SOLVABILITY AND APPROXIMATION OF TWO-SIDE CONSERVATIVE FRACTIONAL DIFFUSION PROBLEMS WITH VARIABLE-COEFFICIENT BASED ON LEAST-SQUARES *

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Abstract. In this article, we investigate the solvability theory and numerical simulation for two-side conservative fractional diffusion equations (CFDE) with variable-coefficient $K(x)$. We introduce $u = -KDp$ as an intermediate variable to isolate $K(x)$ from the nonlocal operator, and then apply the least-squares to formulate a mixed-type variational formulation for the unknown and the intermediate variable. Correspondingly, the admissible space for the solution is decomposed as a direct sum of a regular fractional Sobolev space and a space spanned by two known kernel-dependent singular functions by proving that the two-side fractional derivative operator is a bijective mapping from its admissible space to $L^2(\Omega)$. The solution p and u then are represented as a sum of a regular part and a kernel-dependent singular part with two undetermined constant coefficients, which can be expressed by prescribed boundary conditions and derived orthogonal decomposition for $L^2(\Omega)$ respectively. Thus, a new regularity theory for the solution is established in terms of the right-hand side only, which extends those regularity results of one side CFDE in [17, 39], and of fractional Laplace operator corresponding to $\theta = \frac{1}{2}$ in [1, 14] to more general CFDE with variable diffusive coefficients for $0 < \theta < 1$. Then, we design a kernel-independent least-squares mixed finite element procedure (LSMFE). Theoretical analysis and numerical experiments conducted in this article demonstrate that the LSMFE can capture the singular part of the solution exactly, approximate the solution with optimal-order accuracy, and be easily implemented.

Key words. Fractional Diffusion Equation; Variable-Coefficient; Space Decomposition; Solvability and Regularity; Least-Squares; Mixed Finite Element; Convergence

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1. Introduction.

We consider the following two-side conservative fractional diffusion equation(CFDE) with variable coefficient $K(x)$ and $0 < \beta < 1$,

$$(1.1)\qquad \begin{aligned} &-\{\theta_0 D_x^{1-\beta} - (1-\theta)_x D_1^{1-\beta}\}(K(x)Dp) = f(x), x \in \Omega := (0,1),\\ &p(0) = p(1) = 0, \end{aligned}$$

where $D = \frac{d}{dx}$ represents the first-order spatial derivative; θ $(0 \le \theta \le 1)$ indicates the relative weight of forward versus backward transition probability; $K(x)$ is a diffusive coefficient with positive upper and lower bounds, i.e.,

$$0 < K_* \le K(x) \le K^*,$$

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and $f(x)$ refers to the source or sink term. ${}_0D_x^{1-\beta}$, ${}_xD_1^{1-\beta}$ denote the left and right Riemann-Liouville fractional derivatives of order $1-\beta$ defined by [29, 30]

$$
\begin{aligned}
&{}_0D_x^{1-\beta}v = \tfrac{d}{dx}({}_0I_x^{\beta}v), \quad {}_0I_x^{\beta}v = \tfrac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}v(t)dt,\\
&{}_xD_1^{1-\beta}v = -\tfrac{d}{dx}({}_xI_1^{\beta}v), \quad {}_xI_1^{\beta}v = \tfrac{1}{\Gamma(\beta)}\int_x^1 (t-x)^{\beta-1}v(t)dt,
\end{aligned}
\tag{1.2}
$$

with $\Gamma(\cdot)$ being the Euler gamma function.

The CFDE (1.1), derived by incorporating Darcy's law with the fractional mass balance [10, 37], describes a physical phenomena exhibiting anomalous diffusion in subsurface fluid flow [3] and solute transport [26], and thus has attracted considerable attention. Many numerical methods, mainly based on the finite difference framework, have been developed (see [7, 18, 20, 22, 25, 27, 28, 36, 38, 40, 42] and the references cited therein).

As the mathematical structure and properties of the CFDE (1.1) is not completely understood, it is reasonable to develop its Galerkin variational formulation for predicting its mathematical properties and also inducing an easily-computed finite element procedure. In this line, Ervin and Roop [11, 12, 13] proposed a theoretical framework for the Galerkin finite element approximation to the fractional diffusion equation with constant diffusion coefficient. Subsequently, the discontinuous Galerkin method [9], Petrov-Galerkin method [17, 19, 32, 33] and mixed finite element [6, 16] were developed successively for its constant coefficient version. However, there has been little research on direct finite element simulations for the CFDE with variable coefficient $K(x)$.

The difficulties in finite element simulation for the variable-coefficient CFDE (1.1) lie in that the coercivity of the Galerkin weak formulation may not be true, and thus the finite element solution may not converge as pointed out in [32, 33]. To overcome these difficulties, [34] developed a Petrov-Galerkin procedure with an optimal-order convergence for sufficiently smooth solutions. However, it needs to construct a $K(x)-$dependent and nonlocal test function space from its trial function space by a nonlocal transform to ensure the LBB constraint, which may increase its computing burden.

Recently, we developed a series of mixed finite element procedures in as in our previous work [6, 16, 23, 31, 39] for the CFDE (1.1) corresponding to $\theta = 1$ based on the intuitional decomposition for the admissible space of one-side differential operator as a direct sum of this subspace and the kernel space. But these results can not be generalized in parallel to two-side case and the solution structure of (1.1) is far from clear, which becomes a major impediment for better understanding and simulation to these diffusion processes governed by (1.1).

In this article, we introduce $u = -KDp$ as an intermediate variable to isolate $K(x)$ from the nonlocal operator and combine the least-squares technique to propose a mixed-type Galerkin formulation and corresponding mixed finite element procedure. The main objectives are to: (1) prove the two-side differential operator of order $1-\beta$ is a surjective mapping from admissible space of the operator to $L^2(\Omega)$. Especially, the surjective mapping becomes bijective as the kernel function is wiped out from the admissible space to ensure the existence and uniqueness of the solution to (1.1); (2) decompose accordingly the admissible space as a direct sum of a regular fractional Sobolev space and a space spanned by two known kernel-dependent singular functions and express the solution p, u as a sum of a regular part and a kernel-dependent singular part, respectively, with two undetermined constant coefficients which can be calculated from prescribed boundary conditions and derived orthogonal decomposition for L^2 space, respectively. Thus, a new regularity theory for the solution of (1.1) is established in terms of the right-hand side only, which clarifies the structure of the solution to (1.1), and also extends those results of one side CFDE in [17, 39] and of fractional Laplace operator in [1, 14] to general fractional diffusion operator (1.1) with variable coefficient and $0 < \theta < 1$; (3) design a least-squares mixed finite element procedure (LSMFE), from which, the regular part of the solution can be computed with an optimal order convergence rate and the singular part can be calculated almost exactly combined with the expression of the two undetermined constant coefficients. This guarantees that the procedure can capture the singular part of the solution exactly and keep the convergence order optimal whatever the solution to (1.1) is. In addition, the corresponding algorithm is easily implemented since the approximation spaces involved are typical finite element spaces; (4) prove rigorously the optimal convergence for the LSMFE and conduct numerical experiments to verify the robustness of the proposed finite element procedure.

The outline of this article is as follows. In Section 2, we review the definitions and properties of fractional operators and fractional spaces, and then prove the equivalence between two-side fractional derivative space and classical Sobolev space. In Section 3, we discuss kernel space, boundness and invertibility of two-side fractional differential operator, prove the two-side differential operator of order $1-\beta$ is a surjective mapping from its admissible space to $L^2(\Omega)$ and also a bijective as the kernel function is wiped out from the admissible space, so to clarify solution structure of fractional equation of order $1-\beta$ to be a regular part and a kernel-dependent singular part with two unknown constant coefficients, and then determine the unknown coefficient by giving a orthogonal decomposition of $L^2(\Omega)$. In Section 4, we develop a least-squares mixed formulation for the CFDE (1.1) by introducing the intermediate variable u, in which the admissible space of the solution u is decomposed into a direct sum of fractional Sobolev space and the known kernel-spanned space, so as to express the solution (u, p) by its regular part and singular part with undetermined coefficient. Subsequently, we demonstrate the solvability and regularity of the solution to the CFDE (1.1). In Section 5, we propose a least-squares mixed finite element method (LSMFE) and demonstrate its solvability and optimal-order convergence. In Section 6, we apply numerical experiments to verify our theoretical results.

2. Fractional Derivative Space and Fractional Sobolev Space. In this section, we shall review the definitions and properties of the fractional derivative spaces and then discuss their equivalences to the fractional Sobolev spaces.

We first recall the definitions of fractional Sobolev space. For any $s \geq 0$, we denote space $H^s(\Omega)$ to be the Sobolev space of order s endowed with the norm $\|\cdot\|_{H^s(\Omega)}$ and seminorm $|\cdot|_{H^s(\Omega)}$, and $H_0^s(\Omega)$, a subspace of $H^s(\Omega)$, to be the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{H^s(\Omega)}$, where $C_0^\infty(\Omega)$ denotes the set of all functions in $C^\infty(\Omega)$ with compact support in Ω. Define $H_L^s(\Omega)$ and $H_R^s(\Omega)$ to be the subspace of fractional Sobolev space $H^s(\Omega)$ as

$$H_L^s(\Omega) = \{v \in H^s(\Omega) : v^{(k)}(0) = 0, \text{ for nonnegative integer } k < s - \tfrac{1}{2}\},$$
$$H_R^s(\Omega) = \{v \in H^s(\Omega) : v^{(k)}(1) = 0, \text{ for nonnegative integer } k < s - \tfrac{1}{2}\},$$

respectively.

For the space $H_0^s(\Omega)$, we have the following lemma [24].

LEMMA 2.1. *[24] Assume that Ω is bounded. Then, $C_0^\infty(\Omega)$ is dense in $H^s(\Omega)$ for $0 \leq s < \frac{1}{2}$. In this case, $H^s(\Omega) = H_0^s(\Omega)$.*

Next, we present the definitions of fractional derivative spaces. We introduce the one-side fractional derivative spaces as follows.

DEFINITION 2.2. *[11, 39] Define the left fractional derivative space $J_L^s(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ for $0 \le s < \frac{1}{2}$, and the closure of $C^\infty(\Omega)$ for $\frac{1}{2} < s \le 1$, under the norm*

$$\|\cdot\|_{J_L^s(\Omega)} = \{\|\cdot\|^2 + \|{}_0D_x^s \cdot\|^2\}^{\frac{1}{2}}. \tag{2.1}$$

Similarly, the right fractional derivative space $J_R^s(\Omega)$ can be defined.

For the left derivative space $J_L^s(\Omega)$, we have the following decomposition lemma proved in [39].

LEMMA 2.3. *[39] Let $0 \le s \le 1$. Then*

$$J_L^s(\Omega) = \begin{cases} H^s(\Omega), & \text{for } \ 0 \le s < \frac{1}{2}, \\ H_L^s(\Omega) \oplus \text{span}\{x^{s-1}\}, & \text{for } \ \frac{1}{2} < s \le 1. \end{cases}$$

Further, for any $v \in H_L^s(\Omega)$, there exist positive constants C_1 and C_2 such that

$$C_1|v|_{J_L^s(\Omega)} \le |v|_{H^s(\Omega)} \le C_2|v|_{J_L^s(\Omega)}. \tag{2.2}$$

The proof as in Lemma 2.3 can be applied to derive the decomposition for the right fractional derivative space $J_R^s(\Omega)$.

LEMMA 2.4. *Let $0 \le s \le 1$. Then*

$$J_R^s(\Omega) = \begin{cases} H^s(\Omega), & \text{for } \ 0 \le s < \frac{1}{2}, \\ H_R^s(\Omega) \oplus \text{span}\{(1-x)^{s-1}\}, & \text{for } \ \frac{1}{2} < s \le 1. \end{cases}$$

Further, for any $v \in H_R^s(\Omega)$, there exist positive constants C_1 and C_2 such that

$$C_1|v|_{J_R^s(\Omega)} \le |v|_{H^s(\Omega)} \le C_2|v|_{J_R^s(\Omega)}. \tag{2.3}$$

To study the two-side fractional diffusion equation (1.1), we define the corresponding two-side fractional derivative spaces as follows.

DEFINITION 2.5. *[11] For $0 < \theta < 1$, define the two-side fractional derivative spaces $J_\theta^s(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ for $0 < s < \frac{1}{2}$, and the closure of $C^\infty(\Omega)$ for $\frac{1}{2} < s < 1$, under the norm*

$$\|\cdot\|_{J_\theta^s(\Omega)} = \{\|\cdot\|^2 + |\cdot|^2_{J_\theta^s(\Omega)}\}^{\frac{1}{2}}, \tag{2.4}$$

with semi-norm

$$|\cdot|_{J_\theta^s(\Omega)} = \{\theta^2\|{}_0D_x^s \cdot\|^2 + (1-\theta)^2\|{}_xD_1^s \cdot\|^2\}^{\frac{1}{2}}. \tag{2.5}$$

REMARK 2.1. *For $s = 0$ and 1, the fractional derivative in this definition reduces to an identity operator and first-order derivative [30] respectively, and thus the space $J_\theta^s(\Omega)$ reduces to $L^2(\Omega)$ and $H^1(\Omega)$, respectively, and the corresponding CFDE simplifies to the classical first-order and second-order differential equation, which have been extensively studied. In addition, for $\theta = 0$ and 1, (1.1) simplifies to the left-side* *or right-side fractional differential equation, respectively, which have been studied in [6, 16, 39]. Therefore, we focus only on the case* $0 < s < 1$ *and* $0 < \theta < 1$*, or equivalently,* $0 < \beta < 1$ *and* $0 < \theta < 1$ *in the CFDE* (1.1).

Next, we present the relationship between the fractional derivative space $J_\theta^s(\Omega)$ and the fractional Sobolev space $H^s(\Omega)$. Analogous to Lemma 2.3 and Lemma 2.4 we have the following theorem.

THEOREM 2.6. *Assume* $0 < \theta < 1$*,* $0 < s < 1$ *and* $s \neq \frac{1}{2}$*. Then* $J_\theta^s(\Omega)$ *is equivalent to* $H_0^s(\Omega)$ *and with equivalent semi-norm and norm.*

Proof. As $0 < s < \frac{1}{2}$, we directly obtain the equivalence between the fractional derivative spaces $J_\theta^s(\Omega)$ and the fractional Sobolev space $H_0^s(\Omega)$ by combining the the definition of the two-side fractional derivative spaces $J_\theta^s(\Omega)$, Lemma 2.3 and Lemma 2.4.

