, $\big(\rho^{(\alpha-\beta\,,\,\beta)}(x)\big)^j$, such that $D^j\big((q(x) - c_{-1} x \, z(x))/\rho^{(\alpha-\beta,\beta)}(x)\big) \in L^2_{\rho^{(\alpha-\beta+j\,,\,\beta+j)}}(0,1)$, increases with each derivative of $(q(x) - c_{-1} x \, z(x))/\rho^{(\alpha-\beta,\beta)}(x)$. This observation leads to the following definition of weighted Sobolev spaces [14].

$$
H^s_{\rho^{(a\,,\,b)}}(0,1) \ := \ \Big\{ v \,|\, v \text{ is measurable and } \|v\|_{s,\,\rho^{(a\,,\,b)}} < \infty \Big\} \ , \quad s \in \mathbb{N}_0 \,,
$$

with associated norm and semi-norm

$$
\|v\|_{s,\,\rho^{(a\,,\,b)}} \ := \ \left(\sum_{j=0}^{s} \|D^j v\|^2_{\rho^{(a+j\,,b+j)}} \right)^{1/2} , \quad |v|_{s,\,\rho^{(a\,,\,b)}} \ := \ \|D^s v\|_{\rho^{(a+s\,,b+s)}} \,.
$$

For $s \in \mathbb{R}^+ \backslash \mathbb{N}_0$, $H^s_{\rho^{(a\,,\,b)}}(0,1)$ is defined by the K-method of interpolation, and for $s \in \mathbb{R}^-$, $H^s_{\rho^{(a\,,\,b)}}(0,1)$ is defined by duality.

Let $\mathcal{P}_N$ denote the space of polynomials of degree $\le N$, and introduce the orthogonal projection $P_{N,a,b} \, : \, L^2_{\rho^{(a\,,\,b)}}(0,1) \to \mathcal{P}_N$ defined by

$$
\left(v \ - \ P_{N,a,b} v \ , \ \phi\right)_{\rho^{(a\,,\,b)}} \ = \ 0 \,, \quad \forall \phi \in \mathcal{P}_N \,.
$$

Then from [14] we have the following theorem.

Theorem 4.2 *[14, Theorem 2.1] For $\mu \in \mathbb{N}_0$ and $v \in H^s_{\rho^{(a\,,\,b)}}(0,1)$, with $0 \leq \mu \leq s$, there exists a constant C, independent of N, α and β such that*

$$\|v(x) \,-\, P_{N,a,b}v(x)\|_{\mu,\,\rho^{(a\,,\,b)}} \;\leq\; C\,(N\,(N+a+b))^{\frac{\mu-s}{2}}\,\|v\|_{s,\,\rho^{(a\,,\,b)}}\,. \tag{4.6}$$

Remark: In [14] (4.6) is stated for $s \in \mathbb{N}_0$. The result extends to $s \in \mathbb{R}^+$ using interpolation.

We can apply (4.6) to obtain an improved error estimate for $\big((q(x) \,-\, q_N(x))/\rho^{(\alpha-\beta,\beta)}(x)\big)$.

Corollary 4.1 *For $\mu \in \mathbb{N}_0$ and $f \in H^s_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$, with $0 \leq \mu \leq s$ there exists $C > 0$ (independent of N and α) such that*

$$\|(q \,-\, q_N)/\rho^{(\alpha-\beta,\beta)}\|_{\mu,\,\rho^{(\alpha-\beta\,,\,\beta)}} \;\leq\; C\,N^{-(\alpha-1)}\,(N\,(N+\alpha-2))^{\frac{\mu-s}{2}}\,\|f\|_{s,\,\rho^{(\beta-1\,,\,\alpha-\beta-1)}}\,. \tag{4.7}$$

