Regularity of the solution to fractional diffusion, advection, reaction equations in weighted Sobolev spaces

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Abstract

In this article we investigate the regularity of the solution to the fractional diffusion, advection, reaction equation on a bounded domain in $\mathbb{R}^1$. The analysis is performed in the weighted Sobolev spaces, $H^s_{(a,b)}(\mathrm{I})$. Three different characterizations of $H^s_{(a,b)}(\mathrm{I})$ are presented, together with needed embedding theorems for these spaces. The analysis shows that the regularity of the solution is bounded by the endpoint behavior of the solution, which is determined by the parameters α and r defining the fractional diffusion operator. Additionally, the analysis shows that for a sufficiently smooth right hand side function, the regularity of the solution to fractional diffusion reaction equation is lower than that of the fractional diffusion equation. Also, the regularity of the solution to fractional diffusion advection reaction equation is two orders lower than that of the fractional diffusion reaction equation.

Key words. Fractional diffusion equation, regularity, weighted Sobolev spaces

AMS Mathematics subject classifications. 35R11, 35B65, 46E35

1 Introduction

Of interest in this report is the regularity of the solution of the fractional diffusion equation

$$\mathcal{L}_r^{\alpha} u(x) \ := \ - \left(r D^{\alpha} \, + \, (1-r) D^{\alpha *} \right) u(x) = \ f(x) \, , \quad x \in \mathrm{I} \, , \tag{1.1}$$

$$\text{subject to } u(0) \, = \, u(1) = \, 0 \, , \tag{1.2}$$

and the regularity of the solution of the fractional diffusion, advection, reaction equation

$$\mathcal{L}_r^{\alpha} u(x) \ + \ b(x) D u(x) \ + \ c(x) u(x) = \ f(x) \, , \quad x \in \mathrm{I} \, , \tag{1.3}$$

$$\text{subject to } u(0) \, = \, u(1) = \, 0 \, , \tag{1.4}$$

where $\mathrm{I} \ := \ (0,1)$, $1 < \alpha < 2$, $0 \leq r \leq 1$, $c(x) \, - \, \frac{1}{2} D b(x) \geq 0$, D denotes the usual derivative operator, D^{α} the α-order left fractional derivative operator, and $D^{\alpha *}$ the α-order right fractional

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derivative operator, defined by:

$$D^{\alpha}u(x) := \; D\,\frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{1}{(x-s)^{\alpha-1}}\,Du(s)\,ds\,, \tag{1.5}$$

$$D^{\alpha*}u(x) := \; D\,\frac{1}{\Gamma(2-\alpha)}\int_x^1 \frac{1}{(s-x)^{\alpha-1}}\,Du(s)\,ds\,. \tag{1.6}$$

The regularity of the solution to a differential equation plays a fundamental role in designing optimal approximation schemes for the solution.

In recent years fractional order differential equations have received increased attention due to their application in the modeling of physical phenomena such as in contaminant transport in ground water flow [7, 12], viscoelasticity [26], image processing [5, 10, 17, 18], turbulent flow [25, 32], and chaotic dynamics [35].

The diffusion operator, $\mathcal{L}_r^{\alpha}$, arises in a random walk process in which the jumps have an unbounded variance (Lévy process) [7, 30].

For the case $r = 1/2$, $\mathcal{L}_{1/2}^{\alpha}$ represents the *integral* fractional Laplacian operator [2]. The existence, uniqueness and regularity of the solution to the fractional Laplacian equation has been investigated by a number of authors, in $\mathbb{R}^1$ see [3], in $\mathbb{R}^{n\geq 2}$ [4, 11, 20, 31]. Recently the regularity results for the fractional Laplacian equation was extended by Hao and Zhang in [23, 36] to the fractional Laplacian equation with a constant advection and reaction term (i.e. (1.3),(1.4), for $r = 1/2$, $b(x) = b$, $c(x) = c$, $b,\, c \in \mathbb{R}$).

Fewer results on the regularity of the solution to the general fractional diffusion, advection, reaction equation have been established. In [15] Ervin and Roop established existence and uniqueness of solution, $u \in H_0^{\alpha/2}(\mathrm{I})$, for $f \in H^{-\alpha/2}(\mathrm{I})$. More recently in [14, 24] precise regularity results were obtained for the solution of (1.1),(1.2) for $f \in H^s_{(a,b)}(\mathrm{I})$, where $H^s_{(a,b)}(\mathrm{I})$ denotes an appropriated weighted Sobolev space. In [22] Hao, Guang and Zhang obtained regularity estimates for the solution of (1.3) for $b(x) = 0$, $c(x) = c$. Their numerical experiments indicated that their regularity estimates were not optimal.

In this article we present the general regularity results for (1.3),(1.4), in appropriately weighted Sobolev spaces. The analysis establishes that the presence of a reaction term (i.e. $c(x) \neq 0$) limits the regularity of the solution, regardless of the smoothness of the right hand side function, $f(x)$. This reduction in regularity is greater (by a factor of 2) when an advective term (i.e. $b(x) \neq 0$) appears in (1.3). This behavior of the solution is in sharp contrast to that for the integer order ($\alpha = 2$) diffusion, advection, reaction equation. In that case, assuming $b(x)$ and $c(x)$ are sufficiently regular, for the right hand side function $f \in H^s(\mathrm{I})$ the solution lies in $H^{s+2}(\mathrm{I})$.

The results we present herein extend those in [14] for the fractional diffusion equation, and those in [23] for the fractional Laplacian equation with a constant advection and reaction term. The proofs given are significantly different that those used in [22, 23].

The analysis of (1.3),(1.4) is most appropriately performed in weighted Sobolev spaces (due to the singular behavior of the solution at the endpoints). There are different ways to define the weighted Sobolev spaces: (i) using interpolation, (ii) using an appropriate basis, (iii) using an explicit definition for the fractional order norms. Each of these representations have their advantages, and are used in the analysis.

This paper is organized as follows. In the next section we present some preliminary definitions and results. Section 3 presents the three different (but equivalent) definitions of the weighted Sobolev spaces, along with some useful properties of the space $H^s_{(a,b)}(\mathrm{I})$. (For example, which $H^s_{(a,b)}(\mathrm{I})$ space the function $f(x) = x^\mu$ lies in, and for which $H^s_{(a,b)}(\mathrm{I})$ we have the embedding $H^s_{(a,b)}(\mathrm{I}) \subset C^k(\mathrm{I})$.) The regularity of the solution to the fractional diffusion problem (1.1),(1.2), is discussed in Section 4. The regularity of the solution to the fractional diffusion, advection, reaction problem (1.3),(1.4) is presented in Section 5. In Section 6 we relate the regularity results obtained in weighted Sobolev spaces to the usual (unweighted) Sobolev spaces. In the final section, Section 7, we give the proof of a key result used in Section 4.

2 Notation and Properties

Jacobi polynomial have an important connection with fractional order diffusion equations [3, 14, 28, 27]. We briefly review their definition and some of their important properties [1, 33].

Usual Jacobi Polynomials, $P_n^{(a,b)}(t)$, on $(-1\,,\,1)$.
Definition: $P_n^{(a,b)}(t) \ := \ \sum_{m=0}^{n} p_{n,m}\,(t-1)^{(n-m)}(t+1)^m$, where

$$p_{n,m} \ := \ \frac{1}{2^n} \left(\begin{array}{c} n+a \\ m \end{array} \right) \left(\begin{array}{c} n+b \\ n-m \end{array} \right) . \tag{2.1}$$

Orthogonality:

$$\int_{-1}^{1} (1-t)^a (1+t)^b \, P_j^{(a,b)}(t)\, P_k^{(a,b)}(t)\, dt \ = \ \begin{cases} 0, & k \neq j \\ |||P_j^{(a,b)}|||^2\,, & k = j \end{cases} ,$$

$$\text{where} \quad |||P_j^{(a,b)}||| \ = \ \left(\frac{2^{(a+b+1)}}{(2j\,+\,a\,+\,b\,+1)} \frac{\Gamma(j+a+1)\,\Gamma(j+b+1)}{\Gamma(j+1)\,\Gamma(j+a+b+1)} \right)^{1/2} . \tag{2.2}$$

In order to transform the domain of the family of Jacobi polynomials to $[0,1]$, let $t \to 2x-1$ and introduce $G_n^{(a,b)}(x) \ = \ P_n^{(a,b)}(t(x))$. From (2.2),

$$\begin{aligned} \int_{-1}^{1} (1-t)^a (1+t)^b \, P_j^{(a,b)}(t)\, P_k^{(a,b)}(t)\, dt &= \int_0^1 2^a\,(1-x)^a\, 2^b\, x^b\, P_j^{(a,b)}(2x-1)\, P_k^{(a,b)}(2x-1)\, 2\, dx \\ &= \ 2^{a+b+1} \int_0^1 (1-x)^a\, x^b\, G_j^{(a,b)}(x)\, G_k^{(a,b)}(x)\, dx \\ &= \ \begin{cases} 0, & k \neq j\,, \\ 2^{a+b+1}\, |||G_j^{(a,b)}|||^2\,, & k = j\,. \end{cases} \\ \text{where} \quad |||G_j^{(a,b)}||| &= \left(\frac{1}{(2j\,+\,a\,+\,b\,+1)} \frac{\Gamma(j+a+1)\,\Gamma(j+b+1)}{\Gamma(j+1)\,\Gamma(j+a+b+1)} \right)^{1/2} . \end{aligned} \tag{2.3}$$

$$\text{Note that} \quad |||G_j^{(a,b)}||| \ = \ |||G_j^{(b,a)}|||\,. \tag{2.4}$$