For $\frac{1}{2} < s < 1$ and $v \in J_\theta^s(\Omega)$, the definition of $J_\theta^s(\Omega)$ implies that $v \in J_L^s(\Omega)$ and $v \in J_R^s(\Omega)$. Then, for $0 < \theta < 1$, v can be decomposed as, by Lemma 2.3 and Lemma 2.4,

$$
\begin{aligned}
&v = v_1 + c_v^1 x^{s-1},\\
&v = v_2 + c_v^2 (1-x)^{s-1},
\end{aligned}
$$

where $v_1 \in H_L^s(\Omega)$, $v_2 \in H_R^s(\Omega)$ and c_v^1, c_v^2 are constants. Subtracting these two equalities, we have

$$
v_1 - v_2 = c_v^2 (1-x)^{s-1} - c_v^1 x^{s-1}.
$$

Sobolev imbedding theorem $H^s(\Omega) \hookrightarrow C^0(\overline{\Omega})$, $s > \frac{1}{2}$, implies that $v_1 - v_2$ is continuous over $\overline{\Omega}$, which forces $c_v^1 = c_v^2 = 0$ and thus $v_1 = v_2 = v$. It follows that $v \in H_L^s(\Omega) \cap H_R^s(\Omega) = H_0^s(\Omega)$.

Conversely, for any $v \in H_0^s(\Omega)$, then $v \in H_L^s(\Omega)$ and $v \in H_R^s(\Omega)$. Applying Lemma 2.3 and Lemma 2.4 again, we obtain that $v \in J_\theta^s(\Omega)$. This completes the proof. ◻

As a straightforward result from Theorem 2.6, we have the following corollary concerning the fractional operators.

COROLLARY 2.7. *Assume* $0 < \theta < 1$*,* $0 < s < 1$ *and* $s \neq \frac{1}{2}$*. Then, the following properties hold, for any* $v \in J_\theta^s(\Omega)$

$$
\begin{aligned}
&{}_0I_x^s\, {}_0D_x^s v = {}_0D_x^s\, {}_0I_x^s v = v,\\
&{}_xI_1^s\, {}_xD_1^s v = {}_xD_1^s\, {}_xI_1^s v = v.
\end{aligned} \tag{2.6}
$$

3. The properties of two-side fractional derivative operator $\mathcal{D}_\theta^{1-\beta}$. In this section, we shall discuss the mathematical properties of the two-side fractional differential operator $\mathcal{D}_\theta^{1-\beta}$ defined by, for $0 < \beta < 1$ and $0 < \theta < 1$,

$$
\mathcal{D}_\theta^{1-\beta} w := \theta\, {}_0D_x^{1-\beta} w - (1-\theta)\, {}_xD_1^{1-\beta} w, \tag{3.1}
$$

which are closely related to the solvability of the fractional equation

$$
\mathcal{D}_\theta^{1-\beta} w = g \quad \text{in } L^2(\Omega). \tag{3.2}
$$

For the kernel space of the operator $\mathcal{D}_\theta^{1-\beta}$, we have the following characterization (from (4.14) and (4.15) of Lemma 4.3 in [10]).

Lemma 3.1. *([10])* $\mathrm{Ker}(\mathcal{D}_\theta^{1-\beta})$, *the kernel space of the operator* $\mathcal{D}_\theta^{1-\beta}$, *is given by*

$$\mathrm{Ker}(\mathcal{D}_\theta^{1-\beta}) = \mathrm{span}\{\kappa(x)\}, \tag{3.3}$$

where $\kappa(x) := x^{r_1}(1-x)^{r_2}$, r_1, r_2 *and* β *satisfy*

$$\begin{array}{ll} (a) & r_1 + r_2 = -\beta, \quad \text{with } -\beta \le r_1,\ r_2 \le 0, \\ (b) & \theta \sin(\pi(-r_2)) = (1-\theta)\sin(\pi(-r_1)). \end{array} \tag{3.4}$$

Remark 3.1. *Although* $\mathcal{D}_\theta^{1-\beta}\kappa(x) = 0$ *for* $0 < \theta < 1$, $\kappa(x) \notin J_\theta^{1-\beta}(\Omega)$ *since*

$$ {}_0D_x^{1-\beta}\kappa(x) = \frac{\Gamma(r_1+1)}{\Gamma(-r_2)} x^{-r_2-1}(1-x)^{-r_1-1} \notin L^2(\Omega).$$

3.1. Properties of $\mathcal{D}_\theta^{1-\beta}$ and Solution Structure of (3.2). In this subsection, we shall decompose the admissible space of (3.2) as a sum of $H_0^{1-\beta}(\Omega)$ and the kernel-spanned space, and prove that $\mathcal{D}_\theta^{1-\beta}$ is a bijective mapping from its admissible space to $L^2(\Omega)$. Thus, the solution structure can be clarified.

For this purpose, we begin by showing the boundness of the operator $\mathcal{D}_\theta^{1-\beta}$ over the space $J_\theta^{1-\beta}(\Omega)$ or $H_0^{1-\beta}(\Omega)$ in the following two lemmas.

Lemma 3.2. *Assume* $0 < \theta < 1$, $0 < \beta < \frac{1}{2}$. *For any* $v \in J_\theta^{1-\beta}(\Omega)$ *(or* $H_0^{1-\beta}(\Omega)$*), there exists constants* $M > 0$ *such that*

$$\|\mathcal{D}_\theta^{1-\beta} v\|^2 \ge M\|v\|^2_{J_\theta^{1-\beta}(\Omega)} \qquad (\textit{or } \|\mathcal{D}_\theta^{1-\beta} v\|^2 \ge M\|v\|^2_{H_0^{1-\beta}(\Omega)}). \tag{3.5}$$

Proof. It suffices to show the first inequality (3.5) since $H_0^{1-\beta}(\Omega)$ is equivalent to $J_\theta^{1-\beta}(\Omega)$.

Since

$$\begin{aligned} &\|\mathcal{D}_\theta^{1-\beta} v\|^2 = \|\theta {}_0D_x^{1-\beta} v - (1-\theta) {}_xD_1^{1-\beta} v\|^2 \\ =\ & \theta^2 \|{}_0D_x^{1-\beta} v\|^2 + (1-\theta)^2\|{}_xD_1^{1-\beta} v\|^2 - 2\theta(1-\theta)({}_0D_x^{1-\beta} v, {}_xD_1^{1-\beta} v) \\ =\ & |v|^2_{J_\theta^{1-\beta}(\Omega)} - 2\theta(1-\theta)\cos(\pi(1-\beta))\|{}_{-\infty}D_x^{1-\beta}\widetilde{v}\|^2_{L^2(\mathbb{R})}, \end{aligned} \tag{3.6}$$

where

$$({}_0D_x^{1-\beta} v, {}_xD_1^{1-\beta} v) = ({}_{-\infty}D_x^{1-\beta}\tilde{v}, {}_xD_{+\infty}^{1-\beta}\tilde{v})_{L^2(\mathbb{R})} = \cos(\pi(1-\beta))\|{}_{-\infty}D_x^{1-\beta}\tilde{v}\|_{L^2(\mathbb{R})},$$

is used and $\tilde{v}$ is the extension of v by zeros outside of Ω (See Lemma 2.4, [11]).

Since $0 < \beta < \frac{1}{2}$, $\cos(\pi(1-\beta)) \le 0$, so

$$\|\theta {}_0D_x^{1-\beta} v - (1-\theta) {}_xD_1^{1-\beta} v\|^2 \ge |v|^2_{J_\theta^{1-\beta}(\Omega)}, \tag{3.7}$$

which implies the first inequality (3.5). □

Lemma 3.3. *Let* $0 < \theta < 1$, $0 < \beta < \frac{1}{2}$. *The operator* $\mathcal{D}_\theta^{1-\beta}$ *is a bounded linear operator from* $H_0^{1-\beta}(\Omega)$ *onto its range* $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$. *Further, the range is a closed subspace of* $L^2(\Omega)$ *and the operator* $\mathcal{D}_\theta^{1-\beta}$ *has a bounded inverse on the range.*

Proof. We begin the proof by applying Theorem 2.6 and the inequality

$$\|\mathcal{D}_\theta^{1-\beta}\phi\| \le \theta\|_0D_x^{1-\beta}\phi\| + (1-\theta)\|_xD_1^{1-\beta}\phi\| \le C\|\phi\|_{H_0^{1-\beta}(\Omega)} \tag{3.8}$$

to conclude that $\mathcal{D}_\theta^{1-\beta}$ is a well-defined and bounded linear operator from $H_0^{1-\beta}(\Omega)$ into $L^2(\Omega)$, and thus onto $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$.

The remaining is to prove the closeness and invertibility, which is split into the following three steps.

I. The operator $\mathcal{D}_\theta^{1-\beta}$ is one-to-one from $H_0^{1-\beta}(\Omega)$ to $L^2(\Omega)$. In fact, for $g \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$, if there exist $\varphi_1, \varphi_2 \in H_0^{1-\beta}(\Omega)$ satisfying

$$\mathcal{D}_\theta^{1-\beta}\varphi_1 = g, \qquad \mathcal{D}_\theta^{1-\beta}\varphi_2 = g,$$

then,

$$\mathcal{D}_\theta^{1-\beta}(\varphi_1 - \varphi_2) = 0,$$

which implies that $\varphi_1 - \varphi_2$ is a kernel function and also lies in $H_0^{1-\beta}(\Omega)$. This forces $\varphi_1 - \varphi_2 = 0$ by Remark 3.1, that is, $\varphi_1 = \varphi_2$. Then, $\mathcal{D}_\theta^{1-\beta}$ is a one-to-one mapping from $H_0^{1-\beta}(\Omega)$ to $L^2(\Omega)$.

II. The range $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$ is a closed subspace of $L^2(\Omega)$. It suffices to prove the range's closeness since one can easily verify that $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$ is a subspace in $L^2(\Omega)$.

Let $\{g_n\}_{n=1}^\infty \subset \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$ be a sequence that converges to $g \in L^2(\Omega)$. By the definition of $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$, there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset H_0^{1-\beta}(\Omega)$ such that $\mathcal{D}_\theta^{1-\beta}\varphi_n = g_n$. Using (3.5), we have

$$\frac{1}{\sqrt{M}}\|\varphi_n - \varphi_m\|_{H_0^{1-\beta}(\Omega)} \le \|\mathcal{D}_\theta^{1-\beta}\varphi_n - \mathcal{D}_\theta^{1-\beta}\varphi_m\| = \|g_n - g_m\|,$$

which implies that $\{\varphi_n\}_{n=1}^\infty$ is a Cauchy sequence in $H_0^{1-\beta}(\Omega)$, and thus φ_n converges to a $\varphi \in H_0^{1-\beta}(\Omega)$. Further, as n tends to ∞

$$\begin{aligned} \|\mathcal{D}_\theta^{1-\beta}\varphi - g\| &\le \|\mathcal{D}_\theta^{1-\beta}\varphi - \mathcal{D}_\theta^{1-\beta}\varphi_n\| + \|\mathcal{D}_\theta^{1-\beta}\varphi_n - g\| \\ &\le \|\varphi - \varphi_n\|_{H_0^{1-\beta}(\Omega)} + \|g_n - g\| \to 0, \end{aligned}$$

which implies $\mathcal{D}_\theta^{1-\beta}\varphi = g$ and $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$ is a closed subspace of $L^2(\Omega)$.

III. $\mathcal{D}_\theta^{1-\beta}$ has a bounded inverse on the range. The invertibility of the operator $\mathcal{D}_\theta^{1-\beta}$ follows immediately from the operator being one-to-one and onto from $H_0^{1-\beta}(\Omega)$ to $\mathcal{R}(\mathcal{D}_\theta^{1-\beta})$, given in Step I and at the very beginning of the proof. Then, using the definition of the operator norm and (3.5), we derive the boundness of its inverse,

$$\begin{aligned} \|(\mathcal{D}_\theta^{1-\beta})^{-1}\| &= \sup_{g\in\mathcal{R}(\mathcal{D}_\theta^{1-\beta})} \frac{\|(\mathcal{D}_\theta^{1-\beta})^{-1}g\|_{H_0^{1-\beta}(\Omega)}}{\|g\|} \le \sup_{g\in\mathcal{R}(\mathcal{D}_\theta^{1-\beta})} \frac{\frac{1}{\sqrt{M}}\|\mathcal{D}_\theta^{1-\beta}(\mathcal{D}_\theta^{1-\beta})^{-1}g\|}{\|g\|} \\ &\le \sup_{g\in\mathcal{R}(\mathcal{D}_\theta^{1-\beta})} \frac{\frac{1}{\sqrt{M}}\|g\|}{\|g\|} = \frac{1}{\sqrt{M}}. \end{aligned}$$

Combining the conclusions of I, II and III, we complete the proof. □

Checking the proof of Lemma 3.3, we notice that $\mathcal{D}_\theta^{1-\beta} : H_0^{1-\beta}(\Omega) \to L^2(\Omega)$ is not surjective, since, for $x\kappa(x) \notin H_0^{1-\beta}(\Omega)$, $0 < \beta < \frac{1}{2}$

$$\mathcal{D}_\theta^{1-\beta}(x\kappa(x)) = -C_*^*\Gamma(2-\beta) := C_* \in L^2(\Omega) \quad \text{with } C_*^* = \frac{\sin(\pi\beta)}{\sin(\pi r_1)+\sin(\pi r_2)}. \tag{3.9}$$

This shows that the admissible space for (3.2) is larger than $H_0^{1-\beta}(\Omega)$ and should contain, at least, the function $x\kappa(x)$. Fortunately, when $x\kappa(x)$ is added, the operator $\mathcal{D}_\theta^{1-\beta}$ becomes a bijective mapping from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ to $L^2(\Omega)$. This is the following theorem.

THEOREM 3.4. *Let $0 \le \beta < \frac{1}{2}, 0 < \theta < 1$. Then, the operator $\mathcal{D}_\theta^{1-\beta}$ is bijective from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ to $L^2(\Omega)$.*

Proof. Applying Theorem 2.6 and (3.9), we conclude that $\mathcal{D}_\theta^{1-\beta}$ is well-defined on $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$. We next prove the operator is injective and also surjective mapping, respectively.