Proof: Noting that $f_N(x) \;=\; P_{N,\,\beta-1\,,\,\alpha-\beta-1}f(x)$, from (4.5), taking the limit as $M \to \infty$, we have

$$\begin{aligned}
\|D^\mu\big((q \,-\, q_N)/\rho^{(\alpha-\beta,\beta)}\big)\|_{\rho^{(\alpha-\beta+\mu\,,\,\beta+\mu)}} &\leq C\,N^{-(\alpha-1)}\,\|D^\mu(f \,-\, f_N)\|_{\rho^{(\beta+\mu-1\,,\,\alpha-\beta+\mu-1)}} \\
&\leq C\,N^{-(\alpha-1)}\,\|f \,-\, f_N\|_{\mu,\,\rho^{(\beta-1\,,\,\alpha-\beta-1)}} \\
&\leq C\,N^{-(\alpha-1)}\,(N\,(N+\alpha-2))^{\frac{\mu-s}{2}}\,\|f\|_{s,\,\rho^{(\beta-1\,,\,\alpha-\beta-1)}}\,,
\end{aligned} \tag{4.8}$$

where, in the last step we have used (4.6). ∎

5 Convergence of $u(x) - u_N(x)$

In this section we use the error estimates derived above for $q(x){-}q_N(x)$ to investigate the convergence of $u(x) - u_N(x)$.

From (2.1), $u(x)$ is given by

$$u(x) \;=\; -\int_0^x \frac{\tilde{q}(s)}{K(s)}\,ds \;=\; -\,c_{-2}\int_0^x \frac{z(s)}{K(s)}\,ds \;-\; \int_0^x \frac{q(s)}{K(s)}\,ds\,. \tag{5.1}$$

Hence,

$$\begin{aligned}
|u(x) - u_N(x)| &\leq\; |c_{-2} - c_{-2,N}|\,\left|\int_0^x \frac{z(s)}{K(s)}\,ds\right| \;+\; \left|\int_0^x \frac{q(s) - q_N(s)}{K(s)}\,ds\right| \\
&\leq\; |c_{-2} - c_{-2,N}|\,\left|\int_0^1 \frac{z(s)}{K(s)}\,ds\right| \\
&\qquad +\; \left(\int_0^1 (1-s)^{-(\alpha-\beta)}\,s^{-\beta}\,(q(s) - q_N(s))^2\,ds\right)^{1/2}\,\left(\int_0^1 \frac{(1-s)^{\alpha-\beta}\,s^\beta}{K^2(s)}\,ds\right)^{1/2} \\
&\leq\; C_{11}\,C_{12}\,\|q - q_N\|_{\rho^{(-(\alpha-\beta),-\beta)}}\,C_{11}^{-1} \;+\; C_{12}\,\|q - q_N\|_{\rho^{(-(\alpha-\beta),-\beta)}} \\
&=\; 2\,C_{12}\,\|q - q_N\|_{\rho^{(-(\alpha-\beta),-\beta)}} \;=\; \|(q \,-\, q_N)/\rho^{(\alpha-\beta,\beta)}\|_{\rho^{(\alpha-\beta\,,\,\beta)}}\,.
\end{aligned} \tag{5.2}$$

Combining (5.2) and (4.7) we obtain an L^∞ error estimate for $u(x) - u_N(x)$.

Lemma 5.1 *For $f \in H^s_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$ there exists $C > 0$ such that*

$$\|u \,-\, u_N\|_{L^\infty} \;\leq\; C\,N^{-\alpha+1-s}\,\|f\|_{s,\,\rho^{(\beta-1\,,\,\alpha-\beta-1)}}\,. \tag{5.3}$$

The above analysis is very coarse. We improve on this estimate with a more detailed error analysis in the next section.

5.1 Convergence of $\|u - u_N\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}}$

In this section we investigate the convergence of $u_N(x)$ to $u(x)$ in the weighted L^2 norm. To improve upon the analysis used to obtain (5.2) an additional regularity assumption is needed for $K(x)$.