From [27, equation (2.19)] we have that

$$\frac{d^k}{dt^k} P_n^{(a,b)}(t) \;=\; \frac{\Gamma(n+k+a+b+1)}{2^k\,\Gamma(n+a+b+1)} P_{n-k}^{(a+k\,,\,b+k)}(t)\,. \tag{2.5}$$

Hence,

$$\frac{d^k}{dx^k} G_n^{(a,b)}(x) = \frac{\Gamma(n+k+a+b+1)}{\Gamma(n+a+b+1)} G_{n-k}^{(a+k\,,\,b+k)}(x)\,. \tag{2.6}$$

Also, from [27, equation (2.15)],

$$\frac{d^k}{dt^k} \Big\{(1-t)^{a+k}\,(1+t)^{b+k}\,P_{n-k}^{(a+k\,,\,b+k)}(t)\Big\} \;=\; \frac{(-1)^k\,2^k\,n!}{(n-k)!}\,(1-t)^a\,(1+t)^b\,P_n^{(a\,,\,b)}(t)\,,\ n \ge k \ge 0\,, \tag{2.7}$$

from which it follows that

$$\frac{d^k}{dx^k} \Big\{\,(1\,-\,x)^{a+k}\,x^{b+k}\,G_{n-k}^{(a+k\,,\,b+k)}(x)\Big\} \;=\; \frac{(-1)^k\,n!}{(n-k)!}\,(1\,-\,x)^a\,x^b\,G_n^{(a\,,\,b)}(x)\,. \tag{2.8}$$

For compactness of notation we introduce

$$\rho^{(a,b)} \;=\; \rho^{(a,b)}(x) \;:=\; (1-x)^a\,x^b\,. \tag{2.9}$$

We let $\mathbb{N}_0 := \mathbb{N} \cup 0$ and use $y_n \sim n^p$ to denote that there exists constants c and $C > 0$ such that, as $n \to \infty$, $c\,n^p \le |y_n| \le C\,n^p$. Additionally, we use $a \lesssim b$ to denote that there exists a constant C such that $a \le C\,b$.

For $s \in \mathbb{R}$, $\lfloor s \rfloor$ is used to denote the largest integer that is less than or equal to s, and $\lceil s \rceil$ is used to denote the smallest integer that is greater than or equal to s.

Note, from Stirling's formula we have that

$$\lim_{n\to\infty} \frac{\Gamma(n+\sigma)}{\Gamma(n)\,n^\sigma} \;=\; 1\,, \text{ for } \sigma \in \mathbb{R}. \tag{2.10}$$

Definition: **Condition A**
For α and r given, satisfying $1 < \alpha < 2$, $0 \le r \le 1$, let β be determined by $\alpha - 1 \le \beta \le 1$ and

$$r \;=\; \frac{\sin(\pi\,\beta)}{\sin(\pi(\alpha-\beta))\,+\,\sin(\pi\,\beta)}\,. \tag{2.11}$$

Furthermore, introduce the constant c_*^* defined by

$$c_*^* \;=\; \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta))\,+\,\sin(\pi\beta)}\,. \tag{2.12}$$

Function space $L^2_\omega(\mathrm{I})$.
For $\omega(x) > 0,\ x \in (0,1)$, let

$$L^2_\omega(\mathrm{I}) \;:=\; \{f(x)\,:\, \int_0^1 \omega(x)\,f(x)^2\,dx \;<\; \infty\}\,. \tag{2.13}$$

Associated with $L^2_\omega(0,1)$ is the inner product, $\langle\cdot,\cdot\rangle_\omega$, and norm, $\|\cdot\|_\omega$, defined by

$$\langle f\,,\,g\rangle_\omega \;:=\; \int_0^1 \omega(x)\,f(x)\,g(x)\,dx\,, \quad \text{and} \quad \|f\|_\omega \;:=\; (\langle f\,,\,f\rangle_\omega)^{1/2}\,.$$

The set of orthogonal polynomials $\{G_j^{(a,b)}\}_{j=0}^\infty$ form an orthogonal basis for $L^2_{\rho^{(a,b)}}(\mathrm{I})$.

3 Weighted Sobolev Spaces $H^s_{(a,b)}(\mathrm{I})$

In this section we introduce the weighted Sobolev spaces $H^s_{(a,b)}(\mathrm{I})$. These spaces differ from the usual $H^s(\mathrm{I})$ spaces in that the associated norms apply a polynomial weight at each endpont of I, namely, x^b and $(1-x)^a$. These weights increase with the order of the derivative. We give three equivalent definitions for the $H^s_{(a,b)}(\mathrm{I})$ spaces. In the first definition the spaces $H^s_{(a,b)}(\mathrm{I})$, for $s > 0 \not\in \mathbb{N}$, are defined by the K- method of interpolation. The second definition is based on the decay rate of the Jacobi coefficients of a function expanded in terms of the Jacobi polynomials $G_j^{(a,b)}(x)$. The third definition uses an explicit Sobolev-Slobodeckij type definition for the semi-norm, $|\cdot|_{H^s_{(a,b)}(\mathrm{I})}$. All three definitions are useful, and used in the analysis below.

Definition: Using Interpolation
Following Babuška and Guo [6], and Guo and Wang [21], we introduce the weighted Sobolev spaces $H^s_{\rho^{(a,b)}}(\mathrm{I})$.

Definition 3.1 *Let $s, a, b \in \mathbb{R}$, $s \geq 0$, $a, b > -1$. Then, for $n \in \mathbb{N}_0$,*

$$H^n_{\rho^{(a,b)}}(\mathrm{I}) := \Big\{ v \,:\, \sum_{j=0}^{n} \big\|D^j v\big\|^2_{\rho^{(a+j,b+j)}} < \infty \Big\}, \tag{3.1}$$

with associated norm $\|v\|_{n,\rho^{(a,b)}} := \Big(\sum_{j=0}^{n} \big\|D^j v\big\|^2_{\rho^{(a+j,b+j)}}\Big)^{1/2}$.

Definition (3.1) is extended to $s \in \mathbb{R}^+$ using the K- method of interpolation. For $s < 0$ the spaces are defined by (weighted) L^2 duality.

Babuška and Guo used the $H^s_{\rho^{(a,b)}}(\mathrm{I})$ spaces in establishing the optimal convergence properties of the p- version of the finite element method. They also related the definition (3.1) to the decay property of the coefficients of the Jacobi polynomials of the expansion of the function v. In [21] Guo and Wang derived approximation properties of Jacobi polynomials for functions in the weighted Sobolev spaces (3.1).

Definition: Using the decay rate of Jacobi coefficients
Next we define function spaces in terms of the decay property of the Jacobi coefficients of their member functions.

Introduce $\widetilde{G}_j^{(a,b)}(x) \;=\; G_j^{(a,b)}(x)/|||G_j^{(a,b)}|||$ as an orthonormal basis for $L^2_{\rho^{(a,b)}}(\mathrm{I})$.

Given v, let

$$v_j \;=\; \int_0^1 \rho^{(a,b)}(x)\,v(x)\,\widetilde{G}_j^{(a,b)}(x)\,dx\,. \tag{3.2}$$

Note that for $v \in L^2_{\rho^{(a,b)}}(\mathrm{I})$,

$$v(x) \;=\; \sum_{j=0}^{\infty} v_j \, \widetilde{G}_j^{(a,b)}(x)\,. \tag{3.3}$$

Definition 3.2 *Let $s, a, b \in \mathbb{R}$, $a, b > -1$, $L^2_{(a,b)}(\mathrm{I}) := L^2_{\rho^{(a,b)}}(\mathrm{I})$, and v_j be given by (3.2). Then, define*

$$H^s_{(a,b)}(\mathrm{I}) \;:=\; \{v \,:\, \sum_{j=0}^{\infty}(1+j^2)^s\, v_j^2 \;<\; \infty\}\,, \tag{3.4}$$

with associated norm $\|v\|_{s,(a,b)} \;:=\; \left(\sum_{j=0}^{\infty}(1+j^2)^s\, v_j^2\right)^{1/2}$, as the (a,b)-weighted Sobolev space of order s.

Theorem 3.1 *The spaces $H^s_{(a,b)}(\mathrm{I})$ and $H^s_{\rho^{(a,b)}}(\mathrm{I})$ coincide, and their corresponding norms are equivalent.*

Proof: See [13].

With the structure of the $H^s_{(a,b)}(\mathrm{I})$ spaces, and properties (2.6) and (2.3), it is straight forward to show that D is a bounded mapping from $H^s_{(a,b)}(\mathrm{I})$ onto $H^{s-1}_{(a+1\,,\,b+1)}(\mathrm{I})$.

Lemma 3.1 *For $s, a, b \in \mathbb{R}$, $a, b > -1$, the differential operator D is a bounded mapping from $H^s_{(a,b)}(\mathrm{I})$ onto $H^{s-1}_{(a+1\,,\,b+1)}(\mathrm{I})$.*

Proof: See [13].