The operator $\mathcal{D}_\theta^{1-\beta}$ is an injective mapping from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ into $L^2(\Omega)$. In fact, for a given $g \in L^2(\Omega)$, suppose that there exist $\varphi_1, \varphi_2 \in H_0^{1-\beta}(\Omega)$ and constants C_1, C_2 such that

$$\mathcal{D}_\theta^{1-\beta}(\varphi_1 + C_1 x\kappa(x)) = g, \qquad \mathcal{D}_\theta^{1-\beta}(\varphi_2 + C_2 x\kappa(x)) = g.$$

Then, by applying (3.9), we get

$$\mathcal{D}_\theta^{1-\beta}(\varphi_1 - \varphi_2) = (C_2 - C_1)C_*.$$

By Lemma 2.1 in [5], we obtain

$$\varphi_1 - \varphi_2 = (C_2 - C_1)x\kappa(x) + C_3\kappa(x),$$

where C_3 is an arbitrary constant. Noting that $\varphi_1 - \varphi_2 \in H_0^{1-\beta}(\Omega)$ and $\kappa(x), x\kappa(x) \notin H_0^{1-\beta}(\Omega)$, we obtain $\varphi_1 = \varphi_2, C_2 = C_1$, which implies that the operator $\mathcal{D}_\theta^{1-\beta}$ is an injective mapping from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ into $L^2(\Omega)$.

The operator $\mathcal{D}_\theta^{1-\beta}$ is a bijective mapping from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ onto $L^2(\Omega)$. Let $\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta})$ be the range of the operator $\mathcal{D}_\theta^{1-\beta}$ to $H_0^{1-\beta}(\Omega)\oplus\operatorname{span}\{x\kappa(x)\}$. Obviously, $\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta}) \subset L^2(\Omega)$.

Now we are to show that $\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta}) = L^2(\Omega)$. It suffices to show that if $g \in (\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta}))^\perp$, the orthogonal complement of $\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta})$ in $L^2(\Omega)$, then $g = 0$. In fact, for any $\varphi \in H_0^{1-\beta}(\Omega)$ and any constant C,

$$(g, \mathcal{D}_\theta^{1-\beta}(\varphi + Cx\kappa(x))) = 0,$$

which implies

$$(g, \mathcal{D}_\theta^{1-\beta}\varphi) + CC_*(g, 1) = 0,$$

by applying (3.9). Taking $\varphi = 0$ and $C = 0$, respectively, in above formula, we obtain

$$(g, 1) = 0, \tag{3.10}$$

and

$$(g, \mathcal{D}_\theta^{1-\beta}\varphi) = 0, \quad \forall \varphi \in H_0^{1-\beta}(\Omega). \tag{3.11}$$

Let $\varphi_n = x(1-x)\kappa(x)x^n, n = 0, 1, 2, \cdots$. Then, by simple calculation, we get $\varphi_n \in H_0^{1-\beta}(\Omega)$ and

$$\mathcal{D}_\theta^{1-\beta}\varphi_n = \sum_{k=0}^{n+1} b_{n,k}x^k := p_{n+1}(x), \tag{3.12}$$

where

$$b_{n,k} = (-1)^n C_*^* \Gamma(r_2+2) \frac{(-1)^{k+1}\Gamma(2-\beta+k)}{\Gamma(r_2-n+k+1)\Gamma(n-k+2)\Gamma(k+1)}. \tag{3.13}$$

Taking $\varphi = \varphi_n$ in (3.11) and applying (3.10), we obtain

$$(g, p_n(x)) = 0, \quad n = 0, 1, 2, \cdots, \tag{3.14}$$

where $p_0(x) = 1$. The separability of $L^2(\Omega)$ implies that $\{p_n(x)\}_0^\infty$ is dense in $L^2(\Omega)$, and then, any $g \in L^2(\Omega)$ can be expressed as $g(x) = \sum_{k=0}^{\infty} C_k p_k(x)$. Applying (3.14), we obtain

$$(g, g) = (g, \sum_{k=0}^{\infty} C_k p_k(x)) = \sum_{k=0}^{\infty} C_k (g, p_k(x)) = 0,$$

which implies $g = 0$.

This shows that $(\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta}))^\perp = \{0\}$ and $\widetilde{\mathcal{R}}(\mathcal{D}_\theta^{1-\beta}) = L^2(\Omega)$.

Combined the above two conclusions, we obtain $\mathcal{D}_\theta^{1-\beta}$ is bijective from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ onto $L^2(\Omega)$. □

REMARK 3.2. *Although the operator $\mathcal{D}_\theta^{1-\beta}$ is bijective from $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ to $L^2(\Omega)$, we can not select $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{x\kappa(x)\}$ as the admissible space of solution to Equation (3.2) since the kernel function $\kappa(x)$ multiplied by any constants may also be its solution. Therefore, the admissible space of solution to Equation (3.2) should be $H_0^{1-\beta}(\Omega) \oplus \operatorname{span}\{\kappa(x), x\kappa(x)\}$.*

Consequently, its solution can be decomposed as

$$w = \widetilde{w} + C_{\kappa 1}x\kappa(x) + C_\kappa \kappa(x), \tag{3.15}$$

and Equation (3.2) is reduced to

$$\mathcal{D}_\theta^{1-\beta}\widetilde{w} = \widetilde{g} := g - C_{\kappa 1}C_*. \tag{3.16}$$

where $\widetilde{w} \in H_0^{1-\beta}(\Omega)$ can be solved by the variational formulation of (3.16) when the coefficients $C_{\kappa 1}$ and C_κ are known. In fact, C_κ can be determined by the prescribed boundary conditions (see Section 4), and $C_{\kappa 1}$ is expressed by the following L^2-orthogonal decomposition theorem.

3.2. Decomposition of $L^2(\Omega)$. In this subsection, we shall give a decomposition of space $L^2(\Omega)$, and then, the coefficient $C_{\kappa 1}$ can be determined.

THEOREM 3.5. *Let $0 < \theta < 1$ and $0 < \beta < \frac{1}{2}$. Then, $L^2(\Omega)$ can be orthogonally decomposed as*

$$L^2(\Omega) = \mathcal{R}(\mathcal{D}_\theta^{1-\beta}) \oplus \operatorname{span}\{1 - r(x)\}, \tag{3.17}$$

where $r(x) = \mathcal{D}_\theta^{1-\beta}\widetilde{v}_2$ *and* $\widetilde{v}_2 \in H_0^{1-\beta}(\Omega)$ *is decided by*

$$(\mathcal{D}_\theta^{1-\beta}\widetilde{v}_2, \mathcal{D}_\theta^{1-\beta}v) = (1, \mathcal{D}_\theta^{1-\beta}v), \qquad \forall v \in H_0^{1-\beta}(\Omega). \tag{3.18}$$

Proof. Noting (3.9) and applying Theorem 3.4, we obtain the direct sum decomposition,

$$L^2(\Omega) = \mathcal{R}(\mathcal{D}_\theta^{1-\beta}) \oplus \mathrm{span}\{1\}. \tag{3.19}$$

However, this decomposition is not orthogonal since $(1, \mathcal{D}_\theta^{1-\beta}v) \neq 0$ for some $v \in H_0^{1-\beta}(\Omega)$.

By the definition of $r(x)$, we have,

$$(1 - r, \mathcal{D}_\theta^{1-\beta}v) = 0, \qquad \forall\ v \in H_0^{1-\beta}(\Omega),$$

which indicates that the decomposition (3.17) is orthogonal.

We next prove that

$$L^2(\Omega) = \mathcal{R}(\mathcal{D}_\theta^{1-\beta}) \oplus \mathrm{span}\{1 - r\}.$$

For any $g(x) \in L^2(\Omega)$, there exist a function $\varphi_g \in H_0^{1-\beta}(\Omega)$ and a constant C_g, by Theorem 3.4, such that

$$\begin{aligned} g &= \mathcal{D}_\theta^{1-\beta}\varphi_g + C_g \\ &= \mathcal{D}_\theta^{1-\beta}\varphi_g + C_g r(x) + C_g(1 - r(x)) \\ &= \mathcal{D}_\theta^{1-\beta}(\varphi_g + C_g\widetilde{v}_2) + C_g(1 - r(x)) \\ &\in \mathcal{R}(\mathcal{D}_\theta^{1-\beta}) \oplus \mathrm{span}\{1 - r(x)\}, \end{aligned} \tag{3.20}$$

and

$$(\mathcal{D}_\theta^{1-\beta}(\varphi_g + C_g\widetilde{v}_2), 1 - r(x)) = 0,$$

by noting $\varphi_g + C_g\widetilde{v}_2 \in H_0^{1-\beta}(\Omega)$. This implies that the decomposition is L^2-orthogonal decomposition, and thus completes the proof. □

Remark 3.3. *Based on the orthogonal decomposition, we can determine the constant* $C_{\kappa 1}$ *for given* $g \in L^2(\Omega)$ *as follows:*

$$C_{\kappa 1} = \frac{(g, 1 - r)}{C_*(1, 1 - r)}, \tag{3.21}$$

where C_* *see (3.9) and* $r = \mathcal{D}_\theta^{1-\beta}\widetilde{v}_2$ *with* $\widetilde{v}_2 \in H_0^{1-\beta}(\Omega)$ *is decided by (3.18). In fact, by applying (3.20), we can get*

$$(g - C_g(1 - r), 1 - r) = 0,$$

which and $(r, 1 - r) = 0$ *(since* $r \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$*) indicate that*

$$C_g = \frac{(g, 1 - r)}{(1, 1 - r)}, \qquad \text{and} \qquad g - C_g \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta}). \tag{3.22}$$

The fact that $\widetilde{g} \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$ *in (3.16) forces* $C_* C_{\kappa 1} = C_g$*, which and above formula imply (3.21).*

4. Variational Formulation for (1.1) Based on Least Squares. In this section we shall establish a least squares mixed Galerkin variational formulation independent of the kernel function, prove its solvability over $H_L^1(\Omega) \times H_0^{1-\beta}(\Omega)$ and show its equivalence to (1.1).

Let $u = -KDp$. Then, the CFDE (1.1) is split as

$$
\begin{aligned}
&(a) \quad -KDp = u,\\
&(b) \quad \mathcal{D}_\theta^{1-\beta} u = f,\\
&(c) \quad p(0) = p(1) = 0.
\end{aligned}
\tag{4.1}
$$

From the analysis in §3, we known that the admissible space of solution u to the second equation of (4.1) is $H_0^{1-\beta}(\Omega) \oplus \mathrm{span}\{\kappa(x), x\kappa(x)\}$, and then, the solution u can be expressed as

$$
u = \widetilde{u} + C_{\kappa 1} x\kappa(x) + C_\kappa \kappa(x),
$$

where $\widetilde{u} \in H_0^{1-\beta}(\Omega)$ and $C_{\kappa 1}$ and C_κ be undetermined constants. However, $x\kappa(x)$ and $\kappa(x)$ may result in a singularity at $x = 0$ and 1, and may heavily influence the convergence rate of the corresponding numerical procedure if the variational formulation is formed based upon (4.1) directly.

Let

$$
\widetilde{u} = u - u_{s1} - u_s, \quad \widetilde{p} = p - p_{s1} - p_s,
\tag{4.2}
$$

where

$$
\begin{aligned}
u_{s1} &= C_{\kappa 1} x\kappa(x), & p_{s1} &= -\int_0^x K^{-1} u_{s1} d\xi,\\
u_s &= C_\kappa \kappa(x), & p_s &= -\int_0^x K^{-1} u_s d\xi,
\end{aligned}
\tag{4.3}
$$

Equation (4.1) can be rewritten as

$$
\begin{aligned}
&-KD\widetilde{p} = \widetilde{u},\\
&\mathcal{D}_\theta^{1-\beta} \widetilde{u} = \widetilde{f},\\
&\widetilde{p}(0) = 0,
\end{aligned}
\tag{4.4}
$$

with $\widetilde{f} = f - C_{\kappa 1} C_* \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$.

The undetermined constant $C_{\kappa 1}$ can be derived by (3.3) and the analysis of §3

$$
C_{\kappa 1} = \frac{(f, 1-r)}{C_*(1, 1-r)},
\tag{4.5}
$$

where C_* see (3.9) and $r = \mathcal{D}_\theta^{1-\beta} \widetilde{v}_2$ with $\widetilde{v}_2 \in H_0^{1-\beta}(\Omega)$ is decided by (3.18). The other undetermined constant C_κ can be determined by the second boundary condition $p(1) = 0$ and $\widetilde{p}$, or specifically by the following formula

$$
\widetilde{p}(1) - \int_0^1 K^{-1}(u_{s1} + u_s) d\xi = 0,
$$

i.e.,

$$
(4.6)\qquad \begin{aligned} C_\kappa &= \frac{\widetilde{p}(1) - C_{\kappa 1}\int_0^1 K^{-1} x\kappa(x)dx}{\int_0^1 K^{-1}\kappa(x)dx} \\ &= \frac{-\int_0^1 K^{-1}\widetilde{u}dx - C_{\kappa 1}\int_0^1 K^{-1} x\kappa(x)dx}{\int_0^1 K^{-1}\kappa(x)dx}. \end{aligned}
$$

As a conclusion, solving (4.1) is equivalent to solving (4.4) over the admissible space $H_L^1(\Omega)\times H_0^{1-\beta}(\Omega)$ with $\widetilde{p} = p - p_{s_1} - p_s$ and $\widetilde{u} = u - u_{s_1} - u_s$. This allows us to discuss the mathematical and numerical issues of (4.1) (or (1.1)) through (4.4) and (p_{s_1}, u_{s_1}), (p_s, u_s)−term in the coming sections.