Theorem 5.1 *For $f \in H^s_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$ and $K^{-1} \in W^1_\infty(0,1)$, then there exists $C > 0$ (independent of N and α) such that*

$$\|u \,-\, u_N\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \;\leq\; C\,N^{-\alpha}\,(N\,(N+\alpha-1))^{-\frac{s}{2}}\,\|f\|_{s,\,\rho^{(\beta-1,\alpha-\beta-1)}}. \tag{5.4}$$

Proof: Recall that $\tilde{q}(x) \;=\; c_{-2}z(x) \;+\; q(x)$, where c_{-2} is determined by (2.2). Then, with (5.1), we have that

$$u(x) - u_N(x) = \frac{\int_0^x \frac{z(s)}{K(s)}\,ds}{\int_0^1 \frac{z(s)}{K(s)}\,ds}\int_0^1 \frac{q(s)-q_N(s)}{K(s)}ds - \int_0^x \frac{q(s)-q_N(s)}{K(s)}ds. \tag{5.5}$$

From (2.12) it follows that

$$\int_0^x \rho^{(\alpha-\beta,\beta)}(s)\,G_n^{(\alpha-\beta,\beta)}(s)\,ds \;=\; \frac{-1}{n}\rho^{(\alpha-\beta+1\,,\,\beta+1)}(x)\,G_{n-1}^{(\alpha-\beta+1\,,\,\beta+1)}(x)\,, \quad n \geq 1\,. \tag{5.6}$$

Using (3.7) and integration by parts in (5.5), together with (5.6), we obtain

$$\begin{aligned} u(x) \,-\, u_N(x) = {} & \frac{\int_0^x \frac{z(s)}{K(s)}\,ds}{\int_0^1 \frac{z(s)}{K(s)}\,ds}\sum_{i=N}^{\infty}\frac{c_i}{i}\int_0^1 \rho^{(\alpha-\beta+1\,,\,\beta+1)}(s)\,G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(s)D\left(\frac{1}{K(s)}\right)ds \\ & + \sum_{i=N}^{\infty}\frac{c_i}{i}\frac{1}{K(x)}\rho^{(\alpha-\beta+1\,,\,\beta+1)}(x)\,G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(x) \\ & - \sum_{i=N}^{\infty}\frac{c_i}{i}\int_0^x \rho^{(\alpha-\beta+1\,,\,\beta+1)}(s)\,G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(s)D\left(\frac{1}{K(s)}\right)ds \\ := {} & R_1 + R_2 + R_3. \end{aligned} \tag{5.7}$$

Applying $\|\cdot\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}}$ on both sides of (5.7) yields

$$\|u \,-\, u_N\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \leq \sum_{j=1}^{3}\|R_j\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}}.$$

We bound R_1 by

$$\begin{aligned}
&\|R_1\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \\
&= \left\| \frac{\int_0^x \frac{z(s)}{K(s)}\,ds}{\int_0^1 \frac{z(s)}{K(s)}\,ds} \sum_{i=N}^{\infty} \frac{c_i}{i} \int_0^1 \rho^{(\alpha-\beta+1\,,\,\beta+1)}(s)\, G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(s) D\left(\frac{1}{K(s)}\right) ds \right\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \\
&\le \|1\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \int_0^1 \left| D\left(\frac{1}{K(s)}\right) \sum_{i=N}^{\infty} \frac{c_i}{i} \rho^{(\alpha-\beta+1\,,\,\beta+1)}(s)\, G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(s) \right| ds \\
&\le C\|K^{-1}\|_{W^1_\infty} \int_0^1 \left| \rho^{(\alpha-\beta+1)/2\,,\,(\beta+1)/2}(s) \rho^{(\alpha-\beta+1)/2\,,\,(\beta+1)/2}(s) \sum_{i=N}^{M-1} \frac{c_i}{i} G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(s) \right| ds \\
&\le C\|K^{-1}\|_{W^1_\infty} \|1\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \left\| \sum_{i=N}^{\infty} \frac{c_i}{i} G_{i-1}^{(\alpha-\beta+1\,,\,\beta+1)}(s) \right\|_{\rho^{(\alpha-\beta+1\,,\,\beta+1)}} \\
&= C\|K^{-1}\|_{W^1_\infty} \left(\sum_{i=N+1}^{\infty} \frac{1}{(i-1)^2} \frac{f_i^2}{\lambda_{i-1}^2 \,|||G_i^{(\beta-1\,,\,\alpha-\beta-1)}|||^4} |||G_{i-2}^{(\alpha-\beta+1\,,\,\beta+1)}|||^2 \right)^{1/2} \text{(using (3.8)).}
\end{aligned} \tag{5.8}$$