Definition: Using a Sobolev-Slobodeckij type semi-norm
For $s > 0$, let $s \;=\; \lfloor s \rfloor \,+\, r$, where $0 < r < 1$. Let the semi-norm, $|\cdot|_{H^s_{(a,b)}(\mathrm{I})}$, and norm, $\|\cdot\|_{H^s_{(a,b)}(\mathrm{I})}$ be defined as

$$|f|^2_{H^s_{(a,b)}(\mathrm{I})} \;:=\; \iint_{\widetilde{\Lambda}} (1-x)^{a+s}\, x^{b+s}\, \frac{|D^{\lfloor s \rfloor} f(x) \;-\; D^{\lfloor s \rfloor} f(y)|^2}{|x \;-\; y|^{1\,+\,2(s-\lfloor s \rfloor)}} dy\, dx$$

$$\text{and} \quad \|f\|^2_{H^s_{(a,b)}(\mathrm{I})} \;:=\; \begin{cases} \sum_{j=0}^{s} \|D^j f\|^2_{L^2_{(a+j\,,\,b+j)}(\mathrm{I})}\,, & \text{for } s \in \mathbb{N}_0 \\ \sum_{j=0}^{\lfloor s \rfloor} \|D^j f\|^2_{L^2_{(a+j\,,\,b+j)}(\mathrm{I})} \;+\; |f|^2_{H^s_{(a,b)}(\mathrm{I})}\,, & \text{for } s \in \mathbb{R}^+ \backslash \mathbb{N}_0 \end{cases}\,,$$

where (see Figure 3.1)

$$\widetilde{\Lambda} \;:=\; \left\{(x,y) \,:\, \frac{2}{3}x < y < \frac{3}{2}x,\, 0 < x < \frac{1}{2}\right\} \cup \left\{(x,y) \,:\, \frac{3}{2}x - \frac{1}{2} \;< y <\; \frac{2}{3}x + \frac{1}{3},\, 1/2 \le x < 1\right\}. \tag{3.5}$$

Definition 3.3 *Let $s, a, b \in \mathbb{R}$, $s \ge 0$, $a, b > -1$. Then*

$$\widetilde{H}^s_{(a,b)}(\mathrm{I}) := \left\{v \,:\, \|v\|_{H^s_{(a,b)}(\mathrm{I})} < \infty\right\}. \tag{3.6}$$

Definition (3.6) is extended to $s < 0$ by (weighted) L^2 duality.

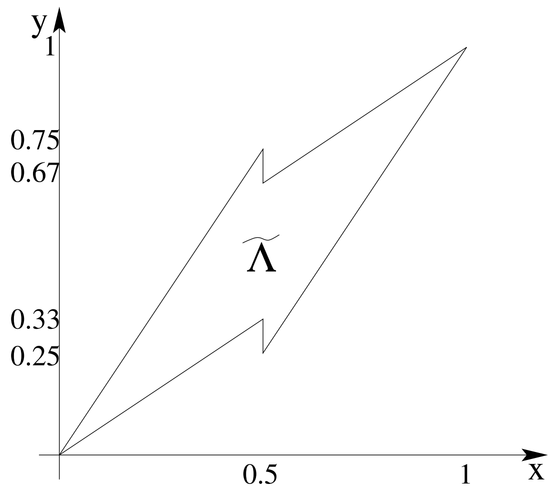

Figure 3.1: Domain $\widetilde{\Lambda}$.

Lemma 3.2 *[16] Let $s \geq 0$ with $s \neq 1+a$ if $a \in (-1,0)$ and $s \neq 1+b$ if $b \in (-1,0)$. Then, $\widetilde{H}^s_{(a,b)}(\mathrm{I}) = H^s_{(a,b)}(\mathrm{I})$.*

Proof: See [13].

For convenience, from hereon we use $H^s_{(a,b)}(\mathrm{I})$ to represent the spaces $H^s_{\rho^{(a,b)}}(\mathrm{I})$, $H^s_{(a,b)}(\mathrm{I})$, and $\widetilde{H}^s_{(a,b)}(\mathrm{I})$.

Some insight into the structure of the weighted Sobolev spaces is provided by the following two lemmas.

Lemma 3.3 *Let $u(x) = x^{\mu}$. Then, $u \in H^s_{(a,b)}(\mathrm{I})$ for $s < 2\mu + b + 1$.*

Proof: See [13].

Theorem 3.2 *(See [3, Theorem 4.14]) Let $a, b > -1$, $k \in \mathbb{N}_0$ and $s > k + 1 + \max\{a + k\,, b + k\,, -1/2\}$. Then we have a continuous embedding $H^s_{(a,b)}(\mathrm{I}) \subset C^k(\mathrm{I})$ of $H^s_{(a,b)}(\mathrm{I})$ into the Banach space $C^k(\mathrm{I})$ with the norm $\|v\|_{C^k} = \|v\|_{\infty} + \|v^{(k)}\|_{\infty}$.*

Proof: See [13].

4 Regularity of the solution to the fractional diffusion equation

In this section we give sharp regularity results for the fractional diffusion equation (1.1),(1.2).

Theorem 4.1 *For $\mathcal{L}_r^\alpha(\cdot)$ defined for $1 < \alpha < 2$, $0 \le r \le 1$, let β be determined by* ***Condition A****. Then the mapping $\mathcal{L}_r^\alpha(\cdot) \,:\, \rho^{(\alpha-\beta,\beta)}(x) \otimes H^{s+\alpha}_{(\alpha-\beta\,,\beta)}(\mathrm{I}) \to H^s_{(\beta\,,\alpha-\beta)}(\mathrm{I})$ is bijective, continuous, and has a continuous inverse.*

Proof: From [14, 24],

$$\mathcal{L}_r^\alpha\left(\rho^{(\alpha-\beta\,,\,\beta)}\,\tilde{G}_k^{(\alpha-\beta\,,\,\beta)}\right)(x) \;=\; \lambda_k\,\tilde{G}_k^{(\beta\,,\,\alpha-\beta)}(x)\,, \quad \text{where} \quad \lambda_k \;=\; -c_*^*\frac{\Gamma(k+1+\alpha)}{\Gamma(k+1)}\,,\ k=0,1,2,\ldots,$$

and c_*^* given by (2.12).

Using Stirling's formula,

$$\begin{aligned} \lambda_k^2 \;&=\; (c_*^*)^2\frac{\Gamma(k+1+\alpha)^2}{\Gamma(k+1)^2} \sim\; (k+1)^{2\alpha} \quad \text{as } k\to\infty\,, \\ &\text{i.e.,}\quad c\,(1+k^2)^\alpha \le\; \lambda_k^2 \;\le\; C\,(1+k^2)^\alpha\,. \end{aligned} \tag{4.1}$$

Let $\phi(x) \;=\; \sum_{k=0}^\infty \phi_k\tilde{G}_k^{(\alpha-\beta\,,\,\beta)}(x) \;\in\; H^{s+\alpha}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$, and $\phi_N(x) \;=\; \sum_{k=0}^N \phi_k\tilde{G}_k^{(\alpha-\beta\,,\,\beta)}(x)$. Then $\{\phi_n\}_{n=1}^\infty$ is a Cauchy sequence in $H^{s+\alpha}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$, with $\lim_{n\to\infty}\phi_n \;=\; \phi$.

Consider,

$$f_N(x) \;=\; \mathcal{L}_\alpha^r\left(\rho^{(\alpha-\beta\,,\,\beta)}\phi_N\right)(x) \;=\; \sum_{k=0}^N \lambda_k\,\phi_k\,\tilde{G}_k^{(\beta\,,\,\alpha-\beta)}(x) \;\in H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,.$$

Then using (4.1),

$$\begin{aligned} \|f_N - f_M\|^2_{s,(\beta\,,\,\alpha-\beta)} \;&=\; \sum_{k=M}^N (1+k^2)^s\;(\lambda_k\,\phi_k)^2 \;\le\; C\,\sum_{k=M}^N (1+k^2)^s\,(1+k^2)^\alpha\,\phi_k^2 \\ &=\; C\,\|\phi_N - \phi_M\|^2_{s+\alpha\,,\,(\alpha-\beta\,,\,\beta)}\,. \end{aligned}$$

Thus $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$. It then follows that $f \,:=\, \lim_{n\to\infty} f_n$ satisfies

$$\begin{aligned} f \;&=\; \mathcal{L}_r^\alpha\left(\rho^{(\alpha-\beta\,,\,\beta)}\,\phi\right)(x) \;=\; \mathcal{L}_r^\alpha\left(\rho^{(\alpha-\beta\,,\,\beta)}\,\sum_{k=0}^\infty \phi_k\tilde{G}_k^{(\alpha-\beta\,,\,\beta)}\right)(x) \\ &=\; \sum_{k=0}^\infty \phi_k\,\mathcal{L}_r^\alpha\left(\rho^{(\alpha-\beta\,,\,\beta)}\tilde{G}_k^{(\alpha-\beta\,,\,\beta)}\right)(x) \\ &=\; \sum_{k=0}^\infty \lambda_k\,\phi_k\,\tilde{G}_k^{(\beta\,,\,\alpha-\beta)}(x)\,. \end{aligned}$$

Also, using (4.1),

$$\|f\|^2_{s,(\beta\,,\,\alpha-\beta)} \le\; \sum_{k=0}^\infty (1+k^2)^s\,\lambda_k^2\,\phi_k^2 \;\le\; C\,\sum_{k=0}^\infty (1+k^2)^{s+\alpha}\,\phi_k^2 \;=\; C\,\|\phi\|^2_{s+\alpha\,,\,(\alpha-\beta\,,\beta)}\,. \tag{4.2}$$

Hence $\mathcal{L}_r^\alpha(\cdot)$ is a one-to-one, continuous mapping from $\rho^{(\alpha-\beta,\beta)}(x) \otimes H^{s+\alpha}_{(\alpha-\beta\,,\beta)}(\mathrm{I})$ into $H^s_{(\beta\,,\alpha-\beta)}(\mathrm{I})$.

Next, for $f(x) = \sum_{k=0}^{\infty} f_k \tilde{G}_k^{(\beta\,,\,\alpha-\beta)}(x) \in H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$, let $\phi(x) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k} f_k \tilde{G}_k^{(\alpha-\beta\,,\,\beta)}(x)$. Note that $\mathcal{L}_r^{\alpha}\left(\rho^{(\alpha-\beta,\beta)}\,\phi\right)(x) = f(x)$, and using a similar argument to that in (4.2), $\|\phi\|_{s+\alpha\,,\,(\alpha-\beta\,,\,\beta)} \le C\,\|f\|_{s,(\beta\,,\,\alpha-\beta)}$, from which the stated result then follows.