4.1. Variational formulation. Define the functional

$$
(4.7)\qquad \mathcal{L}([q,v]) = \tfrac{1}{2}\{\textstyle\int_\Omega |KDq + v|^2 dx + \int_\Omega |\mathcal{D}_\theta^{1-\beta} v - \widetilde{f}|^2 dx\},
$$

which is to be minimized over $H_L^1(\Omega)$ and $H_0^{1-\beta}(\Omega)$. Define the bilinear form $\mathcal{B}([\cdot,\cdot];[\cdot,\cdot]) : (H_L^1(\Omega)\times H_0^{1-\beta}(\Omega))\times(H_L^1(\Omega)\times H_0^{1-\beta}(\Omega)) \mapsto \mathbb{R}$ as

$$
(4.8)\qquad \mathcal{B}([\widetilde{p},\widetilde{u}];[q,v]) = (KD\widetilde{p} + \widetilde{u}, KDq + v) + (\mathcal{D}_\theta^{1-\beta}\widetilde{u}, \mathcal{D}_\theta^{1-\beta} v),
$$

and the linear functional $F([\cdot,\cdot]) : H_L^1(\Omega)\times H_0^{1-\beta}(\Omega) \mapsto \mathbb{R}$ as

$$
(4.9)\qquad F([q,v]) = \int_\Omega \widetilde{f}\mathcal{D}_\theta^{1-\beta} v dx.
$$

The minimization leads to the following variational formulation: find $[\widetilde{p},\widetilde{u}] \in H_L^1(\Omega)\times H_0^{1-\beta}(\Omega)$ such that

$$
(4.10)\qquad \mathcal{B}([\widetilde{p},\widetilde{u}];[q,v]) = F([q,v]), \quad [q,v] \in H_L^1(\Omega)\times H_0^{1-\beta}(\Omega).
$$

THEOREM 4.1. *The minimizing problem* $\mathcal{L}([\widetilde{p},\widetilde{u}]) = \min\limits_{[q,v]\in H_L^1(\Omega)\times H_0^{1-\beta}(\Omega)} \mathcal{L}([q,v])$ *is equivalent to the variational formulation (4.10).*

Proof. For any $s \geq 0$, we define a quadratic function

$$
\Phi(s) = \mathcal{L}([\widetilde{p} + sq, \widetilde{u} + sv]).
$$

Noting that

$$
\begin{aligned}
\Phi(s) &= \mathcal{L}([\widetilde{p} + sq, \widetilde{u} + sv]) \\
&= \tfrac{1}{2}\{\|KD(\widetilde{p} + sq) + (\widetilde{u} + sv)\|^2 + \|\mathcal{D}_\theta^{1-\beta}(\widetilde{u} + sv) - \widetilde{f}\|^2\} \\
&= \tfrac{1}{2}\{(KD\widetilde{p} + \widetilde{u}, KD\widetilde{p} + \widetilde{u}) + (\mathcal{D}_\theta^{1-\beta}\widetilde{u} - \widetilde{f}, \{\mathcal{D}_\theta^{1-\beta}\widetilde{u} - \widetilde{f})\} \\
&\quad + s\{(KD\widetilde{p} + \widetilde{u}, KDq + v) + (\mathcal{D}_\theta^{1-\beta}\widetilde{u} - \widetilde{f}, \mathcal{D}_\theta^{1-\beta} v)\} \\
&\quad + \tfrac{s^2}{2}\{(KDq + v, KDq + v) + (\mathcal{D}_\theta^{1-\beta} v, \mathcal{D}_\theta^{1-\beta} v)\}.
\end{aligned}
$$

Obviously, if $[\widetilde{p},\widetilde{u}]$ minimizes the functional $\mathcal{L}$, then $\Phi'(0) = (KD\widetilde{p} + \widetilde{u}, KDq + v) + (\mathcal{D}_\theta^{1-\beta}\widetilde{u} - \widetilde{f}, \mathcal{D}_\theta^{1-\beta} v) = 0$. This shows that $[\widetilde{p},\widetilde{u}]$ solves the variational formulation (4.10).

Conversely, if $[\widetilde{p}, \widetilde{u}]$ solves the variational formulation (4.10), then for any $s > 0$,

$$\Phi(s) = \Phi(0) + \frac{s^2}{2}\{\|KDq + v\|^2 + \|\mathcal{D}_\theta^{1-\beta} v\|^2\} > \Phi(0),$$

which implies $\Phi(0) = \min_{s\geq 0} \Phi(s)$, and so $[\widetilde{p}, \widetilde{u}]$ minimizes the functional $\mathcal{L}$. □

With the help of Lemma 3.2 we can prove the solvability of the variational formulation.

THEOREM 4.2. *For $\beta \neq \frac{1}{2}$, there exists a unique solution $[\widetilde{p}, \widetilde{u}] \in H_L^1(\Omega) \times H_0^{1-\beta}(\Omega)$ to variational formulation (4.10).*

Proof. For the coerciveness, we first note that from the definition of the bilinear form $\mathcal{B}([\cdot,\cdot];[\cdot,\cdot])$ and Lemma 3.2,

$$\begin{aligned} &\mathcal{B}([q,v];[q,v]) \geq (KDq + v, KDq + v), \\ &\mathcal{B}([q,v];[q,v]) \geq \|\mathcal{D}_\theta^{1-\beta} v\|^2 \geq M\|v\|^2_{J_\theta^{1-\beta}(\Omega)}, \end{aligned} \tag{4.11}$$

which leads to

$$\begin{aligned} &(1 + \tfrac{1}{M})\mathcal{B}([q,v];[q,v]) \\ &\quad\geq (KDq + v, KDq + v)\| + \|v\|^2_{J_\theta^{1-\beta}} \\ &\quad\geq (KDq, KDq) + 2(KDq, v) + 2\|v\|^2 \\ &\quad\geq (KDq, KDq) - 2\|KDq\|\|v\| + 2\|v\|^2 \\ &\quad= \tfrac{1}{2}(KDq, KDq) + \{\tfrac{1}{2}(KDq, KDq) - 2\|KDq\|\|v\| + 2\|v\|^2\} \\ &\quad= \tfrac{1}{2}(KDq, KDq) + \{\tfrac{1}{\sqrt{2}}\|KDq\| - \sqrt{2}\|v\|\}^2 \\ &\quad\geq \tfrac{1}{2}\|KDq\|^2. \end{aligned}$$

Applying the equivalence of the semi-norm and norm over $H_L^1(\Omega)$ and Theorem 2.6, we obtain, with constant $M_0 > 0$, the final coercivity of the bilinear form

$$\begin{aligned} 2\mathcal{B}([q,v];[q,v]) &\geq \tfrac{M}{2(1+M)}\|KDq\|^2 + M^2\|v\|^2_{J_\theta^{1-\beta}} \\ &\geq M_0\{\|q\|_1^2 + \|v\|^2_{H^{1-\beta}(\Omega)}\}. \end{aligned} \tag{4.12}$$

The continuity of the bilinear form follows from an application of Cauchy-Schwarz inequality and the triangle inequality

$$\begin{aligned} &\mathcal{B}([\widetilde{p},\widetilde{u}];[q,v]) = (KD\widetilde{p} + \widetilde{u}, KDq + v) + (\mathcal{D}_\theta^{1-\beta}\widetilde{u}, \mathcal{D}_\theta^{1-\beta} v) \\ \leq\ &(\|KD\widetilde{p}\| + \|\widetilde{u}\|)(\|KDq\| + \|v\|) + \|\mathcal{D}_\theta^{1-\beta}\widetilde{u}\|\|\mathcal{D}_\theta^{1-\beta} v\| \\ \leq\ &2\max\{K^*, 1\}(\|\widetilde{p}\|_1^2 + \|\widetilde{u}\|^2)^{\frac{1}{2}}(\|q\|_1^2 + \|v\|^2)^{\frac{1}{2}} \\ &+\sqrt{2}(\theta^2\|{}_0D_x^{1-\beta}\widetilde{u}\|^2 + (1-\theta)^2\|{}_xD_1^{1-\beta}\widetilde{u}\|^2)^{\frac{1}{2}} \\ &\cdot\sqrt{2}(\theta^2\|{}_0D_x^{1-\beta}v\|^2 + (1-\theta)^2\|{}_xD_1^{1-\beta}v\|^2)^{\frac{1}{2}} \\ \leq\ &2\max\{K^*, 1\}(\|\widetilde{p}\|_1^2 + \|\widetilde{u}\|^2)^{\frac{1}{2}}(\|q\|_1^2 + \|v\|)^{\frac{1}{2}} + 2\|\widetilde{u}\|_{J_\theta^{1-\beta}(\Omega)}\|v\|_{J_\theta^{1-\beta}(\Omega)} \\ \leq\ &M_2(\|\widetilde{p}\|_1^2 + \|\widetilde{u}\|^2_{H^{1-\beta}})^{\frac{1}{2}}(\|q\|_1^2 + \|v\|^2_{H^{1-\beta}(\Omega)})^{\frac{1}{2}}. \end{aligned} \tag{4.13}$$

The continuity of the linear form $F([q,v]) = (\widetilde{f}, \mathcal{D}_\theta^{1-\beta} v)$ over the space $H_L^1(\Omega) \times H_0^{1-\beta}(\Omega)$ is obvious.

Applying the Lax-Milgram lemma, we obtain the existence and uniqueness of the variational form. □

Now we shall clarify the equivalence between the variational formulation and the CFDE (1.1).

THEOREM 4.3. *Assume that* $f \in L^2(\Omega)$ *and* $0 < \beta < \frac{1}{2}$. *If* $[\widetilde{p}, \widetilde{u}]$ *solves the variational formulation (4.10), then* $p = \widetilde{p} + p_{s_1} + p_s \in H_0^1(\Omega)$ *is the solution of the FDE (1.1) and* $u = -KDp$. *Conversely, if* $p \in H_0^1(\Omega)$ *solves (1.1), then* $\widetilde{p} = p - p_{s_1} - p_s \in H_L^1(\Omega)$ *and* $\widetilde{u} = -KD\widetilde{p} \in H_0^{1-\beta}(\Omega)$ *are the solution of the variational formulation (4.10).*

Proof. The analysis at the beginning of this section and the derivation of the variational formulation presented in Theorem 4.1 and Theorem 4.2 imply the second assertion of this theorem. Therefore we only need to show that the first assertion holds.

The key idea of proof is to decompose the variational formulation (4.10) into a system for $\widetilde{p}$ and a system for $\widetilde{u}$ by carefully selecting test functions.

Taking $v = 0$ to obtain the decomposed equation for p

$$(KD\widetilde{p} + \widetilde{u}, KDq) = 0, \quad \forall q \in H_L^1(\Omega). \tag{4.14}$$

For any $w \in L^2(\Omega)$, let $q_w = \int_0^x K^{-1} w d\xi \in H_L^1(\Omega)$. Substituting it into (4.14), we derive

$$(KD\widetilde{p} + \widetilde{u}, w) = 0 \quad \forall w \in L^2(\Omega),$$

which leads to that $\widetilde{p} \in H^1(\Omega)$ satisfies

$$-KD\widetilde{p} = \widetilde{u} \text{ in } L^2(\Omega) - \text{sence}. \tag{4.15}$$

To obtain the decomposed equation for $\widetilde{u}$, we substitute (4.15) into (4.10) to derive

$$(\mathcal{D}_\theta^{1-\beta}\widetilde{u}, \mathcal{D}_\theta^{1-\beta} v) = (\widetilde{f}, \mathcal{D}_\theta^{1-\beta} v), \qquad \forall v \in H_0^{1-\beta}(\Omega). \tag{4.16}$$

This equation is viewed as another variational formulation, in which the bilinear form defined by

$$(\mathcal{D}_\theta^{1-\beta}\widetilde{u}, \mathcal{D}_\theta^{1-\beta} v) : H_0^{1-\beta}(\Omega) \times H_0^{1-\beta}(\Omega) \to \mathbb{R}$$

is continuous and coercive from Lemma 3.2, and the linear form

$$(\widetilde{f}, \mathcal{D}_\theta^{1-\beta} v) : H_0^{1-\beta}(\Omega) \to \mathbb{R}$$

is continuous. Using Lax-Milgram lemma again, we know that there exists a unique solution $\widetilde{u} \in H_0^{1-\beta}(\Omega)$ to (4.16).

Next we shall show that

$$\mathcal{D}_\theta^{1-\beta}\widetilde{u} = \widetilde{f}. \tag{4.17}$$

Nothing that $\widetilde{f} \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$, i.e., there exists a function $\rho(x) \in H_0^{1-\beta}(\Omega)$ such that $\mathcal{D}_\theta^{1-\beta}\rho(x) = \widetilde{f}$. Taking $v = \widetilde{u} - \rho$ in (4.16), we obtain $\widetilde{u} = \rho$, which implies (4.17).

Substituting (3.9), (4.3) into (4.15) and (4.17), we have

$$-KDp = u, \tag{4.18}$$

$$\mathcal{D}_\theta^{1-\beta} u = f, \tag{4.19}$$

which lead to the first equation of (1.1).

Finally, we verify the boundary condition. Applying (4.15), we solve $\widetilde{p}$ as

$$\widetilde{p} = -\int_0^x K^{-1}\widetilde{u}d\xi. \tag{4.20}$$

which, (4.3), the expression $C_{\kappa 1}$ (4.5) and the expression C_κ (4.6) lead to

$$\begin{aligned} p(0) &= 0, \\ p(1) &= \widetilde{p}(1) + p_{s1}(1) + p_s(1) \\ &= -\int_0^1 K^{-1}\widetilde{u}dx - C_{\kappa 1}\int_0^1 K^{-1}x\kappa(x)dx - C_\kappa \int_0^1 K^{-1}\kappa(x)dx = 0. \end{aligned}$$

This completes the proof. □

From Theorem 4.2 and the proof of Theorem 4.3, we obtain the following corollary immediately.

Corollary 4.4. *Under the condition of Theorem* 4.3, *variational formulation* (4.10) *is equivalent to* (4.14) *and* (4.16).

Suppose that $C_{\kappa 1} = C_\kappa = 0$ in Theorem 4.3. Then $u_{s1} = u_s = 0$ and $p_{s1} = p_s = 0$, which leads to the following corollary

Corollary 4.5. *Suppose* $0 < \beta < \frac{1}{2}$, $0 < \theta < 1$, $f \in L^2(\Omega)$ *and* $C_{\kappa 1} = C_\kappa = 0$. *If* $[\widetilde{p}, \widetilde{u}] \in H_L^1(\Omega) \times H_0^{1-\beta}(\Omega)$ *solves variational formulation* (4.10), *then* $p = \widetilde{p} \in H^{2-\beta}(\Omega) \cap H_0^1(\Omega)$ *satisfies fractional diffusion* (1.1). *Conversely, if* $p \in H^{2-\beta}(\Omega) \cap H_0^1(\Omega)$ *satisfies* (1.1), *then* $[p, u = -KDp] \in H_L^1(\Omega) \times H_0^{1-\beta}(\Omega)$ *solves* (4.10).