Similar to Lemma 3.1, we have

$$\begin{aligned}
\frac{|||G_{i-2}^{(\alpha-\beta+1\,,\,\beta+1)}|||^2}{|||G_i^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} &= \frac{1}{2i+\alpha-1} \frac{\Gamma(i+\alpha-\beta)\,\Gamma(i+\beta)}{\Gamma(i-1)\Gamma(i+\alpha+1)} \frac{2i+\alpha-1}{1} \frac{\Gamma(i+1)\,\Gamma(i+\alpha-1)}{\Gamma(i+\beta)\,\Gamma(i+\alpha-\beta)} \\
&= \frac{i(i-1)}{(i+\alpha)(i+\alpha-1)} \le 1\,.
\end{aligned} \tag{5.9}$$

Using (5.8) and (5.9), together with (2.15) we then obtain

$$\begin{aligned}
\|R_1\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} &\le C\|K^{-1}\|_{W^1_\infty} \left(\sum_{i=N+1}^{\infty} \frac{1}{(i-1)^2} \frac{f_i^2}{\lambda_{i-1}^2 \,|||G_i^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} \right)^{1/2} \\
&\le \frac{C\|K^{-1}\|_{W^1_\infty}}{N\,\lambda_N} \left(\int_0^1 \rho^{(\beta-1\,,\,\alpha-\beta-1)}(x) \left(\sum_{i=N+1}^{\infty} \frac{f_i}{|||G_i^{(\beta-1\,,\,\alpha-\beta-1)}|||^2} G_i^{(\beta-1\,,\,\alpha-\beta-1)}(x) \right)^2 dx \right)^{1/2} \\
&\le \frac{C\|K^{-1}\|_{W^1_\infty}}{N\,(N+1)^{\alpha-1}} \|f - f_N\|_{\rho^{(\beta-1,\alpha-\beta-1)}}.
\end{aligned}$$

Then we bound R_2 and R_3 in a similar manner and apply the approximating property (4.6) to obtain

$$\|u - u_N\|_{\rho^{(-(\alpha-\beta+1)\,,\,-(\beta+1))}} \le C\|K^{-1}\|_{W^1_\infty} N^{-1}\,(N+1)^{-(\alpha-1)}\,(N\,(N+\alpha-1))^{-\frac{s}{2}}\,\|f\|_{s,\,\rho^{(\beta-1,\alpha-\beta-1)}}.$$

■

Remark: For the discussion in Sections 3, 4 and 5, we have assumed that $1/2 \le r \le 1$, which lead to our representation of $q(x)$ as $q(x) = c_{-1}x\,z(x) + \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{i=0}^{\infty} c_i\, G_i^{(\alpha-\beta\,,\,\beta)}(x)$. As previously commented, since $(1-x)\,z(x)$ is a more regular function than $x\,z(x)$ for $r < 1/2$, for $0 \le r < 1/2$ we would use as the representation for $q(x)$, $q(x) = c_{-1}(1-x)\,z(x) + \rho^{(\alpha-\beta\,,\,\beta)}(x) \sum_{i=0}^{\infty} c_i\, G_i^{(\alpha-\beta\,,\,\beta)}(x)$. Apart from the obvious change, $q(x) - c_{-1}x\,z(x) \to q(x) - c_{-1}(1-x)\,z(x)$, the above analysis also applies directly for $0 \le r < 1/2$.

6 Numerical experiments

In this section we present two numerical examples to validate our approximation scheme, and to compare the experimental rate of convergence of the approximation with the theoretically predicated rate. The numerical examples were computed using MATLAB R2016a, on a computer with an Intel(R) Core(TM) i5-4570 CPU @ 3.20 GHz, and with 8 GB of RAM. To evaluate the intermediate integrals, such as the computations of the coefficients (3.6) and the final transformation (2.1) for $u(x)$, we used MATLAB's high-order global adaptive quadrature routine 'quadgk', with the default error tolerances.

Within Example 1 we consider three numerical experiments corresponding to different values of α and r. For this example we choose $K(x) = 1$ which permits us to compare the theoretically predicted rate of convergence of u_N to u in the $L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}$ norm with its experimental rate.