■

Using this theorem we obtain the following regularity result for (1.1),(1.2).

Corollary 4.1 *For* $f \in H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$ *there exists a unique solution* u *to* (1.1),(1.2)*, which can be expressed as* $u(x) = \rho^{(\alpha-\beta,\beta)}(x)\phi(x)$*, where* $\phi(x) \in H^{s+\alpha}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$*, with* $\|\phi\|_{s+\alpha\,,\,(\alpha-\beta\,,\,\beta)} \le C\,\|f\|_{s,(\beta\,,\,\alpha-\beta)}$.

■

Combining Theorem 3.2 and Corollary 4.1 we obtain a sufficient condition for the solution of (1.1),(1.2) to be continuous on I.

Corollary 4.2 *Let* $1 < \alpha < 2$*,* $0 \le r \le 1$*, and* β *be determined by* ***Condition A****. Then for* $f \in H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$ *where*

$$s \;>\; (1-\alpha) \,+\, \max\{\alpha-\beta\,,\,\beta\}\,, \tag{4.3}$$

the solution of (1.1),(1.2) *is continuous on* I.

■

Note: For $r = 0$ ($r = 1$) Corollary 4.2 implies for $f \in H^s_{(1\,,\,\alpha-1)}(\mathrm{I})$ $\left(f \in H^s_{(\alpha-1\,,\,1)}(\mathrm{I})\right)$, where $s > 2-\alpha$, that $u \in C(\mathrm{I})$.
For $r = 1/2$ Corollary 4.2 implies for $f \in H^s_{(\alpha/2\,,\,\alpha/2)}(\mathrm{I})$, where $s > 1-\alpha/2$, that $u \in C(\mathrm{I})$.

5 Regularity of the solution to the fractional diffusion, advection, reaction equation (1.3)

The $H^s_{(a,b)}(\mathrm{I})$ space a function f lies in is determined by its behavior at: (i) the left endpoint ($x = 0$), (ii) the right endpoint ($x = 1$), and (iii) away from the endpoints. In order to separate the consideration of the endpoint behaviors, following [9], we introduce the following function space $H^s_{(\gamma)}(\mathrm{J})$. Let $\mathrm{J} := (0,\,3/4)$, and

$$\begin{aligned}\Lambda^* &:= \left\{(x,y)\,:\,\frac{2}{3}x < y < \frac{3}{2}x,\ 0 < x < \frac{1}{2}\right\} \cup \left\{(x,y)\,:\,\frac{3}{2}x - \frac{1}{2} < y < \frac{2}{3}x + \frac{1}{3},\ 1/2 \le x < 3/4\right\}\\ &:= \Lambda \cup \Lambda_1 \quad \text{(see Figure 5.1)}\,.\end{aligned}$$

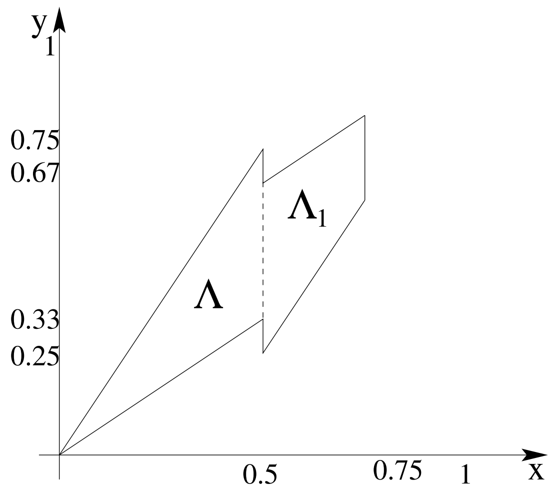

Figure 5.1: Domain $\Lambda^* = \Lambda \cup \Lambda_1$.

Introduce the semi-norm and norm

$$|f|^2_{H^s_{(\gamma)}(\mathrm{J})} := \iint_{\Lambda} x^{\gamma+s} \frac{|D^{\lfloor s \rfloor} f(x) - D^{\lfloor s \rfloor} f(y)|^2}{|x - y|^{1+2(s-\lfloor s \rfloor)}} dy\, dx + \iint_{\Lambda_1} x^{\gamma+s} \frac{|D^{\lfloor s \rfloor} f(x) - D^{\lfloor s \rfloor} f(y)|^2}{|x - y|^{1+2(s-\lfloor s \rfloor)}} dy\, dx \tag{5.1}$$

$$:= |f|^2_{H^s_{(\gamma)}(\Lambda)} + |f|^2_{H^s_{(\gamma)}(\Lambda_1)}, \tag{5.2}$$

(5.3)

$$\text{and } \ \|f\|^2_{H^s_{(\gamma)}(\mathrm{J})} := \begin{cases} \sum_{j=0}^{s} \|D^j f\|^2_{L^2_{(\gamma+j)}(\mathrm{J})}, & \text{for } s \in \mathbb{N}_0 \\ \sum_{j=0}^{\lfloor s \rfloor} \|D^j f\|^2_{L^2_{(\gamma+j)}(\mathrm{J})} + |f|^2_{H^s_{(\gamma)}(\mathrm{J})}, & \text{for } s \in \mathbb{R}^+ \backslash \mathbb{N}_0 \end{cases}, \tag{5.4}$$

(5.5)

where $\|g\|^2_{L^2_{(\gamma)}(\mathrm{J})} := \int_{\mathrm{J}} x^{\gamma} g^2(x)\, dx$.

Then, $H^s_{(\gamma)}(\mathrm{J}) := \{f : f \text{ is measurable and } \|f\|_{H^s_{(\gamma)}(\mathrm{J})} < \infty\}$.

Note: A function $f(x)$ is in $H^s_{(a,b)}(\mathrm{I})$ if and only if $f(\frac{3}{4}x) \in H^s_{(b)}(\mathrm{J})$ and $f(\frac{3}{4}(1-x)) \in H^s_{(a)}(\mathrm{J})$.

The following lemma and discussion allows us to further focus our analysis for determining the regularity of $(1-x)^{\alpha-\beta} x^{\beta} \phi(x)$.

Lemma 5.1 *Let $s \geq 0$, $\psi \in H^s_{(\gamma)}(\mathrm{J})$, and $g \in C^{\lceil s \rceil}(\mathrm{J})$. Then*

$$\|g\,\psi\|^2_{H^s_{(\gamma)}(\mathrm{J})} \lesssim \|g\|^2_{C^{\lceil s \rceil}(\mathrm{J})} \|\psi\|^2_{H^s_{(\gamma)}(\mathrm{J})}.$$

Proof: For $s = 0$,

$$\|g\,\psi\|^2_{H^s_{(\gamma)}(\mathrm{J})} = \|g\,\psi\|^2_{L^2_{(\gamma)}(\mathrm{J})} \leq \|g\|^2_{L^\infty(\mathrm{J})} \|\psi\|^2_{L^2_{(\gamma)}(\mathrm{J})}. \tag{5.6}$$

Then, for $s = 1$,

$$\begin{aligned}
\|g\,\psi\|^2_{H^s_{(\gamma)}(\mathrm{J})} \le\ & \|g\,\psi\|^2_{L^2_{(\gamma)}(\mathrm{J})} \ +\ \|D(g\,\psi)\|^2_{L^2_{(\gamma+1)}(\mathrm{J})} \\
\lesssim\ & \|g\|^2_{L^\infty(\mathrm{J})}\,\|\psi\|^2_{L^2_{(\gamma)}(\mathrm{J})} \ +\ \|\psi\,Dg\|^2_{L^2_{(\gamma+1)}(\mathrm{J})} \ +\ \|g\,D\psi\|^2_{L^2_{(\gamma+1)}(\mathrm{J})} \\
\lesssim\ & \|g\|^2_{L^\infty(\mathrm{J})}\,\|\psi\|^2_{L^2_{(\gamma)}(\mathrm{J})} \ +\ \|Dg\|^2_{L^\infty(\mathrm{J})}\,\|\psi\|^2_{L^2_{(\gamma)}(\mathrm{J})} \ +\ \|g\|^2_{L^\infty(\mathrm{J})}\,\|D\psi\|^2_{L^2_{(\gamma+1)}(\mathrm{J})} \\
\lesssim\ & \|g\|^2_{C^1(\mathrm{J})}\,\|\psi\|^2_{H^1_{(\gamma)}(\mathrm{J})}\,. \qquad (5.7)
\end{aligned}$$

Next, for $0 < s < 1$, consider the mapping $\mathcal{F} : H^s_{(\gamma)}(\mathrm{J}) \longrightarrow H^s_{(\gamma)}(\mathrm{J})$, defined by $\mathcal{F}(\psi) := g\,\psi$. From (5.6) and (5.7) we have that $\mathcal{F}$ is a bounded mapping for $s = 0$ and $s = 1$. As $H^s_{(\gamma)}(\mathrm{J})$ is a family of interpolation spaces, it follows that $\mathcal{F}$ is a bounded mapping for $0 < s < 1$, with $\|\mathcal{F}\| \lesssim \|g\|_{C^1(\mathrm{J})}$.

For $s > 1$ the above argument extends in a straight forward manner.