Remark 4.1. *Theorem 4.3 implies that for the solution of equation (1.1) (or (4.1)), we only need to solve variational formulation (4.10) and then add the $C_{\kappa 1}, C_\kappa$-term to express the solution as*

$$\begin{aligned} p &= \widetilde{p} - C_{\kappa 1}\int_0^x K^{-1}x\kappa(\xi)d\xi - C_\kappa \int_0^x K^{-1}\kappa(\xi)d\xi, \\ u &= \widetilde{u} + C_{\kappa 1}x\kappa(x) + C_\kappa \kappa(x), \end{aligned} \tag{4.21}$$

where $C_{\kappa 1}$ and C_κ are determined by (4.5) and (4.6), respectively.

4.2. Regularity. In this subsection, we shall discuss the regularity using (4.15), and (4.17), and the properties of integral operator $\mathcal{D}_\theta^{1-\beta}$ proved in Section 3.

Theorem 4.6. *Assume that $f \in L^2(\Omega)$, $0 < \beta < \frac{1}{2}$ and $0 < \theta < 1$. For $0 < \gamma < \frac{1}{2}$, the solution p of the fractional diffusion (1.1) (or (4.1)) satisfies the following regularity estimates in terms of the right-hand side f*

(*a*) *Assume $C_{\kappa 1} = C_\kappa = 0$. Then $p \in H^{2-\beta}(\Omega) \cap H_0^1(\Omega)$ and*
$\|p\|_{H^{2-\beta}(\Omega)} \le C\|u\|_{H^{1-\beta}(\Omega)} \le C\|f\|_{L^2(\Omega)}$;

(*b*) *Assume $C_{\kappa 1} \ne 0, C_\kappa = 0$. Then $p \in H^{2-\beta}(\Omega) \oplus H_0^{r_2+1+\gamma}(\Omega)$ and*
$\|p\|_{H^{r_2+1+\gamma}(\Omega)} \le C\|f\|_{L^2(\Omega)}$;

(*c*) *Assume $C_{\kappa 1}, C_\kappa \ne 0$. Then $p \in H^{2-\beta}(\Omega) \oplus H_0^{\min\{r_1,r_2\}+1+\gamma}(\Omega)$ and*
$\|p\|_{H^{\min\{r_1,r_2\}+1+\gamma}(\Omega)} \le C\|f\|_{L^2(\Omega)}$;

(*d*) *Assume $C_{\kappa 1} \ne 0,\ C_\kappa \ne 0$. Then, $\widetilde{p} \in H^{2-\beta}(\Omega) \cap H_0^1(\Omega)$ and*
$\|\widetilde{p}\|_{H^{2-\beta}(\Omega)} \le C\|\widetilde{u}\|_{H^{1-\beta}(\Omega)} \le C\|\widetilde{f}\|_{L^2(\Omega)}$,

where $\widetilde{p} = p - p_{s1} - p_s$, $\widetilde{u} = u - u_{s1} - u_s$.

Proof. Using (4.1), the solution can be rewritten as

$$\begin{array}{rcl} u &=& C_{\kappa 1} x\kappa(x) + C_\kappa \kappa(x) + (\mathcal{D}_\theta^{1-\beta})^{-1}\widetilde{f}, \\ p &=& -\int_0^x K^{-1} u d\xi. \end{array} \tag{4.22}$$

and $\widetilde{f} \in \mathcal{R}(\mathcal{D}_\theta^{1-\beta})$.

Assume $C_{\kappa 1} = C_\kappa = 0$ and $f \in L^2(\Omega)$. In this case, $\widetilde{f} = f$, and $u = (\mathcal{D}_\theta^{1-\beta})^{-1}\widetilde{f}$ by (4.22). Then, applying Theorem 3.3, we can derive the regularity of u: $u \in H_0^{1-\beta}(\Omega)$, and the regularity of p: $p \in H^{2-\beta}(\Omega) \cap H_0^1(\Omega)$. Applying the boundness of integral operator $(\mathcal{D}_\theta^{1-\beta})^{-1} : \mathcal{R}(\mathcal{D}_\theta^{1-\beta}) \subset L^2(\Omega) \to H_0^{1-\beta}(\Omega)$ and $\int_0^x : H_0^{1-\beta}(\Omega) \to H^{2-\beta}(\Omega)$, we obtain

$$\|p\|_{H^{2-\beta}(\Omega)} \le C\|u\|_{H^{1-\beta}(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Next, assume $C_{\kappa 1} \neq 0, C_\kappa = 0$. In this case, $u = C_{\kappa 1} x\kappa(x) + (\mathcal{D}_\theta^{1-\beta})^{-1}\widetilde{f}$ by (4.22). The solution u contains singular parts $x\kappa(x) \in H^{r_2+\gamma}(\Omega)$ and a regular part $(\mathcal{D}_\theta^{1-\beta})^{-1}\widetilde{f} \in H_0^{1-\beta}(\Omega)$. This implies $u \in H_0^{1-\beta}(\Omega) \oplus H^{r_2+\gamma}(\Omega)$, and thus $p \in H^{2-\beta}(\Omega) \oplus H^{r_2+1+\gamma}(\Omega)$. The appearance of singular parts $x\kappa(x)$ and $H^{2-\beta}(\Omega) \subset H^{r_2+1+\gamma}(\Omega)$, since $r_2 + 1 + \gamma \le 2 - \beta$, implies that $p \in H^{r_2+1+\gamma}(\Omega)$, no matter how regular the right term f is. By a simple computation we obtain $|C_{\kappa 1}| \le C\|f\|_{L^2(\Omega)}$, and

$$\begin{array}{rcl} \|u\|_{H^{r_2+\gamma}(\Omega)} &\le& C|C_{\kappa 1}| + C\|\widetilde{f}\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}, \\ \|p\|_{H^{r_2+1+\gamma}(\Omega)} &\le& C\|u\|_{H^{r_2+\gamma}(\Omega)} \le C\|f\|_{L^2(\Omega)}. \end{array}$$

Assume $C_{\kappa 1}, C_\kappa \neq 0$. The solution u contains singular parts $x\kappa(x) \in H^{r_2+\gamma}(\Omega)$, $\kappa(x) \in H^{\min\{r_1,r_2\}+\gamma}(\Omega)$ and a regular part $(\mathcal{D}_\theta^{1-\beta})^{-1}\widetilde{f} \in H_0^{1-\beta}(\Omega)$. This implies $u \in H_0^{1-\beta}(\Omega) \oplus H^{\min\{r_1,r_2\}+\gamma}(\Omega)$, and thus $p \in H^{2-\beta}(\Omega) \oplus H^{\min\{r_1,r_2\}+1+\gamma}(\Omega)$. The appearance of singular parts $x\kappa(x)$, $\kappa(x)$ and $H^{2-\beta}(\Omega) \subset H^{\min\{r_1,r_2\}+1+\gamma}(\Omega)$, since $\min\{r_1,r_2\} + 1 + \gamma \le 2 - \beta$, implies that $p \in H^{\min\{r_1,r_2\}+1+\gamma}(\Omega)$, no matter how regular the right term f is. By a simple computation we obtain $|C_{\kappa 1}| \le C\|f\|_{L^2(\Omega)}$, $|C_\kappa| \le C\|\widetilde{u}\|_{L^2(\Omega)} \le C\|\widetilde{f}\|_{L^2(\Omega)}$, and

$$\begin{array}{rcl} \|u\|_{H^{\min\{r_1,r_2\}+\gamma}(\Omega)} &\le& C\{|C_{\kappa 1}| + |C_\kappa| + \|\widetilde{f}\|_{L^2(\Omega)}\} \le C\|f\|_{L^2(\Omega)}, \\ \|p\|_{H^{\min\{r_1,r_2\}+1+\gamma}(\Omega)} &\le& \|u\|_{H^{\min\{r_1,r_2\}+\gamma}(\Omega)} \le C\|f\|_{L^2(\Omega)}. \end{array}$$

Similar to the proof of (a), we derive the result of (d). This completes the proof. ☐

REMARK 4.2. *It is easily seen that when* $\theta = \frac{1}{2}$ *and* $K =$ constant, *the fractional diffusion operator on the left hand side of (1.1) is equivalent to the fractional Laplace operator defined by Riesz potential. In this sense, Theorem 4.6 extends the regularity result proved in [1, 14] for fractional Laplace operator, corresponding to* $\theta = \frac{1}{2}$ *and* $K(x) =$ constant, *to general fractional diffusion operator (1.1) for* $0 < \theta < 1$ *and variable coefficient* $K(x)$.

5. Finite Element Approximation. In this section, we shall develop a least squared mixed finite element scheme over those typically used finite element spaces based on the variational formulation (4.10) and then conduct the optimal convergence analysis.

Let $\Im_h$ be a uniform partition of the domain Ω

$$\Im_h = \{I_j = (x_{j-1}, x_j); j = 1, 2, \cdots, N\}$$

with $x_j = jh$, $j = 0, 1, 2, \cdots, N$, and the mesh step $h = \frac{1}{N}$. Let $\mathbb{H}_h$ and $\mathbb{V}_h$ be the finite dimensional subspaces of $H_L^1(\Omega)$ and $H_0^{1-\beta}(\Omega)$:

$$\begin{aligned} &\mathbb{H}_h = \{q_h \in H_L^1(\Omega) \cap C^0(\bar{\Omega}) : q_h|_{I_i} \in P_k(I_i), i = 1, 2, \cdots, N\}, \quad k \geq 1, \\ &\mathbb{V}_h = \{v_h \in H_0^{1-\beta}(\Omega) : v_h|_{I_i} \in P_l(I_i), i = 1, 2, \cdots, N\}, \quad l \geq 1. \end{aligned} \tag{5.1}$$

Here $P_k(I_i)$ denotes the set of polynomials of degree not larger than k on I_i.

Then, the mixed finite element formulation based on least squares techniques is to find $[\widetilde{p}_h, \widetilde{u}_h] \in \mathbb{H}_h \times \mathbb{V}_h$ such that

$$\mathcal{B}([\widetilde{p}_h, \widetilde{u}_h]; [q_h, v_h]) = F_h([q_h, v_h]), \quad \forall [q_h, v_h] \in (\mathbb{H}_h \times \mathbb{V}_h). \tag{5.2}$$

where $F_h([q_h, v_h]) = (\widetilde{f}_h, \mathcal{D}_\theta^{1-\beta} v_h), \widetilde{f}_h = f - C_{\kappa 1}^h C_*$, C_* see (3.9), $C_{\kappa 1}^h$ is determined by (5.3). The solvability of the problem (5.2) is guaranteed by the coercivity and continuity of the bilinear form $\mathcal{B}([:, :]; [:, :])$ and presented by the following theorem.

THEOREM 5.1. *For $0 < \beta < \frac{1}{2}$, there exists a unique solution $[\widetilde{p}_h, \widetilde{u}_h] \in \mathbb{H}_h \times \mathbb{V}_h$ to the least squares mixed finite element formulation (5.2).*

Proof. Since $\mathbb{H}_h \times \mathbb{V}_h$ are subspaces of $H_L^1(\Omega) \times H_0^{1-\beta}(\Omega)$, by the analogous argument as in Theorem 4.2 and Lax-Milgram lemma, we obtain the conclusion of this theorem. □

REMARK 5.1. *In the implementation we can use (4.16) and (4.14) to design a decoupled scheme to compute $\widetilde{u}_h$ and $\widetilde{p}_h$, and then to compute u_h and p_h.*

Step 1. Determine $C_{\kappa 1}^h$.

$$C_{\kappa 1}^h = \frac{(f, 1 - r_h)}{C_*(1, 1 - r_h)}, \tag{5.3}$$

where C_* satisfies (3.9) and $r_h = \mathcal{D}_\theta^{1-\beta} \widetilde{v}_{2,h}$ with $\widetilde{v}_{2,h} \in \mathbb{V}_h$ is determined by

$$(\mathcal{D}_\theta^{1-\beta} \widetilde{v}_{2,h}, \mathcal{D}_\theta^{1-\beta} v_h) = (1, \mathcal{D}_\theta^{1-\beta} v_h), \quad \forall v_h \in \mathbb{V}_h. \tag{5.4}$$

Step 2. Determine $\widetilde{u}_h$ satisfying

$$(\mathcal{D}_\theta^{1-\beta} \widetilde{u}_h, \mathcal{D}_\theta^{1-\beta} v_h) = (\widetilde{f}_h, \mathcal{D}_\theta^{1-\beta} v_h), \quad \forall v_h \in \mathbb{V}_h, \tag{5.5}$$

where $\widetilde{f}_h = f - C_{\kappa 1}^h C_*$.

Step 3. Determine $\widetilde{p}_h$ satisfying

$$(KD\widetilde{p}_h, KDq_h) = -(\widetilde{u}_h, KDq_h), \quad \forall q_h \in \mathbb{H}_h. \tag{5.6}$$

Step 4. Determine p_h, u_h. we define

$$\begin{aligned} p_h &= \widetilde{p}_h - C_{\kappa 1}^h \int_0^x K^{-1} \xi \kappa(\xi) d\xi - C_\kappa^h \int_0^x K^{-1} \kappa(\xi) d\xi, \\ u_h &= \widetilde{u}_h + C_{\kappa 1}^h x \kappa(x) + C_\kappa^h \kappa(x) \end{aligned} \tag{5.7}$$

to approximate p and u of the solution of (1.1) (or (4.1)) with

$$C_\kappa^h = \frac{\widetilde{p}_h(1) - C_{\kappa 1}^h \int_0^1 K^{-1} x\kappa(x)dx}{\int_0^1 K^{-1}\kappa(x)dx}, \tag{5.8}$$

$$\text{or} \quad C_\kappa^h = \frac{-\int_0^1 K^{-1}\widetilde{u}_h dx - C_{\kappa 1}^h \int_0^1 K^{-1} x\kappa(x)dx}{\int_0^1 K^{-1}\kappa(x)dx} \tag{5.9}$$

to be the approximation for C_κ.

REMARK 5.2. *During the computation of $C_{\kappa 1}^h$, we can choose a different h from the discrete formulation (5.5). In order to improve accuracy, the step size h can be chosen sufficiently large for the computation of $\widetilde{v}_{2,h}$ and of $C_{\kappa 1}^h$. So without loss of generality, we assume $\widetilde{v}_{2,h} = \widetilde{v}_2$, which leads to $C_{\kappa 1}^h = C_{\kappa 1}$. In the coming convergence analysis, we assume $C_{\kappa 1}^h = C_{\kappa 1}$.*

EXAMPLE 5.3. *Let $f = 1$, the solution of diffusion equation (1.1) is*

$$p = -C_{\kappa 1}\int_0^x K^{-1}\xi\kappa(\xi)d\xi - C_\kappa \int_0^x K^{-1}\kappa(\xi)d\xi,$$

$$fractional\ flux: \quad u = -K(x)Dp = C_{\kappa 1}x\kappa(x) + C_\kappa \kappa(x),$$

where $C_{\kappa 1} = \frac{1}{C_*}$, $C_\kappa = -C_{\kappa 1}\frac{\int_0^1 K^{-1}x\kappa(x)dx}{\int_0^1 K^{-1}\kappa(x)dx}$.