In order to determine the theoretical rate of convergence for $\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$ and $\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$ from (4.7) and (5.4), respectively, we need to determine the largest value for j such that $f(x) \in H^j_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$, i.e, the largest j such that $\|D^j f\|^2_{\rho^{(\beta+j-1\,,\,\alpha+j-\beta-1)}} < \infty$. The most singular terms for $f(x)$ in Example 1 are $x^{2-\alpha}$ and $(1-x)^{2-\alpha}$. We focus our attention on $x^{2-\alpha}$.

Note that $D^j x^{2-\alpha} \sim x^{2-\alpha-j}$. Then

$$\begin{aligned}
\|D^j x^{2-\alpha}\|^2_{\rho^{(\beta+j-1\,,\,\alpha+j-\beta-1)}} &\sim \int_0^1 x^{\alpha+j-\beta-1}\,\left(x^{2-\alpha-j}\right)^2\,dx \;=\; \int_0^1 x^{3-\alpha-\beta-j}\,dx \;<\; \infty \\
&\Rightarrow\;\; -1 < \; 3-\alpha-\beta-j \\
&\Rightarrow\;\; j < \; 4-\alpha-\beta\,.
\end{aligned}$$

Then, for experiment 1 in Example 1 ($\alpha = 1.60$, $\beta = 0.85$) $f(x) \in H^j_{\rho^{(\beta-1\,,\,\alpha-\beta-1)}}(0,1)$ for $j < 1.55$, which leads to theoretical asymptotic rates of $\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}} \sim N^{-(\alpha-1+j)} = N^{-2.15}$ and $\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}} \sim N^{-(\alpha+j)} = N^{-3.15}$.

Assuming that $\|\xi - \xi_N\|_{L_\rho} \sim N^{-\kappa}$, the experimental convergence rate is calculated using

$$\kappa \approx \frac{\log(\|\xi - \xi_{N_1}\|_{L_\rho} / \|\xi - \xi_{N_2}\|_{L_\rho})}{\log(N_2/N_1)}\,.$$

Example 1. Let $K(x) = 1$, β be determined by (2.13), and

$$\begin{aligned}
f(x) = {} & \frac{6r}{\Gamma(2-\alpha)\delta}((2\alpha-8)x^{3-\alpha} + (\alpha-3)(\alpha-4)x^{2-\alpha}) \\
& + \frac{6(1-r)}{\Gamma(2-\alpha)\delta}(-(2\alpha-8)(1-x)^{3-\alpha} - (\alpha-3)(\alpha-4)(1-x)^{2-\alpha}),
\end{aligned}$$

where $\delta := \alpha^3 - 9\alpha^2 + 26\alpha - 24$. Then the solution $u(x)$, and the related $q(x)$, are given by

$$u(x) = 3x^2 - 2x^3 - \frac{x^\beta\,{}_2F_1(-\alpha+\beta+1\,,\,\beta\,;\,\beta+1\,,\,x)}{{}_2F_1(-\alpha+\beta+1\,,\,\beta\,;\,\beta+1\,,\,1)}, \quad q(x) = -6x + 6x^2,$$

where ${}_2F_1(a,b;c,x)$ denote the Gauss three-parameter hypergeometric function defined as follows:

$$\begin{aligned} {}_2F_1(a,b;c,x) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 z^{b-1}(1-z)^{c-b-1}(1-zx)^{-a}dz \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n x^n}{(c)_n n!}, \end{aligned}$$

with convergence only if $Re(c) > Re(b) > 0$ and $(s)_n$ is the rising Pochhammer symbol defined by $(s)_n = \Gamma(s+n)/\Gamma(s)$.

A plot of the solution $u(x)$, corresponding to $\alpha = 1.40$, $r = 0.76$ and $\beta = 0.50$, and a plot of the errors for this numerical experiment are presented in Figure 6.1. Near $x = 0$, $u'(x) \sim x^{\beta-1} = x^{-0.5}$, and near $x = 1$, $u'(x) \sim (1-x)^{\alpha-\beta-1} = x^{-0.1}$.