■

Consider $h(x) = g(x)\psi(x)$, where $\psi(x) \in H^s_{(\mu)}(\mathrm{J})$ and $g(x) = \begin{cases} x^\alpha, & \frac{1}{2} \le x \le \frac{3}{4} \\ g_{ext}(x), & 0 \le x \le \frac{1}{2} \end{cases}$, for $g_{ext}(x)$ a $C^{\lceil s \rceil}(0\,,\,\frac{1}{2})$ extension of x^α satisfying $\|g\|_{C^{\lceil s \rceil}(\mathrm{J})} \ \le\ \|x^\alpha\|_{C^{\lceil s \rceil}[\frac{1}{2},\frac{3}{4}]}$.

Note that for $t \le s$, $t \notin \mathbb{N}$,

$$\begin{aligned}
\|x^\alpha\,\psi(x)\|^2_{H^t_{(\sigma)}(\mathrm{J})} \lesssim\ & \sum_{j=0}^{\lfloor t \rfloor} \|D^j(x^\alpha\,\psi(x))\|^2_{L^2_{(\sigma+j)}(\mathrm{J})} \ +\ \iint_\Lambda x^{\sigma+t}\,\frac{|D^{\lfloor t \rfloor}(x^\alpha\,\psi(x)) \,-\, D^{\lfloor t \rfloor}(y^\alpha\,\psi(y))|^2}{|x \,-\, y|^{1\,+\,2(t-\lfloor t \rfloor)}}dy\,dx \\
& +\ \iint_{\Lambda_1} x^{\mu+t}\,\frac{|D^{\lfloor t \rfloor}(x^\alpha\,\psi(x)) \,-\, D^{\lfloor t \rfloor}(y^\alpha\,\psi(y))|^2}{|x \,-\, y|^{1\,+\,2(t-\lfloor t \rfloor)}}dy\,dx \\
\lesssim\ & \sum_{j=0}^{\lfloor t \rfloor} \|D^j(x^\alpha\,\psi(x))\|^2_{L^2_{(\sigma+j)}(\mathrm{J})} \ +\ \iint_\Lambda x^{\sigma+t}\,\frac{|D^{\lfloor t \rfloor}(x^\alpha\,\psi(x)) \,-\, D^{\lfloor t \rfloor}(y^\alpha\,\psi(y))|^2}{|x \,-\, y|^{1\,+\,2(t-\lfloor t \rfloor)}}dy\,dx \\
& +\ \|h\|^2_{H^t_{(\mu)}(\mathrm{J})} \\
\lesssim\ & \sum_{j=0}^{\lfloor t \rfloor} \|D^j(x^\alpha\,\psi(x))\|^2_{L^2_{(\sigma+j)}(\mathrm{J})} \ +\ \iint_\Lambda x^{\sigma+t}\,\frac{|D^{\lfloor t \rfloor}(x^\alpha\,\psi(x)) \,-\, D^{\lfloor t \rfloor}(y^\alpha\,\psi(y))|^2}{|x \,-\, y|^{1\,+\,2(t-\lfloor t \rfloor)}}dy\,dx \\
& +\ \|\psi\|^2_{H^s_{(\mu)}(\mathrm{J})} \quad \text{(using Lemma 5.1).}
\end{aligned}$$

Hence for the analysis of the regularity of $f(x) = (1-x)^{\alpha-\beta}x^\beta\phi(x)$ we can restrict our attention on the analysis of the semi-norm to $|f|_{H^t_{(\sigma)}(\Lambda)}$.

The regularity of the solution to (1.3) can be influenced by the regularity of the coefficients $b(x)$ and $c(x)$. The following two lemmas enables us to insulate the influence of these terms.

Introduce the space $W^{k,\infty}_w(\mathrm{I})$ and its associated norm, defined for $k \in \mathbb{N}_0$, as

$$W^{k,\infty}_w(\mathrm{I}) := \Big\{ f \ :\ (1-x)^{j/2}x^{j/2}D^j f(x) \in L^\infty(\mathrm{I}),\ j = 0, 1, \ldots, k \Big\}\,, \qquad (5.8)$$

$$\|f\|_{W^{k,\infty}_w} := \max_{0 \le j \le k} \|(1-x)^{j/2}x^{j/2}D^j f(x)\|_{L^\infty(\mathrm{I})}\,. \qquad (5.9)$$

The subscript w denotes the fact that $W^{k,\infty}_w(\mathrm{I})$ is a weaker space than $W^{k,\infty}(\mathrm{I})$ in that the derivative of functions in $W^{k,\infty}_w(\mathrm{I})$ may be unbounded at the endpoints of the interval.

Lemma 5.2 *Let $\alpha,\, \beta > -1$, $0 \le s \le k \in \mathbb{N}_0$, and $f \in W^{k,\infty}_w(\mathrm{I})$. Then, for*

$$g \in H^s_{(\alpha,\beta)}(\mathrm{I}) \text{ we have that } \ fg \in H^s_{(\alpha,\beta)}(\mathrm{I}) \,. \tag{5.10}$$

Proof: We establish Lemma 5.2 for $s = 0$, $s = 1$, and $0 < s < 1$. The proof extends in an obvious manner for $s > 1$.
If $s = 0$ then $k \ge 0$, and

$$\begin{aligned} \|fg\|^2_{H^s_{(\alpha,\beta)}(\mathrm{I})} = \ \|fg\|^2_{H^0_{(\alpha,\beta)}(\mathrm{I})} \ &= \int_0^1 (1-x)^\alpha x^\beta \, (f(x)g(x))^2 \, dx \\ &\lesssim \|f\|^2_{L^\infty(\mathrm{I})} \, \|g\|^2_{H^0_{(\alpha,\beta)}(\mathrm{I})} \ = \ \|f\|^2_{W^{0,\infty}_w(\mathrm{I})} \, \|g\|^2_{H^0_{(\alpha,\beta)}(\mathrm{I})} \,. \end{aligned} \tag{5.11}$$

Hence (5.10) is established for $s = 0$.

For $s = 1$ then $k \ge 1$, and

$$\begin{aligned} \|fg\|^2_{H^1_{(\alpha,\beta)}(\mathrm{I})} &= \ \|fg\|^2_{L^2_{(\alpha,\beta)}(\mathrm{I})} \ + \ \|D(fg)\|^2_{L^2_{(\alpha+1\,,\,\beta+1)}(\mathrm{I})} \\ &= \int_0^1 (1-x)^\alpha x^\beta \, \left(f(x)g(x)\right)^2 \, dx \ + \int_0^1 (1-x)^{\alpha+1} x^{\beta+1} \, \left(g(x)\, Df(x) \ + \ f(x)\, Dg(x)\right)^2 \, dx \\ &\le \int_0^1 f(x)^2 \, (1-x)^\alpha x^\beta \, \left(g(x)\right)^2 \, dx \ + \ 2 \int_0^1 (1-x)^1 x^1 \, \left(Df(x)\right)^2 \, (1-x)^\alpha x^\beta \, \left(g(x)\right)^2 dx \\ &\qquad\qquad + \ 2 \int_0^1 f(x)^2 \, (1-x)^{\alpha+1} x^{\beta+1} \, \left(Dg(x)\right)^2 dx \\ &\lesssim \left(\|f\|^2_{L^\infty(\mathrm{I})} \ + \ \|(1-x)^{1/2} x^{1/2} \, Df(x)\|^2_{L^\infty(\mathrm{I})}\right) \left(\|g\|^2_{L^2_{(\alpha,\beta)}(\mathrm{I})} \ + \ \|Dg\|^2_{L^2_{(\alpha+1\,,\,\beta+1)}(\mathrm{I})}\right) \\ &= \ \|f\|^2_{W^{1,\infty}_w(\mathrm{I})} \, \|g\|^2_{H^1_{(\alpha,\beta)}(\mathrm{I})} \,. \end{aligned} \tag{5.12}$$

Hence (5.10) is established for $s = 1$.

Note that for $f \in W^{1,\infty}_w(\mathrm{I})$, from (5.11), $\|fg\|_{H^0_{(\alpha,\beta)}(\mathrm{I})} \ \le \ \|f\|_{W^{1,\infty}_w(\mathrm{I})} \, \|g\|_{H^0_{(\alpha,\beta)}(\mathrm{I})}$. Combining this with (5.12) and the fact that $H^s_{(\alpha,\beta)}(\mathrm{I})$ are interpolation spaces, it follows that for $0 < s < 1$, $\|fg\|_{H^s_{(\alpha,\beta)}(\mathrm{I})} \ \le \ \|f\|_{W^{1,\infty}_w(\mathrm{I})} \, \|g\|_{H^s_{(\alpha,\beta)}(\mathrm{I})}$.

■

Arising in the analysis is the product of a function and a functional (e.g., $b(x)\, Du(x)$). To define such a product it is convenient to consider $L^2_{(\alpha,\beta)}(\mathrm{I}) = H^0_{(\alpha,\beta)}(\mathrm{I})$ as the pivot space for $H^s_{(\alpha,\beta)}(\mathrm{I})$ and $H^{-s}_{(\alpha,\beta)}(\mathrm{I})$, with $H^{-s}_{(\alpha,\beta)}(\mathrm{I})$ characterized as the closure of $L^2_{(\alpha,\beta)}(\mathrm{I})$ with respect to the operator norm

$$|||v||| \ := \ \sup_{h \in H^s_{(\alpha,\beta)}(\mathrm{I})} \frac{\langle v \,,\, h \rangle_{\rho^{(\alpha,\beta)}}}{\|h\|_{H^s_{(\alpha,\beta)}(\mathrm{I})}} \,.$$

Definition: Product of a function and a functional.
Let $g \in H^{-s}_{(\alpha,\beta)}(\mathrm{I})$. Then there exists $\{g_i\}_{i=1}^\infty \subset L^2_{(\alpha,\beta)}(\mathrm{I})$, such that $\lim_{i\to\infty} |||g - g_i||| \ = \ 0$. Thus, for any $h \in H^s_{(\alpha,\beta)}(\mathrm{I})$, $g(h) \ = \ \lim_{i\to\infty} \langle g_i \,,\, h \rangle_{\rho^{(\alpha,\beta)}}$.