In this example, the regular part of p, u is $\widetilde{p} = 0, \widetilde{u} = 0$.

Now we are to verify the feasibility of the numerical scheme by solving Example 5.3 numerically. When $f \equiv 1$, from the expression (5.3), we derive $C_{\kappa 1}^h = \frac{1}{C_*}$, which equals $C_{\kappa 1}$. So $\widetilde{u}_h = 0$ by (5.5) since $\widetilde{f}_h = 0$, and $\widetilde{p}_h = 0$ by (5.6). Substituting these into (5.8), we derive that $C_\kappa^h = -C_{\kappa 1}^h \frac{\int_0^1 K^{-1}x\kappa(x)dx}{\int_0^1 K^{-1}\kappa(x)dx}$, which is just C_κ. Comparing with (4.21) and (5.7), we obtain that p_h, u_h are consistent with the exact solution p, u. i.e., the fraction diffusion equation (1.1) is exactly solved by LSMFE for right term $f = 1$.

Now we shall give the error estimates. The error between C_κ^h and C_κ is estimated as follows.

LEMMA 5.2. *Let C_κ and C_κ^h be defined by (4.6) and (5.8) or (5.9), respectively. Then, the following estimates hold*

$$|C_\kappa - C_\kappa^h| \le \frac{CK^*}{B(1+r_1, 1+r_2)}\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)} \tag{5.10}$$

and

$$|C_\kappa - C_\kappa^h| \le \frac{K^*}{K_* B(1+r_1, 1+r_2)}\|\widetilde{u} - \widetilde{u}_h\|. \tag{5.11}$$

Proof. Since the diffusion coefficient $K(x)$ has positive lower and upper bounds K_* and K^*, we calculate the integral directly to deduce

$$(K^*)^{-1}B(1+r_1, 1+r_2) \le \int_0^1 K^{-1}\kappa(x)dx \le K_*^{-1}B(1+r_1, 1+r_2),$$

where $B(1+r_1, 1+r_2) = \frac{\Gamma(1+r_1)\Gamma(1+r_2)}{\Gamma(2+r_1+r_2)}$ is the Beta function. Therefore, a direct application of the Sobolev imbedding theorem leads to

$$|C_\kappa - C_\kappa^h| = \frac{|\widetilde{p}(1) - \widetilde{p}_h(1)|}{|\int_0^1 K^{-1}\kappa(x)dx|} \le \frac{CK^*}{B(1+r_1, 1+r_2)}\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)}$$

and application of Cauchy-Schwarz inequality ensures,

$$|C_\kappa - C_\kappa^h| = \frac{|\int_0^1 K^{-1}(\widetilde{u}-\widetilde{u}_h)dx|}{|\int_0^1 K^{-1}\kappa(x)dx|} \le \frac{K^*}{K_* B(1+r_1,1+r_2)}\|\widetilde{u}-\widetilde{u}_h\|.$$

□

REMARK 5.4. *This corollary only gives some upper bounds in accordance with* $\|\widetilde{u}-\widetilde{u}_h\|$ *or* $\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)}$. *In fact, the exact bounds,* $\frac{|\int_0^1 K^{-1}(\widetilde{u}-\widetilde{u}_h)dx|}{|\int_0^1 K^{-1}\kappa(x)dx|}$ *or* $\frac{|\widetilde{p}(1)-\widetilde{p}_h(1)|}{|\int_0^1 K^{-1}\kappa(x)dx|}$, *can be much smaller than these estimates.*

In the coming convergence analysis, we shall use two interpolation operators $I_h^k : H^s(\Omega) \to \mathbb{H}_h$ and $I_h^l : H^t(\Omega) \to \mathbb{V}_h$ which satisfy the following approximation properties [4, 8].

LEMMA 5.3. *Let* $q \in H^s(\Omega)$ *and* $w \in H^t(\Omega)$ *with* $s \ge 1$, $t \ge 1-\beta$. *Then, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &(a) \quad \|q - I_h^k q\|_{H^1(\Omega)} \le Ch^{\min\{k,s-1\}}\|q\|_{H^s(\Omega)}, \\ &(b) \quad \|w - I_h^l w\|_{H^{1-\beta}(\Omega)} \le Ch^{\min\{l+\beta,t-1+\beta\}}\|w\|_{H^t(\Omega)}. \end{aligned} \tag{5.12}$$

With the help of Lemma 5.3, we shall conduct the error estimates of $\widetilde{u}-\widetilde{u}_h$ and $\widetilde{p}-\widetilde{p}_h$ in the energy-norm.

THEOREM 5.4. *Assume* $0 < \beta < \frac{1}{2}$, $0 < \theta < 1$ *and* $s \ge 2-\beta$. *Let* $\widetilde{p} \in H^s(\Omega)\cap H_E^1(\Omega)$ *and* $\widetilde{u} \in H^{s-1}(\Omega)\cap H_0^{1-\beta}(\Omega)$. *Let* $[\widetilde{p},\widetilde{u}]$ *and* $[\widetilde{p}_h,\widetilde{u}_h]$ *denote solutions of the variational formulation (4.10) and discrete formulation (5.2), respectively. Then, there exist constants* C *such that*

$$\begin{aligned} &\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)} + \|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)} \\ \le\ & Ch^{\min\{s-2+\beta,k,l+\beta\}}\{\|\widetilde{p}\|_{H^s(\Omega)} + \|\widetilde{u}\|_{H^{s-1}(\Omega)}\}. \end{aligned} \tag{5.13}$$

Proof. Subtracting (5.2) from (4.10), we obtain the error equation

$$\mathcal{B}([\widetilde{p}-\widetilde{p}_h,\widetilde{u}-\widetilde{u}_h];[q_h,v_h]) = 0, \quad \forall [q_h,v_h] \in \mathbb{H}_h \times \mathbb{V}_h. \tag{5.14}$$

Take $q_h = I_h^k\widetilde{p}-\widetilde{p}_h \in \mathbb{H}_h$, $v_h = I_h^l\widetilde{u}-\widetilde{u}_h \in \mathbb{V}_h$ in (5.14) to derive,

$$\mathcal{B}([I_h^k\widetilde{p}-\widetilde{p}_h, I_h^l\widetilde{u}-\widetilde{u}_h];[I_h^k\widetilde{p}-\widetilde{p}_h, I_h^l\widetilde{u}-\widetilde{u}_h]) = \mathcal{B}([I_h^k\widetilde{p}-\widetilde{p}, I_h^l\widetilde{u}-\widetilde{u}];[I_h^k\widetilde{p}-\widetilde{p}_h, I_h^l\widetilde{u}-\widetilde{u}_h]).$$

Applying the coercivity and continuity of the bilinear form $\mathcal{B}([\cdot,\cdot];[\cdot,\cdot])$ and the approximation properties (5.12), we conclude

$$\begin{aligned} &\|I_h^k\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)} + \|I_h^k\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)} \\ \le\ & Ch^{\min\{s-1,k\}}\|\widetilde{p}\|_{H^s(\Omega)} + Ch^{\min\{s-2+\beta,l+\beta\}}\|\widetilde{u}\|_{H^{s-1}(\Omega)}. \end{aligned} \tag{5.15}$$

A direct application of the triangle inequality and (5.12) to this estimate will lead to (5.13), which completes the proof. □

REMARK 5.5. *The estimate of* $\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ *in Theorem 5.4 is optimal only if* $s-2+\beta \le \min\{k, l+\beta\}$. *This arises from the involvement of* $\widetilde{p}$ *and* $\widetilde{u}$ *in the resulting error equation (5.14).*

We can sharpen the estimate for $\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ by directly applying the error equation resulting from (4.16) and (5.5).

COROLLARY 5.5. *Under the assumptions of Theorem* 5.4, *there exists a constant* C *such that the following optimal energy norm estimate for* $\widetilde{u}$ *holds,*

$$\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)} \le Ch^{\min\{l+\beta,s-2+\beta\}}\|\widetilde{u}\|_{H^{s-1}(\Omega)}. \tag{5.16}$$

Proof. Taking $v = v_h \in \mathbb{V}_h$ in (4.16) and subtracting (5.5), we derive the error equation,

$$(\mathcal{D}_\theta^{1-\beta}(\widetilde{u} - \widetilde{u}_h), \mathcal{D}_\theta^{1-\beta} v_h) = 0, \quad \forall v_h \in \mathbb{V}_h. \tag{5.17}$$

Taking $v_h = I_h^l \widetilde{u} - \widetilde{u}_h$ in (5.17), then applying the approximation properties (5.12b), the inequality (3.5) and the triangle inequality, we obtain the optimal estimate (5.16) for $\widetilde{u}$. ◻

What remains is to compute p_h and u_h, and estimate their error bounds.

Since the kernel function $\kappa(x)$ and function $x\kappa(x)$ only belongs to $L^1(\Omega)$, we select the the following metrics for u and p to measure their errors,

$$|||u||| := \|\widetilde{u}\|_{H^{1-\beta}(\Omega)} + \|C_\kappa \kappa(x)\|_{L^1(\Omega)}$$

and

$$|||p||| := \|\widetilde{p}\|_{H^1(\Omega)} + \|C_\kappa \int_0^x K^{-1}\kappa(\xi)d\xi\|_{W^{1,1}(\Omega)}$$

Combining the estimates for $[\widetilde{p}, \widetilde{u}]$ in Theorem 5.4 and for C_κ in Lemma 5.2, we obtain the following error estimates for $[p, u]$.

THEOREM 5.6. *Assume $0 < \beta < \frac{1}{2}$, and $0 < \theta < 1$. Let $[p, u]$ be the solution to (4.1) and $[p_h, u_h]$ to (5.7). Then, there exist constants C such that*

$$|||p - p_h||| + |||u - u_h||| \le Ch^{\min\{s-2+\beta, k, l+\beta\}}\{\|\widetilde{p}\|_{H^s(\Omega)} + \|\widetilde{u}\|_{H^{s-1}(\Omega)}\}. \tag{5.18}$$

Proof. Subtracting (4.21) from (5.7) and taking the metrics $|||\cdot|||$, we have

$$\begin{aligned} |||p - p_h||| &= \|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)} + \|(C_\kappa - C_\kappa^h)\textstyle\int_0^x K^{-1}\kappa(\xi)d\xi\|_{W^{1,1}(\Omega)} \\ &\le \|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)} + |C_\kappa - C_\kappa^h| \; \|\textstyle\int_0^x K^{-1}\kappa(\xi)d\xi\|_{W^{1,1}(\Omega)} \end{aligned}$$

and

$$\begin{aligned} |||u - u_h||| &= \|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)} + \|(C_\kappa - C_\kappa^h)\kappa(x)\|_{L^1(\Omega)} \\ &\le \|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)} + |C_\kappa - C_\kappa^h| \; \|\kappa(x)\|_{L^1(\Omega)}. \end{aligned}$$

Noting that

$$\|\kappa(x)\|_{L^1(\Omega)} = \int_0^1 \kappa(x)dx = B(1 + r_1, 1 + r_2),$$

$$\|\textstyle\int_0^x K^{-1}\kappa(\xi)d\xi\|_{L^1} \quad \le \quad K_*^{-1}B(1 + r_1, 1 + r_2),$$

$$\begin{aligned} \|\textstyle\int_0^x K^{-1}\kappa(\xi)d\xi\|_{W^{1,1}(\Omega)} &= \|\textstyle\int_0^x K^{-1}\kappa(\xi)d\xi\|_{L^1(\Omega)} + \|D\textstyle\int_0^x K^{-1}\kappa(\xi)d\xi\|_{L^1(\Omega)} \\ &\le 2K_*^{-1}B(1 + r_1, 1 + r_2), \end{aligned}$$

we then apply Lemma 5.2 to obtain the estimates

$$\begin{aligned} |||p - p_h||| &\le \|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)} + \tfrac{2K^*}{K_*}\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)} \\ &\le (1 + \tfrac{2K^*}{K_*})\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)}, \\ |||u - u_h||| &\le \|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)} + \tfrac{K^*}{K_*}\|\widetilde{u} - \widetilde{u}_h\| \\ &\le (1 + \tfrac{K^*}{K_*})\|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)}, \end{aligned}$$

which, combining with Theorem 5.4, completes the proof. ☐

Analogous to Corollary 5.5, we immediately obtain a sharper estimate for u.

Corollary 5.7. *Under the assumption of Theorem 5.6, there exists a constant C such that the following optimal error estimate holds,*

$$|||u - u_h||| \leq Ch^{\min\{l+\beta, s-2+\beta\}} \|\widetilde{u}\|_{H^{s-1}(\Omega)}. \tag{5.19}$$

Remark 5.6. *The theoretical results derived in Section 3, 4 and 5 only include the case $0 < \beta < \frac{1}{2}$. This stems from Lemma 3.2 concerning the coerciveness of the fractional operator, which remain an open question as for the case $\frac{1}{2} < \beta < 1$. However, the numerical computations in section Numerical Experiments suggest that the convergence result for $\frac{1}{2} \leq \beta < 1$ are consistent with those for $0 < \beta < \frac{1}{2}$.*

6. Numerical Experiments. In this section, we employ piecewise linear finite element approximation ($k = l = 1$) and conduct three numerical experiments to verify our theoretical findings: the first one is for a regular solution, i.e., $C_{\kappa 1} = C_\kappa = 0$ and the others are for a singular solution, i.e., $C_{\kappa 1} \neq 0$ or $C_\kappa \neq 0$.