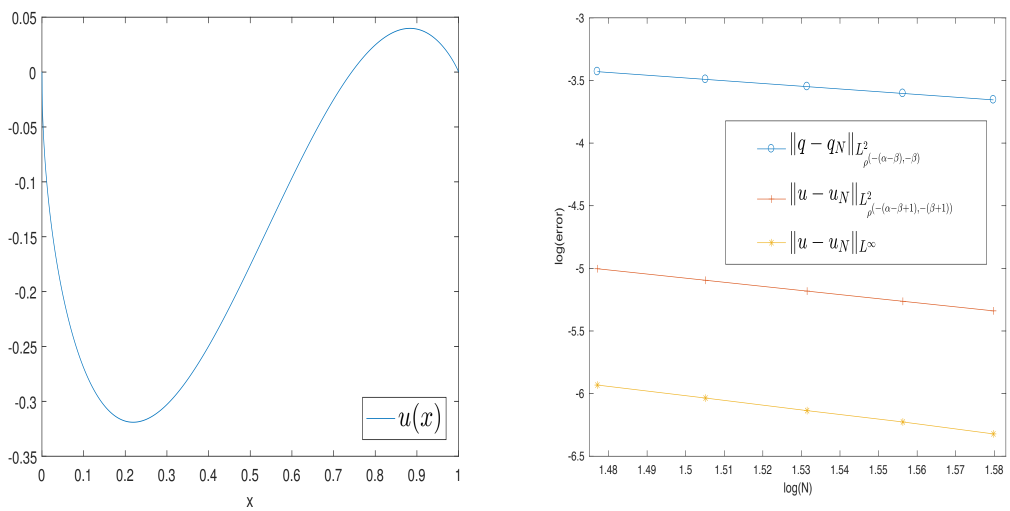

Figure 6.1: The plot of solution $u(x)$ (left), and (right) the $log-log$ plot of the errors for experiment 1 of Example 1.

The experimental convergence rate κ of the error in different norms for Example 1 are shown in Table 6.1, 6.2 and 6.3.

Table 6.1: Example 1 with $\alpha = 1.60$, $r = 0.39$ and $\beta = 0.85$.

N	$\|q-q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$	κ	$\|u-u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$	κ	$\|u-u_N\|_{L^\infty}$	κ
30	5.23E-04		1.40E-05		1.51E-06	
32	4.54E-04	2.18	1.13E-05	3.26	1.17E-06	3.90
34	3.98E-04	2.18	9.29E-06	3.28	9.63E-07	3.27
36	3.51E-04	2.18	7.69E-06	3.30	7.83E-07	3.62
38	3.12E-04	2.18	6.43E-06	3.33	6.46E-07	3.54
Pred.		2.15		3.15		2.15

Table 6.2: Example 1 with $\alpha = 1.40$, $r = 0.50$ and $\beta = 0.70$.

N	$\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$	κ	$\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$	κ	$\|u - u_N\|_{L^\infty}$	κ
30	5.10E-04		1.37E-05		1.59E-06	
32	4.40E-04	2.29	1.10E-05	3.39	1.22E-06	4.16
34	3.83E-04	2.29	8.94E-06	3.42	1.02E-06	2.87
36	3.36E-04	2.29	7.34E-06	3.44	8.28E-07	3.72
38	2.97E-04	2.29	6.09E-06	3.47	6.72E-07	3.86
Pred.		2.30		3.30		2.30

Table 6.3: Example 1 with $\alpha = 1.80$, $r = 0.50$ and $\beta = 0.90$.

N	$\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$	κ	$\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$	κ	$\|u - u_N\|_{L^\infty}$	κ
30	4.21E-04		1.11E-05		1.08E-06	
32	3.69E-04	2.07	9.07E-06	3.16	8.40E-07	3.85
34	3.25E-04	2.07	7.48E-06	3.18	7.08E-07	2.82
36	2.89E-04	2.08	6.23E-06	3.21	5.76E-07	3.60
38	2.58E-04	2.08	5.23E-06	3.24	4.64E-07	4.03
Pred.		2.10		3.10		2.10

The experimental convergence rates for $\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$ and $\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$ are in good agreement with the theoretically predicted rates. Not surprisingly, the theoretically predicted rate for $\|u - u_N\|_{L^\infty}$ in (5.2) appears to be suboptimal.