For $f \in W_w^{k,\infty}(\mathrm{I})$, let fg be defined as

$$fg(h) \ := \ \lim_{i \to \infty} \langle f g_i \, , \, h \rangle_{\rho^{(\alpha,\beta)}} \, . \tag{5.13}$$

Lemma 5.3 *Let α, $\beta > -1$, $0 \le s \le k \in \mathbb{N}_0$, and $f \in W_w^{k,\infty}(\mathrm{I})$. Then, for*

$$g \in H^{-s}_{(\alpha,\beta)}(\mathrm{I}) \ \text{we have that} \ \ fg \in H^{-s}_{(\alpha,\beta)}(\mathrm{I}). \tag{5.14}$$

Proof: To establish that fg is well defined and contained in $H^{-s}_{(\alpha,\beta)}(\mathrm{I})$, we show that $\{f g_i\}_{i=1}^{\infty}$ is a Cauchy sequence with respect to the norm $|||\cdot|||$.

As (5.14) trivially holds for $f = 0$, assume $0 \ne f \in W_w^{k,\infty}(\mathrm{I})$. Then,

$$\begin{aligned}
|||f g_i \, - \, f g_j||| = \ & \sup_{h \in H^s_{(\alpha,\beta)}(\mathrm{I})} \frac{\langle f g_i \, - \, f g_j \, , \, h \rangle_{\rho^{(\alpha,\beta)}}}{\|h\|_{H^s_{(\alpha,\beta)}(\mathrm{I})}} \ = \ \|f\|_{W_w^{k,\infty}(\mathrm{I})} \sup_{h \in H^s_{(\alpha,\beta)}(\mathrm{I})} \frac{\langle g_i - g_j \, , \, f h \rangle_{\rho^{(\alpha,\beta)}}}{\|f\|_{W_w^{k,\infty}(\mathrm{I})} \, \|h\|_{H^s_{(\alpha,\beta)}(\mathrm{I})}} \\
\le \ & \|f\|_{W_w^{k,\infty}(\mathrm{I})} \sup_{h \in H^s_{(\alpha,\beta)}(\mathrm{I})} \frac{\langle g_i - g_j \, , \, f h \rangle_{\rho^{(\alpha,\beta)}}}{\|f h\|_{H^s_{(\alpha,\beta)}(\mathrm{I})}} \quad \text{(using Lemma 5.2)}, \\
\le \ & \|f\|_{W_w^{k,\infty}(\mathrm{I})} \sup_{\tilde{h} \in H^s_{(\alpha,\beta)}(\mathrm{I})} \frac{\langle g_i - g_j \, , \, \tilde{h} \rangle_{\rho^{(\alpha,\beta)}}}{\|\tilde{h}\|_{H^s_{(\alpha,\beta)}(\mathrm{I})}} \\
\le \ & \|f\|_{W_w^{k,\infty}(\mathrm{I})} \, |||g_i - g_j||| \, .
\end{aligned}$$

As $\{g_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $H^{-s}_{(\alpha,\beta)}(\mathrm{I})$, then it follows that $\{f g_i\}_{i=1}^{\infty}$ is also a Cauchy sequence with limit in $H^{-s}_{(\alpha,\beta)}(\mathrm{I})$. Hence, (5.13) defines a linear functional in $H^{-s}_{(\alpha,\beta)}(\mathrm{I})$.

■

A key tool used in the boot strapping argument used to establish the regularity of the solution (1.3) is given in the following theorem. The lengthy proof of this theorem is given in Section 7.

Theorem 5.1 *Let $s \ge 0$, $\mu > -1$, and $\psi \in H^s_{(\mu)}(\mathrm{J})$. Then $x^p \, \psi \in H^t_{(\sigma)}(\mathrm{J})$ provided*

$$0 \le t \le s \, , \quad \sigma \, + \, 2p \, \ge \, \mu \, , \quad \sigma \, + \, 2p \, - t \, > \, -1 \, , \quad \text{and} \quad \sigma \, + \, 2p \, + t \, \ge \, \mu \, + \, s \, . \tag{5.15}$$

Additionally, when (5.15) is satisfied, there exists $C > 0$ (independent of ψ) such that $\|x^p \, \psi\|_{H^t_{(\sigma)}(\mathrm{J})} \le C \, \|\psi\|_{H^s_{(\mu)}(\mathrm{J})}$.

Proof: See Section 7.

We are now in a position to establish the regularity of the solution (1.3)

Theorem 5.2 *Let $s > -1$, β be determined by **Condition A**, $c \in W_w^{\lceil \min\{s \, , \, \alpha \, + \, (\alpha - \beta) \, + \, 1 \, , \, \alpha \, + \, \beta \, + \, 1\} \rceil, \infty}(\mathrm{I})$ satisfying $c(x) \ge 0$ and*

$$f \in H^{-\alpha/2}(\mathrm{I}) \cap H^s_{(\beta \, , \, \alpha - \beta)}(\mathrm{I}). \tag{5.16}$$

Then there exists a unique solution $u(x) = (1 - x)^{\alpha - \beta} \, x^{\beta} \, \phi(x)$, with $\phi(x) \in H^{\alpha \, + \, \min\{s \, , \, \alpha \, + \, (\alpha - \beta) \, + \, 1 - \epsilon \, , \, \alpha \, + \, \beta \, + \, 1 - \epsilon\}}_{(\alpha - \beta \, , \, \beta)}(\mathrm{I})$ for arbitrary $\epsilon > 0$, to

$$\mathcal{L}_r^{\alpha} u(x) \ + \ c(x) \, u(x) \ = \ f(x) \, , \ x \in \mathrm{I}, \quad \text{subject to } u(0) = u(1) = 0 \, . \tag{5.17}$$

Proof: The stated result is established using two steps. In Step 1 existence of a solution $u \in H_0^{\alpha/2}(\mathrm{I})$ to (5.17) is shown. Then, in Step 2 a boot strapping argument is applied to improve the regularity of u.

Step 1: For f satisfying (5.16), from [15], there exists a unique solution $u \in H_0^{\alpha/2}(\mathrm{I})$ to (5.17).

Step 2: For $u \in H_0^{\alpha/2}(\mathrm{I})$, from [19, Theorem 1.2.16], $u(x) \ = \ (1-x)^{\alpha/2}\, x^{\alpha/2}\, g(x)$, where $g \in L^2(\mathrm{I})$. Then, as for $0 < \epsilon \le \min\{\alpha - \beta\,,\, \beta\}$

$$\int_0^1 (1-x)^{-1+\epsilon}\, x^{-1+\epsilon}\, (u(x))^2 \, dx \ = \ \int_0^1 (1-x)^{-1+\epsilon+\alpha}\, x^{-1+\epsilon+\alpha}\, (g(x))^2 \, dx \ < \ \int_0^1 (g(x))^2 \, dx \ < \ \infty\,,$$

it follows that $u \in H^0_{(-1+\epsilon\,,\,-1+\epsilon)}(\mathrm{I}) \subset H^0_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I}) \subset H^0_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$, and using Lemma 5.2 (with the association $g(x) = u(x) \in H^0_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$, $f(x) = c(x) \in W^{0,\infty}_w(\mathrm{I})$)

$$c(x)\, u(x) \in H^0_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,. \tag{5.18}$$

Using (5.18), the solution u of (5.17) satisfies

$$\mathcal{L}_r^{\alpha} u(x) \ = \ f(x) \ - \ c(x)\, u(x) \ := \ f_1(x) \in H^{\min\{s,0\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,.$$

From Corollary 4.1 it follows that

$$u(x) \ = \ (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi_1(x)\,, \ \text{where } \phi_1 \in H^{\alpha+\min\{s\,,\,0\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

Using Theorem 5.1 (with its parameters s, μ, p, σ replaced by $\min\{s{+}\alpha\,,\,\alpha\}$, $\alpha{-}\beta$, $\alpha{-}\beta$, β, respectively; and in the second instance, with its parameters s, μ, p, σ replaced by $\min\{s{+}\alpha\,,\,\alpha\}$, β, β, $\alpha-\beta$, respectively) we have that, for arbitrary $\epsilon > 0$, $u \in H^{\min\{s+\alpha\,,\,\alpha\,,\,\alpha+(\alpha-\beta)+1-\epsilon\,,\,\alpha+\beta+1-\epsilon\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$ and using Lemma 5.2,

$$c(x)\, u(x) \in H^{\min\{s\,,\,\alpha\,,\,\alpha+(\alpha-\beta)+1-\epsilon\,,\,\alpha+\beta+1-\epsilon\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I}) \tag{5.19}$$

Again, using that the solution u of (5.17) satisfies

$$\mathcal{L}_r^{\alpha} u(x) \ = \ f(x) \ - \ c(x)\, u(x) \ := \ f_2(x) \in H^{\min\{s\,,\,\alpha\,,\,\alpha+(\alpha-\beta)+1-\epsilon\,,\,\alpha+\beta+1-\epsilon\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,,$$

and from Corollary 4.1,

$$u(x) \ = \ (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi_2(x)\,, \ \text{where } \phi_2 \in H^{\alpha+\min\{s\,,\,\alpha\,,\,\alpha+(\alpha-\beta)+1-\epsilon\,,\,\alpha+\beta+1-\epsilon\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

Using Theorem 5.1 (with its parameters s, μ, p, σ replaced by $\min\{s+\alpha\,,\,2\alpha\}$, $\alpha-\beta$, $\alpha-\beta$, β, respectively; and in the second instance, with its parameters s, μ, p, σ replaced by $\min\{s+\alpha\,,\,2\alpha\}$, β, β, $\alpha-\beta$, respectively), and Lemma 5.2,

$$c(x)\, u(x) \in H^{\min\{s\,,\,2\alpha\,,\,\alpha+(\alpha-\beta)+1-\epsilon\,,\,\alpha+\beta+1-\epsilon\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,,$$

from which it then follows that

$$u(x) \ = \ (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi_2(x)\,, \ \text{where } \phi_2 \in H^{\alpha+\min\{s\,,\,2\alpha\,,\,\alpha+(\alpha-\beta)+1-\epsilon\,,\,\alpha+\beta+1-\epsilon\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

Noting that $4\alpha \geq \min\{\alpha + (\alpha - \beta) + 1\,,\, \alpha + \beta + 1\}$ for $1 < \alpha < 2$, repeating the boot strapping argument two more times establishes the stated result.