Example 6.1. *Let $K(x) = x + 1$ and $\theta = \frac{\sin(\pi q_1)}{\sin(\pi q_1)+\sin(\pi q_2)}$ with $q_1, q_2 \in (0, 1)$, $q_1 + q_2 = 2 - \beta$. The analytic solution p and the right-hand side function f are prescribed respectively to be*

$$\begin{aligned} p(x) &= x^{q_1+1}(1-x)^{q_2+1}, \\ f(x) &= -(1-\theta)\tfrac{\sin(\pi\beta)}{sin(\pi q_2)}\Gamma(q_2+2)\{-\tfrac{\Gamma(2-\beta)}{6\Gamma(q_2)}((1-\beta)q_2+4\beta-7) \\ &\quad +\tfrac{\Gamma(3-\beta)}{2\Gamma(q_2+1)}((2-\beta)q_2+2\beta-6)x - \tfrac{\Gamma(4-\beta)}{2\Gamma(q_2+2)}((3-\beta)q_2-1)x^2 \\ &\quad +\tfrac{\Gamma(5-\beta)}{6\Gamma(q_2+2)}(4-\beta)x^3\}. \end{aligned}$$

In this example $C_{\kappa 1} = C_\kappa = 0, f \in H^\gamma(\Omega)$, $\widetilde{p} = p \in H_0^s(\Omega) \cap H_L^1(\Omega)$ and $\widetilde{u} = u = -K(x)Dp \in H^{s-1}(\Omega) \cap H_0^{1-\beta}(\Omega)$ with $s = \min\{q_1, q_2\} + 1 + \gamma$, $0 < \gamma < 1/2$.

We take different $\beta = 0.1, 0.25, 0.5,$ 0.75 and 0.9 to investigate the β dependence on the convergence rates.

From Theorem 5.4 and Theorem 5.6, the predicted convergence rate for $\|\widetilde{p} - \widetilde{p}\|_{H^1(\Omega)}$ and $|||p-p_h|||$ are $h^{\min\{1+\beta,1,s-2+\beta\}} = h^{\gamma+\min\{1-q_1,1-q_2\}}$. From Corollary 5.5 and Corollary 5.7, the predicted convergence rate for $\|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u - u_h|||$ are $h^{\min\{1+\beta,s-2+\beta\}} = h^{\gamma+\min\{1-q_1,1-q_2\}}$.

For $\beta = 0.10$, $\theta = 0.60$, $q_1 = 0.94, q_2 = 0.96$. Table 6.1 and Table 6.2 suggest that the convergence rates for $\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)}$ and $|||p - p_h|||$ are h^1, which are higher than that of predicted by Theorem 5.4 and Theorem 5.6, $h^{0.54}$, and in accordance with that of interpolation h^1 (see (5.12)(a)); the convergence rates for $\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u-u_h|||$ are higher than that of predicted $h^{0.54}$, and almost in accordance with that of interpolation $h^{0.54}$ (see (5.12)(b)).

For $\beta = 0.25$, $\theta = 0.60$, $q_1 = 0.85, r_2 = 0.90$. Table 6.1 and Table 6.2 suggest that the convergence rates for $\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)}$ and $|||p - p_h|||$ are h^1, which are higher than that of predicted by Theorem 5.4 and Theorem 5.6, $h^{0.60}$, and in accordance with that of interpolation h^1 (see (5.12)(a)); the convergence rates for $\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u-u_h|||$ are higher than that of predicted $h^{0.60}$, and almost in accordance with that of interpolation $h^{0.60}$ (see (5.12)(b)).

Example 6.2. *Let $K(x) = x + 1, 0 < \theta < 1$. The analytic solution p and the*

Table 6.1
Numerical results for Example 6.1 with $\theta = 0.6$

β, q_1, q_2	h	$\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)}$	rates	$\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$	rates	$\|C_\kappa - C_\kappa^h\|$
	2^{-4}	1.75e-2		1.48e-1		6.35e-4
β=0.10	2^{-5}	8.75e-3	1.00	7.29e-2	1.02	2.62e-4
$q_1 = 0.94$	2^{-6}	4.37e-3	1.00	3.68e-2	0.99	1.16e-4
$q_2 = 0.96$	2^{-7}	2.19e-3	1.00	1.93e-2	0.93	5.44e-5
	2^{-8}	1.09e-3	1.00	1.06e-2	0.86	2.62e-5
Pred. rates			0.54		0.54	
	2^{-4}	1.99e-2		9.70e-2		2.01e-3
β=0.25	2^{-5}	1.00e-2	1.00	4.91e-2	0.98	9.17e-4
$q_1 = 0.85$	2^{-6}	5.00e-3	1.00	2.65e-2	0.89	4.37e-4
$q_2 = 0.90$	2^{-7}	2.50e-3	1.00	1.52e-2	0.80	2.13e-4
	2^{-8}	1.25e-3	1.00	9.21e-3	0.73	1.06e-4
Pred. rates			0.60		0.60	
	2^{-4}	2.56e-2		4.76e-2		4.03e-3
β=0.50	2^{-5}	1.30e-2	0.98	2.48e-2	0.94	1.92e-3
$q_1 = 0.69$	2^{-6}	6.58e-3	0.98	1.38e-2	0.85	9.38e-4
$q_2 = 0.81$	2^{-7}	3.31e-3	0.99	8.00e-2	0.79	4.63e-4
	2^{-8}	1.67e-3	0.99	4.75e-3	0.75	2.29e-4
Pred. rates			0.69		0.69	
	2^{-4}	3.44e-2		2.09e-2		4.79e-5
β=0.75	2^{-5}	1.84e-2	0.90	9.67e-3	1.11	1.91e-5
$q_1 = 0.48$	2^{-6}	9.74e-3	0.92	4.76e-3	1.02	9.87e-6
$q_2 = 0.77$	2^{-7}	5.13e-3	0.93	2.44e-3	0.97	5.39e-6
	2^{-8}	2.69e-3	0.93	1.29e-3	0.92	3.49e-6
Pred. rates			0.73		0.73	
	2^{-4}	6.36e-2		6.84e-2		6.17e-6
β=0.90	2^{-5}	3.62e-2	0.81	3.56e-2	0.94	1.55e-5
$q_1 = 0.26$	2^{-6}	2.09e-2	0.79	1.94e-2	0.88	1.14e-5
$q_2 = 0.84$	2^{-7}	1.22e-2	0.78	1.08e-2	0.84	6.42e-6
	2^{-8}	7.13e-3	0.77	6.18e-3	0.81	3.13e-6
Pred. rates			0.66		0.66	

right-hand side function f are prescribed respectively to be

$$
\begin{aligned}
p(x) &= 3x^2 - 2x^3 - C_\kappa \textstyle\int_0^x K^{-1}\kappa(\xi)d\xi, \\
f(x) &= 6\theta\Big(\frac{6}{\Gamma(3+\beta)}x^{2+\beta} - \frac{1}{\Gamma(1+\beta)}x^{\beta}\Big) \\
&\quad +6(1-\theta)\Big(\frac{6}{\Gamma(3+\beta)}(1-x)^{2+\beta} + \frac{6}{\Gamma(2+\beta)}(1-x)^{1+\beta} + \frac{2}{\Gamma(1+\beta)}(1-x)^{\beta}\Big),
\end{aligned}
$$

where $C_\kappa = (\int_0^1 K^{-1}x^{r_1}(1-x)^{r_2}dx)^{-1}$, r_1 and r_2 are determined by (3.4).

In this example, $C_{\kappa 1} = 0, C_\kappa \neq 0$, $\widetilde{p} = 3x^2 - 2x^3 \in H^{2+\gamma}(\Omega)$ and $\widetilde{u} = 6x^3 - 6x \in H^{1+\gamma}(\Omega)$ with $\gamma \in (0, 1/2)$, which can be selected as possible as close to $\frac{1}{2}$.

We take different $\beta = 0.1, 0.25, 0.5$, 0.75 and 0.9 to investigate the β dependence on the convergence rates. From (3.4), the index of singular kernel are $r_1 = -0.06$, $r_2 = -0.04$, as $\beta = 0.10, \theta = 0.6$; $r_1 = -0.15$, $r_2 = -0.10$, as $\beta = 0.25, \theta = 0.6$; $r_1 =$

Table 6.2
Numerical results for Example 6.1 with $\theta = 0.6$

β, q_1, q_2	h	$\|\|\|p-p_h\|\|\|$	rates	$\|\|\|u-u_h\|\|\|$	rates	$\|C_{\kappa 1}-C_{\kappa 1}^h\|$
	2^{-4}	1.92e-2		1.50e-1		2.54e-3
β=0.10	2^{-5}	9.38e-3	1.03	7.37e-2	1.03	9.28e-4
$q_1 = 0.94$	2^{-6}	4.64e-3	1.02	3.71e-2	0.99	3.63e-4
$q_2 = 0.96$	2^{-7}	2.30e-3	1.01	1.94e-2	0.93	1.53e-4
	2^{-8}	1.14e-3	1.00	1.07e-2	0.86	6.91e-5
Pred. rates			0.54		0.54	
	2^{-4}	2.53e-2		1.04e-1		6.40e-3
β=0.25	2^{-5}	1.23e-2	1.03	5.19e-2	1.00	2.67e-4
$q_1 = 0.85$	2^{-6}	6.08e-3	1.02	2.78e-2	0.90	1.19e-3
$q_2 = 0.90$	2^{-7}	3.02e-3	1.01	1.59e-2	0.81	5.61e-4
	2^{-8}	1.51e-3	1.00	9.52e-3	0.74	2.72e-4
Pred. rates			0.60		0.60	
	2^{-4}	3.94e-2		6.40e-2		1.20e-2
β=0.50	2^{-5}	1.95e-2	1.02	3.24e-3	0.98	5.49e-3
$q_1 = 0.69$	2^{-6}	9.69e-3	1.01	1.75e-2	0.89	2.62e-3
$q_2 = 0.81$	2^{-7}	4.85e-3	1.00	9.80e-3	0.83	1.28e-3
	2^{-8}	2.43e-3	1.00	5.64e-3	0.80	6.36e-4
Pred. rates			0.69		0.69	
	2^{-4}	4.09e-2		3.13e-2		1.09e-2
β=0.75	2^{-5}	2.12e-2	0.95	1.41e-2	1.15	4.69e-3
$q_1 = 0.48$	2^{-6}	1.10e-2	0.94	6.78e-3	1.06	2.12e-3
$q_2 = 0.77$	2^{-7}	5.73e-3	0.94	3.38e-3	1.00	9.92e-4
	2^{-8}	2.97e-3	0.95	1.73e-3	0.96	4.68e-4
Pred. rates			0.73		0.73	
	2^{-4}	9.88e-2		1.23e-1		5.61e-2
β=0.90	2^{-5}	5.38e-2	0.88	6.27e-2	0.97	2.79e-2
$q_1 = 0.26$	2^{-6}	3.01e-2	0.83	3.36e-2	0.90	1.45e-2
$q_2 = 0.84$	2^{-7}	1.72e-2	0.81	1.85e-2	0.86	7.88e-3
	2^{-8}	9.90e-3	0.79	1.05e-2	0.82	4.39e-3
Pred. rates			0.66		0.66	

-0.31, $r_2 = -0.19$, as $\beta = 0.50, \theta = 0.6$; $r_1 = -0.52$, $r_2 = -0.23$, as $\beta = 0.75, \theta = 0.6$; $r_1 = -0.74$, $r_2 = -0.16$, as $\beta = 0.90, \theta = 0.6$.

The value of s in Theorem 5.4 is given by $s = 2 + \gamma \approx 2.5$. From Theorem 5.4 and Theorem 5.6, the predicted convergence rate for $\|\widetilde{p} - \widetilde{p}\|_{H^1(\Omega)}$ and $|||p - p_h|||$ are: $h^{\min\{1+\beta,1,s-2+\beta\}} = h^{\min\{1,2.5-2+\beta\}} = h^{\min\{1,0.5+\beta\}}$. From Corollary 5.5 and Corollary 5.7, the predicted convergence rate for $\|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u - u_h|||$ are: $h^{\min\{1+\beta,s-2+\beta\}} = h^{\min\{2.5-2+\beta\}} = h^{\min\{0.5+\beta\}}$.

For $\beta = 0.1$, Table 6.3 and Table 6.4 suggest that the convergence rates for $\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)}$ and $|||p - p_h|||$ are h^1, which are higher than that of predicted $h^{0.6}$, and in accordance with that of interpolation h^1 (see (5.12)(a)); the convergence rates for $\|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u - u_h|||$ are $h^{1.10}$, which are higher than that of predicted $h^{0.6}$, and in accordance with that of interpolation $h^{1.10}$ (see (5.12)(b)).

For $\beta = 0.25$, Table 6.3 and Table 6.4 suggest that the convergence rates for $\|\widetilde{p} - \widetilde{p}_h\|_{H^1(\Omega)}$ and $|||p - p_h|||$ are h^1, which are higher than that of predicted $h^{0.75}$, and in accordance with that of interpolation h^1 (see (5.12)(a)); the convergence rates for $\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $\|\|u-u_h\|\|$ are $h^{1.25}$, which are higher than that of predicted $h^{0.75}$, and in accordance with that of interpolation $h^{1.25}$ (see (5.12)(b)).