Example 2. With this example we investigate the numerical approximation for the interesting case of a non constant $K(x)$. Let $K(x) = 1 + x^2$ and

$$
\begin{aligned}
f(x) = {} & r\left(-480\frac{x^{6-\alpha}}{\Gamma(7-\alpha)} + 144\frac{x^{5-\alpha}}{\Gamma(6-\alpha)} - 36\frac{x^{4-\alpha}}{\Gamma(5-\alpha)} + 12\frac{x^{3-\alpha}}{\Gamma(4-\alpha)} - 2\frac{x^{2-\alpha}}{\Gamma(3-\alpha)}\right) \\
& - (1-r)\left(480\frac{(1-x)^{6-\alpha}}{\Gamma(7-\alpha)} - 366\frac{(1-x)^{5-\alpha}}{\Gamma(6-\alpha)} + 132\frac{(1-x)^{4-\alpha}}{\Gamma(5-\alpha)} - 32\frac{(1-x)^{3-\alpha}}{\Gamma(4-\alpha)} + 4\frac{(1-x)^{2-\alpha}}{\Gamma(3-\alpha)}\right).
\end{aligned}
$$

Then the solution $u(x)$, and the related $q(x)$, are

$$
u(x) = x^2(1-x)^2, \quad q(x) = -2(1+x^2)x(1-x)(1-2x).
$$

The convergence rate κ of the error in different norms for Example 2 are shown in Table 6.4, 6.5 and 6.6.

Table 6.4: Example 2 with $\alpha = 1.60$, $r = 0.39$ and $\beta = 0.85$.

N	$\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$	κ	$\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$	κ	$\|u - u_N\|_{L^\infty}$	κ
30	3.01E-04		5.57E-06		7.21E-07	
32	2.59E-04	2.28	4.50E-06	3.31	5.54E-07	4.07
34	2.26E-04	2.28	3.68E-06	3.30	4.54E-07	3.28
36	1.98E-04	2.28	3.05E-06	3.28	3.63E-07	3.95
38	1.75E-04	2.27	2.56E-06	3.27	2.92E-07	4.01
Pred.		2.15		3.15		2.15

Table 6.5: Example 2 with $\alpha = 1.40$, $r = 0.50$ and $\beta = 0.70$.

N	$\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$	κ	$\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$	κ	$\|u - u_N\|_{L^\infty}$	κ
30	2.90E-04		5.49E-06		7.73E-07	
32	2.49E-04	2.37	4.40E-06	3.42	6.08E-07	3.72
34	2.16E-04	2.36	3.58E-06	3.41	4.79E-07	3.92
36	1.89E-04	2.36	2.95E-06	3.40	3.82E-07	3.97
38	1.66E-04	2.35	2.45E-06	3.39	3.10E-07	3.83
Pred.		2.30		3.30		2.30

Table 6.6: Example 2 with $\alpha = 1.80$, $r = 0.50$ and $\beta = 0.90$.

N	$\|q - q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$	κ	$\|u - u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$	κ	$\|u - u_N\|_{L^\infty}$	κ
30	2.38E-04		4.40E-06		5.22E-07	
32	2.07E-04	2.14	3.58E-06	3.19	4.10E-07	3.73
34	1.82E-04	2.14	2.95E-06	3.18	3.32E-07	3.49
36	1.61E-04	2.14	2.46E-06	3.17	2.68E-07	3.77
38	1.43E-04	2.14	2.08E-06	3.16	2.16E-07	4.00
Pred.		2.10		3.10		2.10

The experimental convergence rates for $\|q-q_N\|_{L^2_{\rho^{(-(\alpha-\beta),-\beta)}}}$ and $\|u-u_N\|_{L^2_{\rho^{(-(\alpha-\beta+1),-(\beta+1))}}}$ are again in good agreement with the theoretically predicted rates. We again note that the error estimate obtained in (5.2) appears to be suboptimal.

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