■

The inclusion of an advection term can significantly reduced the regularity of the solution.

Theorem 5.3 *Let $s > -1$, β be determined by **Condition A**,* $b \in W_w^{\max\{1, \lceil \min\{s\,,\, \alpha + (\alpha-\beta) - 1\,,\, \alpha + \beta - 1\}\rceil\},\infty}(\mathrm{I})$, $\quad c \in W_w^{\lceil \min\{s\,,\, \alpha + (\alpha-\beta) - 1\,,\, \alpha + \beta - 1\}\rceil,\infty}(\mathrm{I})$ *satisfying* $c(x) - 1/2Db(x) \geq 0$, *and*

$$f \in H^{-\alpha/2}(\mathrm{I}) \cap H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I}). \tag{5.20}$$

Then there exists a unique solution $u(x) = (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi(x)$, *with* $\phi(x) \in H^{\alpha + \min\{s\,,\, \alpha + (\alpha-\beta) - 1 - \epsilon\,,\, \alpha + \beta - 1 - \epsilon\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$ *for arbitrary $\epsilon > 0$, to*

$$\mathcal{L}_r^{\alpha} u(x) \;+\; b(x)\, Du(x) \;+\; c(x)\, u(x) \;=\; f(x)\,,\;\; x \in \mathrm{I}, \quad \textit{subject to } u(0) = u(1) = 0\,. \tag{5.21}$$

Proof: The proof follows the same two steps as in Theorem . Step 1, establishing the existence of a solution is exactly the same. In Step 2 the boot strapping argument is applied m times, where m is the least integer such that $m(\alpha - 1) \geq \min\{\alpha + (\alpha - \beta)\,,\, \alpha + \beta\}$, to obtain the stated result.

Step 2: For $u \in H_0^{\alpha/2}(\mathrm{I})$ and $0 < \epsilon \leq \min\{\alpha - \beta\,,\, \beta\}$, $u \in H^0_{(-1+\epsilon\,,\,-1+\epsilon)}(\mathrm{I}) \subset H^0_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I})$. Then using Lemma 3.1, $Du \in H^{-1}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$. Hence we have using (5.18) and (5.14),

$$c(x)\, u(x) \in H^0_{(\beta\,,\,\alpha-\beta)}(\mathrm{I}) \;\text{ and }\; b(x)\, Du(x) \in H^{-1}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,.$$

This leads to the solution of (5.21) satisfying

$$\mathcal{L}_r^{\alpha} u(x) \;=\; f(x) \;-\; b(x)\, Du(x) \;-\; c(x)\, u(x) \;:=\; f_1(x) \in H^{\min\{s,-1\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,.$$

From Corollary 4.1 it follows that

$$u(x) \;=\; (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi_1(x)\,,\; \text{ where } \phi_1 \in H^{\alpha + \min\{s\,,\,-1\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

Using Theorem 5.1 (with its parameters s, μ, p, σ replaced by $\min\{s + \alpha\,,\, \alpha - 1\}$, $\alpha - \beta$, $\alpha - \beta$, $\beta - 1$, respectively; and in the second instance, with its parameters s, μ, p, σ replaced by $\min\{s + \alpha\,,\, \alpha - 1\}$, β, β, $\alpha - \beta - 1$, respectively) we have that, for arbitrary $\epsilon > 0$, $u \in H^{\min\{s+\alpha\,,\,\alpha-1\,,\,\alpha+(\alpha-\beta)-\epsilon\,,\,\alpha+\beta-\epsilon\}}_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I}) \;=\; H^{\min\{s+\alpha\,,\,\alpha-1\}}_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I})$ and using Lemmas 5.2 and 3.1

$$Du(x) \in H^{-1+\min\{s+\alpha\,,\,\alpha-1\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I}),\; c(x)\, u(x) \in H^{\min\{s\,,\,\alpha-1\}}_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I}) \;\text{ and }\; b(x)\, Du(x) \in H^{\min\{s\,,\,\alpha-2\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,. \tag{5.22}$$

The solution u of (5.17) then must satisfies

$$\mathcal{L}_r^{\alpha} u(x) \;=\; f(x) \;-\; b(x)\, Du(x) \;-\; c(x)\, u(x) \;:=\; f_2(x) \in H^{\min\{s\,,\,\alpha-2\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,,$$

and from Corollary 4.1,

$$u(x) \;=\; (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi_2(x)\,,\; \text{ where } \phi_2 \in H^{\alpha + \min\{s\,,\,\alpha-2\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

Using Theorem 5.1 (with its parameters s, μ, p, σ replaced by $min\{s+\alpha\,,\,2\alpha-2\}$, $\alpha-\beta$, $\alpha-\beta$, $\beta-1$, respectively; and in the second instance, with its parameters s, μ, p, σ replaced by $\min\{s+\alpha\,,\,2\alpha-2\}$, β, β, $\alpha-\beta-1$, respectively), we have that $u \in H^{\min\{s+\alpha\,,\,2\alpha-2\,,\,\alpha+(\alpha-\beta)-\epsilon\,,\,\alpha+\beta-\epsilon\}}_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I})$, and $Du(x) \in H^{-1+\min\{s+\alpha\,,\,2\alpha-2\,,\,\alpha+(\alpha-\beta)-\epsilon\,,\,\alpha+\beta-\epsilon\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$,

$$c(x)\,u(x) \in H^{\min\{s\,,\,2\alpha-2\,,\,\alpha+(\alpha-\beta)-\epsilon\,,\,\alpha+\beta-\epsilon\}}_{(\beta-1\,,\,\alpha-\beta-1)}(\mathrm{I}) \ \text{ and } \ b(x)\,Du(x) \in H^{\min\{s\,,\,2\alpha-3\,,\,\alpha+(\alpha-\beta)-1-\epsilon\,,\,\alpha+\beta-1-\epsilon\}}_{(\beta\,,\,\alpha-\beta)-1}(\mathrm{I})\,. \tag{5.23}$$

The solution u of (5.17) then must satisfies

$$\mathcal{L}_r^{\alpha} u(x) \ = \ f(x) \ - \ b(x)\,Du(x) \ - \ c(x)\,u(x) \ := \ f_2(x) \in H^{\min\{s\,,\,2\alpha-3\,,\,\alpha+(\alpha-\beta)-1-\epsilon\,,\,\alpha+\beta-1-\epsilon\}}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})\,,$$

and from Corollary 4.1,

$$u(x) \ = \ (1-x)^{\alpha-\beta}\,x^{\beta}\,\phi_3(x)\,, \ \text{ where } \phi_3 \in H^{\alpha+\min\{s\,,\,2\alpha-3\,,\,\alpha+(\alpha-\beta)-1-\epsilon\,,\,\alpha+\beta-1-\epsilon\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

Repeatedly applying this boot stepping procedure we obtain after $(m-2)$ additional steps $u(x) = (1-x)^{\alpha-\beta}\,x^{\beta}\,\phi_{m+1}(x)$, where

$$\phi_{m+1} \in H^{\alpha+\min\{s\,,\,m\alpha-(m+1)\,,\,\alpha+(\alpha-\beta)-1-\epsilon\,,\,\alpha+\beta-1-\epsilon\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I}) \ = \ H^{\alpha+\min\{s\,,\,\alpha+(\alpha-\beta)-1-\epsilon\,,\,\alpha+\beta-1-\epsilon\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})\,.$$

■

6 Regularity of the solution to the fractional diffusion, advection, reaction equation in unweighted Hilbert spaces

To connect the regularity results for the solution of (1.3), given in Theorems 5.2 and 5.3 to the usual (unweighted) Hilbert spaces we use four steps. Step 1 uses Theorem 5.1 to determine $H^t_{(\sigma)}(\mathrm{J})$ such that $u_0(x) = x^{\beta}\psi_0(x) \in H^t_{(\sigma)}(\mathrm{J})$, for $\psi_0(x) \in H^{s^*}_{(\beta)}(\mathrm{J})$. Step 2 applies an embedding theorem (Corollary 6.1) to then obtain $u_0 \in H^{v_0}(\mathrm{J})$. Step 3 repeats Steps 1 and 2 for $u_1(x) = x^{\alpha-\beta}\psi_1(x)$, for $\psi_1(x) \in H^{s^*}_{(\alpha-\beta)}(\mathrm{J})$ to obtain $u_1 \in H^{v_1}(\mathrm{J})$. The final step combines Steps 2 and 3 to conclude that $u \in H^{\min\{v_0,v_1\}}(\mathrm{I})$.