TABLE 6.3
Numerical results for Example 6.2 with $\theta = 0.6$

β, r_1, r_2	h	$\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)}$	rates	$\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$	rates	$\lvert C_\kappa - C_\kappa^h\rvert$
	2^{-4}	6.26e-2		2.25e-1		2.11e-3
β=0.10	2^{-5}	3.13e-2	1.00	1.04e-1	1.10	4.47e-4
$r_1=-0.06$	2^{-6}	1.56e-2	1.00	4.87e-2	1.10	9.14e-5
$r_2=-0.04$	2^{-7}	7.81e-3	1.00	2.27e-2	1.10	1.71e-5
	2^{-8}	3.91e-3	1.00	1.06e-2	1.10	2.42e-6
Pred. rates			0.60		0.60	
	2^{-4}	6.25e-2		1.01e-1		5.39e-3
β=0.25	2^{-5}	3.12e-2	1.00	4.21e-2	1.26	7.16e-5
$r_1=-0.15$	2^{-6}	1.56e-2	1.00	1.77e-2	1.26	4.63e-6
$r_2=-0.10$	2^{-7}	7.81e-3	1.00	7.41e-3	1.25	1.96e-6
	2^{-8}	3.91e-3	1.00	3.11e-3	1.25	1.34e-6
Pred. rates			0.75		0.75	
	2^{-4}	6.25e-2		2.45e-2		9.80e-5
β=0.50	2^{-5}	3.13e-2	1.00	8.39e-3	1.54	4.29e-5
$r_1=-0.31$	2^{-6}	1.56e-2	1.00	2.92e-3	1.52	1.36e-5
$r_2=-0.19$	2^{-7}	7.81e-3	1.00	1.03e-3	1.51	3.86e-6
	2^{-8}	3.91e-3	1.00	3.61e-4	1.51	1.06e-6
Pred. rates			1.00		1.00	
	2^{-4}	6.25e-2		6.82e-3		1.54e-4
β=0.75	2^{-5}	3.12e-2	1.00	1.80e-3	1.92	4.01e-5
$r_1=-0.52$	2^{-6}	1.56e-2	1.00	4.87e-4	1.89	1.03e-5
$r_2=-0.23$	2^{-7}	7.81e-3	1.00	1.35e-4	1.86	2.61e-6
	2^{-8}	3.91e-3	1.00	3.78e-5	1.83	6.57e-7
Pred. rates			1.00		1.25	
	2^{-4}	6.31e-2		1.35e-2		9.09e-5
β=0.90	2^{-5}	3.13e-2	1.00	3.08e-3	2.13	2.26e-5
$r_1=-0.74$	2^{-6}	1.56e-2	1.00	7.01e-4	2.14	5.64e-6
$r_2=-0.16$	2^{-7}	7.81e-3	1.00	1.59e-4	2.14	1.41e-6
	2^{-8}	3.91e-3	1.00	3.38e-5	2.23	3.53e-7
Pred. rates			1.00		1.40	

EXAMPLE 6.3. *Let $K(x) = x+1$. The analytic solution p and the right-hand side function f are prescribed respectively to be*

$$\begin{aligned} p(x) &= -\int_0^x K^{-1}(\xi)\xi(1-\xi)\kappa(\xi)d\xi - C_{\kappa 1}\int_0^x K^{-1}(\xi)\xi\kappa(\xi)d\xi - C_\kappa \int_0^x K^{-1}(\xi)\kappa(\xi)d\xi, \\ f(x) &= \theta\frac{\sin(\pi\beta)}{\sin(\pi r_1)}\Gamma(3-\beta)x, \end{aligned}$$

where $C_{\kappa 1} = -(r_2+1), C_\kappa$ is selected to such that $p(1) = 0$.

In this example $C_{\kappa 1} \neq 0, C_\kappa \neq 0$. The regular part of solution p and u are $\widetilde{p} = -\int_0^x K^{-1}(\xi)\xi(1-\xi)\kappa(\xi)d\xi \in H^s(\Omega)\cap H_L^1((\Omega))$ and $\widetilde{u} = x(1-x)\kappa(x) \in H^{s-1}(\Omega)\cap H_0^{1-\beta}(\Omega)$ with $s = \min\{r_1, r_2\}+2+\gamma$, respectively.

Table 6.4
Numerical results for Example 6.2 with $\theta = 0.6$

β, r_1, r_2	h	$\|\|\|p-p_h\|\|\|$	rates	$\|\|\|u-u_h\|\|\|$	rates	$\|C_{\kappa 1}-C_{\kappa 1}^h\|$
	2^{-4}	6.38e-2		2.28e-1		1.97e-3
β=0.10	2^{-5}	3.14e-2	1.02	1.05e-1	1.12	5.64e-4
$r_1 = -0.06$	2^{-6}	1.56e-2	1.01	4.89e-2	1.11	1.62e-4
$r_2 = -0.04$	2^{-7}	7.80e-3	1.00	2.27e-2	1.10	4.80e-5
	2^{-8}	3.90e-3	1.00	1.06e-2	1.10	1.50e-5
Pred. rates			0.60		0.60	
	2^{-4}	6.16e-2		1.03e-1		2.77e-3
β=0.25	2^{-5}	3.09e-2	0.99	4.27e-2	1.28	7.12e-4
$r_1 = -0.15$	2^{-6}	1.55e-2	0.99	1.78e-2	1.26	1.83e-4
$r_2 = -0.10$	2^{-7}	7.79e-3	1.00	7.44e-3	1.26	4.74e-5
	2^{-8}	3.90e-3	1.00	3.12e-3	1.25	1.27e-5
Pred. rates			0.75		0.75	
	2^{-4}	6.14e-2		2.66e-2		2.54e-3
β=0.50	2^{-5}	3.10e-2	0.98	8.92e-3	1.58	5.79e-4
$r_1 = -0.31$	2^{-6}	1.56e-2	0.99	3.05e-3	1.55	1.31e-4
$r_2 = -0.19$	2^{-7}	7.80e-3	1.00	1.06e-3	1.53	2.95e-5
	2^{-8}	3.90e-3	1.00	3.69e-4	1.52	6.71e-6
Pred. rates			1.00		1.00	
	2^{-4}	6.20e-2		9.43e-3		2.38e-3
β=0.75	2^{-5}	3.12e-2	0.99	2.38e-3	1.99	5.07e-4
$r_1 = -0.52$	2^{-6}	1.56e-2	1.00	6.13e-4	1.95	1.07e-5
$r_2 = -0.23$	2^{-7}	7.81e-3	1.00	1.62e-4	1.92	2.26e-5
	2^{-8}	3.91e-3	1.00	4.37e-5	1.89	4.57e-6
Pred. rates			1.00		1.25	
	2^{-4}	5.92e-2		2.53e-2		1.18e-2
β=0.90	2^{-5}	3.05e-2	0.96	5.75e-3	2.14	2.65e-3
$r_1 = -0.74$	2^{-6}	1.54e-2	0.98	1.30e-3	2.14	5.89e-4
$r_2 = -0.16$	2^{-7}	7.77e-3	0.99	2.90e-4	2.16	1.29e-4
	2^{-8}	3.90e-3	1.00	6.03e-5	2.27	2.58e-5
Pred. rates			1.00		1.40	

We take different $\beta = 0.1, 0.25, 0.5,$ 0.75 and 0.9 to investigate the β dependence on the convergence rates.

From Theorem 5.4 and Theorem 5.6, the predicted convergence rate for $\|\widetilde{p} - \widetilde{p}\|_{H^1(\Omega)}$ and $|||p-p_h|||$ are: $h^{\min\{1+\beta,1,s-2+\beta\}} = h^{\gamma+\min\{-r_1,-r_2\}}$. From Corollary 5.5 and Corollary 5.7, the predicted convergence rate for $\|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u-u_h|||$ are: $h^{\min\{1+\beta,s-2+\beta\}} = h^{\gamma+\min\{-r_1,-r_2\}}$.

For $\beta = 0.10$, $\theta = 0.60$, the index of singular kernel is $r_1 = -0.06, r_2 = -0.04$. Table 6.5 and Table 6.6 suggest that the convergence rates for $\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)}$ and $|||p-p_h|||$ are h^1, which are higher than that of predicted by Theorem 5.4 and Theorem 5.6, $h^{0.54}$, and in accordance with that of interpolation h^1 (see (5.12)(a)); the convergence rates for $\|\widetilde{u} - \widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u-u_h|||$ are higher than that of predicted $h^{0.54}$, and almost in accordance with that of interpolation $h^{0.54}$ (see (5.12)(b)).

For $\beta = 0.25$, $\theta = 0.60$, the index of singular kernel is $r_1 = -0.15, r_2 = -0.10$. Table 6.5 and Table 6.6 suggest that the convergence rates for $\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)}$ and $|||p-$

$p_h|||$ are h^1, which are higher than that of predicted by Theorem 5.4 and Theorem 5.6, $h^{0.60}$, and in accordance with that of interpolation h^1 (see (5.12)(a)); the convergence rates for $\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$ and $|||u-u_h|||$ are higher than that of predicted $h^{0.60}$, and almost in accordance with that of interpolation $h^{0.60}$ (see (5.12)(b)).

TABLE 6.5
Numerical results for Example 6.3 with $\theta = 0.6$

β, r_1, r_2	h	$\|\widetilde{p}-\widetilde{p}_h\|_{H^1(\Omega)}$	rates	$\|\widetilde{u}-\widetilde{u}_h\|_{H^{1-\beta}(\Omega)}$	rates	$\|C_\kappa - C_\kappa^h\|$
	2^{-4}	8.13e-3		3.00e-2		6.87e-4
β=0.10	2^{-5}	4.07e-3	1.00	1.58e-2	0.93	2.27e-4
$r_1=-0.06$	2^{-6}	2.03e-3	1.00	8.74e-3	0.85	8.85e-5
$r_2=-0.04$	2^{-7}	1.01e-3	1.00	5.15e-3	0.76	3.58e-5
	2^{-8}	5.08e-4	1.00	3.21e-3	0.68	1.62e-5
Pred. rates			0.54		0.54	
	2^{-4}	9.60e-3		2.61e-2		1.29e-3
β=0.25	2^{-5}	4.81e-3	1.00	1.52e-2	0.78	5.61e-4
$r_1=-0.15$	2^{-6}	2.41e-3	1.00	9.27e-2	0.71	2.60e-4
$r_2=-0.10$	2^{-7}	1.20e-3	1.00	5.85e-3	0.66	1.25e-4
	2^{-8}	6.02e-4	1.00	3.76e-3	0.64	6.08e-5
Pred. rates			0.60		0.60	
	2^{-4}	1.35e-2		1.93e-2		2.39e-3
β=0.5	2^{-5}	6.89e-3	0.97	1.13e-2	0.77	1.15e-3
$r_1=-0.31$	2^{-6}	3.49e-3	0.98	6.75e-3	0.74	5.61e-4
$r_2=-0.19$	2^{-7}	1.76e-3	0.99	4.10e-3	0.72	2.76e-4
	2^{-8}	8.85e-4	0.99	2.51e-3	0.71	1.37e-4
Pred. rates			0.69		0.69	
	2^{-4}	2.07e-2		9.61e-3		3.25e-5
β=0.75	2^{-5}	1.12e-2	0.88	4.97e-3	0.95	1.53e-5
$r_1=-0.52$	2^{-6}	6.00e-2	0.90	2.63e-3	0.92	7.96e-6
$r_2=-0.23$	2^{-7}	3.19e-3	0.91	1.42e-3	0.89	4.20e-6
	2^{-8}	1.68e-3	0.92	7.76e-4	0.87	2.57e-6
Pred. rates			0.73		0.73	
	2^{-4}	4.02e-2		3.18e-2		2.03e-5
β=0.90	2^{-5}	2.45e-2	0.71	1.89e-2	0.75	1.79e-5
$r_1=-0.74$	2^{-6}	1.48e-2	0.73	1.13e-2	0.75	1.03e-5
$r_2=-0.16$	2^{-7}	8.82e-3	0.74	6.68e-3	0.75	5.39e-6
	2^{-8}	5.27e-3	0.74	4.01e-3	0.74	2.55e-6
Pred. rates			0.66		0.66	

All of these numerical results strongly suggest that the proposed LSMFE in this article captures exactly the singular part of the solution and possesses the optimal-order approximation property as the approximation spaces involved that those typically used in the literature of finite element.

7. Concluding Remark. In this work, we have developed a least-squares mixed variational formulation for two-side fractional diffusion equations with variable coefficient, by introducing an intermediate variable to isolate the variable coefficient from the nonlocal operator and by decomposing the admissible spaces of the solution as a direct sum of a regular fractional Sobolev space and a kernel-dependent singular

TABLE 6.6
Numerical results for Example 6.3 with $\theta = 0.6$

β, r_1, r_2	h	$\|\|\|p - p_h\|\|\|$	rates	$\|\|\|u - u_h\|\|\|$	rates	$\|C_{\kappa 1} - C_{\kappa 1}^h\|$
	2^{-4}	8.99e-3		3.08e-2		5.69e-5
β=0.10	2^{-5}	4.35e-3	1.05	1.60e-2	0.94	1.55e-5
$r_1 = -0.06$	2^{-6}	2.14e-3	1.02	8.84e-3	0.86	4.11e-6
$r_2 = -0.04$	2^{-7}	1.06e-3	1.01	5.20e-3	0.77	9.72e-7
	2^{-8}	5.28e-4	1.01	3.23e-3	0.69	1.47e-7
Pred. rates			0.54		0.54	
	2^{-4}	1.15e-2		2.78e-2		9.13e-5
β=0.25	2^{-5}	5.62e-3	1.03	1.59e-2	0.81	1.65e-5
$r_1 = -0.15$	2^{-6}	2.78e-3	1.02	9.60e-3	0.73	1.12e-6
$r_2 = -0.10$	2^{-7}	1.38e-3	1.00	6.01e-3	0.68	3.86e-6
	2^{-8}	6.91e-4	1.00	3.84e-3	0.65	3.17e-6
Pred. rates			0.60		0.60	
	2^{-4}	1.84e-2		2.35e-2		2.22e-4
β=0.50	2^{-5}	9.20e-3	1.00	1.33e-3	0.83	4.58e-5
$r_1 = -0.31$	2^{-6}	4.60e-3	1.00	7.71e-3	0.78	3.32e-7
$r_2 = -0.19$	2^{-7}	2.31e-3	0.99	4.59e-3	0.75	8.80e-6
	2^{-8}	1.16e-3	0.99	2.75e-3	0.73	7.61e-6
Pred. rates			0.69		0.69	
	2^{-4}	2.14e-2		1.05e-2		8.46e-4
β=0.75	2^{-5}	1.16e-2	0.88	5.48e-3	0.94	5.01e-4
$r_1 = -0.52$	2^{-6}	6.25e-3	0.90	2.92e-3	0.91	2.86e-4
$r_2 = -0.23$	2^{-7}	3.32e-3	0.91	1.57e-3	0.89	1.58e-4
	2^{-8}	1.76e-3	0.92	8.61e-4	0.87	8.40e-5
Pred. rates			0.73		0.73	
	2^{-4}	5.57e-2		4.83e-2		1.69e-2
β=0.90	2^{-5}	3.39e-2	0.71	2.90e-2	0.74	1.02e-2
$r_1 = -0.74$	2^{-6}	2.04e-2	0.73	1.73e-2	0.75	6.13e-3
$r_2 = -0.16$	2^{-7}	1.22e-2	0.74	1.03e-2	0.75	3.67e-3
	2^{-8}	7.35e-3	0.73	6.23e-3	0.72	2.27e-3
Pred. rates			0.66		0.66	

space. The existence and uniqueness of the solution to the CFDE (1.1) via the mixed formulation has been proved. A new regularity theory for the solution is established in terms of right-hand side function, which extends the regularity result for fractional Laplace operator to general fractional diffusion operators. From this, we have designed a kernel-independent least-squares mixed finite element procedure (LSMFE). The numerical analysis and numerical experiments conducted in this work strongly suggest that the proposed LSMFE captures exactly the singular part of the solution and possesses the optimal-order approximation property as the approximation spaces involved are those typically used in the literature of finite elements.

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