To begin we introduce the space $W^{s,2}_{(c,d)}(\mathrm{I}) := \{f : f$ is measurable and $\|f\|_{W^{s,2}_{(c,d)}(\mathrm{I})} < \infty\}$, where

$$\|f\|^2_{W^{s,2}_{(c,d)}(\mathrm{I})} := \begin{cases} \sum_{j=0}^{s} \|D^j f\|^2_{L^2_{(c\,,\,d)}(\mathrm{I})}\,, & \text{for } s \in \mathbb{N}_0 \\ \sum_{j=0}^{\lfloor s\rfloor} \|D^j f\|^2_{L^2_{(c\,,\,d)}(\mathrm{I})} \ + \ |f|^2_{W^{s,2}_{(c,d)}(\mathrm{I})}\,, & \text{for } s \in \mathbb{R}^+\backslash\mathbb{N}_0 \end{cases}\,,$$

$$\text{for } \ |f|^2_{W^{s,2}_{(c,d)}(\mathrm{I})} := \iint_{\widetilde{\Lambda}} (1-x)^c\,x^d\,\frac{|D^{\lfloor s\rfloor}f(x) \ - \ D^{\lfloor s\rfloor}f(y)|^2}{|x \ - \ y|^{1+2(s-\lfloor s\rfloor)}}\,dy\,dx\,,$$

and $\widetilde{\Lambda}$ as defined in (3.5).

Following (5.4), also introduce $W^{s,2}_{(\delta)}(\mathrm{J}) := \{f : f \text{ is measurable and } \|f\|_{W^{s,2}_{(\delta)}(\mathrm{J})} < \infty\}$, where

$$\|f\|^2_{W^{s,2}_{(\delta)}(\mathrm{J})} := \begin{cases} \sum_{j=0}^{s} \|D^j f\|^2_{L^2_{(\delta)}(\mathrm{J})}, & \text{for } s \in \mathbb{N}_0 \\ \sum_{j=0}^{\lfloor s\rfloor} \|D^j f\|^2_{L^2_{(\delta)}(\mathrm{J})} \;+\; |f|^2_{W^{s,2}_{(\delta)}(\mathrm{J})}, & \text{for } s \in \mathbb{R}^+\backslash\mathbb{N}_0 \end{cases},$$

$$\text{and } \; |f|^2_{W^{s,2}_{(\delta)}(\mathrm{J})} := \iint_{\Lambda} x^{\delta} \frac{|D^{\lfloor s\rfloor} f(x) - D^{\lfloor s\rfloor} f(y)|^2}{|x-y|^{1+2(s-\lfloor s\rfloor)}} dy\, dx \;+\; \iint_{\Lambda_1} x^{\delta} \frac{|D^{\lfloor s\rfloor} f(x) - D^{\lfloor s\rfloor} f(y)|^2}{|x-y|^{1+2(s-\lfloor s\rfloor)}} dy\, dx\,.$$

From [9] we have the following embedding result.

Theorem 6.1 *[9, See Theorem 1.d.2] Let μ, $\sigma > -1$, and v, w be two real numbers such that $0 \le v \le w$. Then, if*

$$\begin{cases} v - \frac{\sigma}{2} & < & w - \frac{\mu}{2} \\ \;\; or & & \\ v - \frac{\sigma}{2} & = & w - \frac{\mu}{2} \quad with \quad w - \frac{\mu}{2} - \frac{1}{2} \notin \mathbb{N} \end{cases}, \tag{6.1}$$

we have $W^{w,2}_{(\mu)}(\mathrm{J}) \subset W^{v,2}_{(\sigma)}(\mathrm{J})$.

Corollary 6.1 *Let $\gamma > -1$, and v, w be two real numbers such that $0 \le v \le w$. Then, if*

$$\begin{cases} v & < & \frac{w-\gamma}{2} \\ \;\; or & & \\ v & = & \frac{w-\gamma}{2} \quad with \quad \frac{w-\gamma}{2} - \frac{1}{2} \notin \mathbb{N} \end{cases}, \tag{6.2}$$

we have $H^w_{(\gamma)}(\mathrm{J}) \subset H^v(\mathrm{J})$.

Proof: From [29, Theorem 3.3], it follows that $H^s_{(a,b)}(\mathrm{I})$ and $W^{s,2}_{(a+s,b+s)}(\mathrm{I})$ are equivalent spaces, as are $H^s_{(\gamma)}(\mathrm{J})$ and $W^{s,2}_{(\gamma+s)}(\mathrm{J})$. Using that $H^r(\mathrm{I})$ and $W^{r,2}_{(0,0)}(\mathrm{I})$ are equivalent spaces, the stated result follows from Theorem 6.1 for $\sigma = 0$ and $\mu = w + \gamma$.

∎

Corollary 6.2 (See Corollary 4.1.) *Let $s \ge -\alpha$, $f \in H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$, and $s^* := s + \alpha$. Then the unique solution to* (1.1),(1.2)*, satisfies for any $\epsilon > 0$*

$$u \in H^{\min\{\frac{s^*+(\alpha-\beta)}{2},\, \frac{s^*+\beta}{2},\, (\alpha-\beta)+\frac{1}{2}-\epsilon,\, \beta+\frac{1}{2}-\epsilon\}}(\mathrm{I})\,. \tag{6.3}$$

In particular, for $s > -\alpha + 1 + \min\{(\alpha-\beta),\, \beta\}$,

$$u \in H^{\min\{(\alpha-\beta),\, \beta\} + \frac{1}{2} - \epsilon}(\mathrm{I})\,. \tag{6.4}$$

Proof: Proceeding as described at the beginning of this section, consider $u_0(x) = x^{\beta}\psi_0(x)$, for $\psi_0(x) \in H^{s^*}_{(\beta)}(\mathrm{J})$. Using Theorem 5.1, the most regular (i.e., "nicest") weighted Sobolev space that u_0 lies in is given by the largest value for t and the smallest value for σ such that the conditions stated in (5.15) are satisfied. To apply Theorem 5.1 in this case we have: $s \to s^*$, $\mu \to \beta$, $p \to \beta$. Equation (5.15) then require that σ and t satisfy

$$0 \le t \le s^*, \quad \sigma \ge -\beta, \quad \sigma > t - 2\beta - 1, \quad \sigma \ge s^* - t - \beta\,. \tag{6.5}$$

Two cases arise for consideration.
Case 1. If $s^* < \beta + 1$ then t and σ satisfying (6.5) are determined by: $0 \le t \le s^*$, and $\sigma \ge -\beta$ (see Figure 6.1).
With the choices $t = s^*$, $\sigma = -\beta$, using Corollary 6.1 we obtain

$$u_0 \in H^{s^*}_{(-\beta)}(\mathrm{J}) \subset H^{\frac{s^*+\beta}{2}}(\mathrm{J})\,. \tag{6.6}$$

Case 2. If $s^* \ge \beta + 1$ then t and σ satisfying (6.5) are determined by: $0 \le t \le s^*$, $\sigma > t - 2\beta - 1$, and $\sigma \ge s^* - t - \beta$ (see Figure 6.2). With $t \le s^*$ and $t - \sigma < 2\beta + 1$, using Corollary 6.1 we obtain, for $\epsilon > 0$,

$$u_0 \in H^{t}_{(\sigma)}(\mathrm{J}) \subset H^{\beta+\frac{1}{2}-\epsilon}(\mathrm{J})\,. \tag{6.7}$$

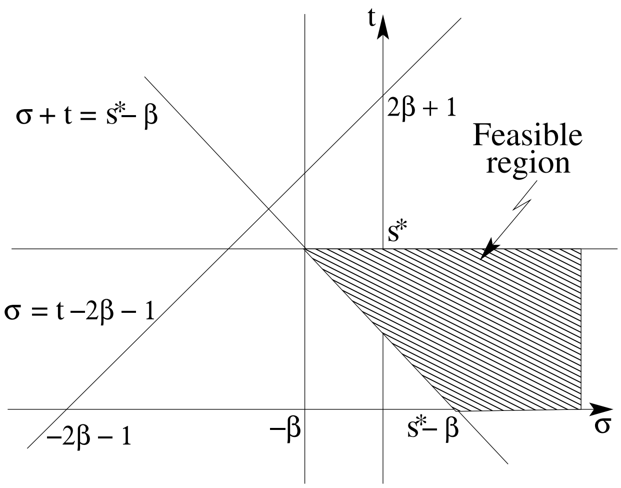

Figure 6.1: Illustration of Case 1 for constraints (6.5).

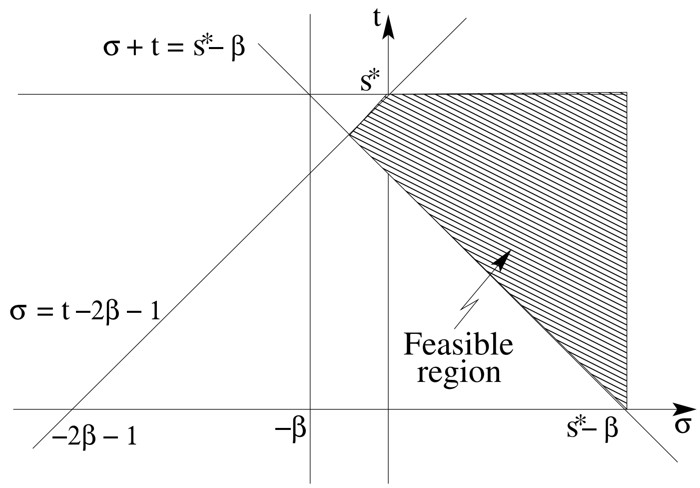

Figure 6.2: Illustration of Case 2 for constraints (6.5).

Combining (6.6) and (6.7) yields

$$u_0 \in H^{\min\{\frac{s^*+\beta}{2}\,,\,\beta+\frac{1}{2}-\epsilon\}}(\mathrm{J})\,. \tag{6.8}$$

For $u_1(x) = x^{\alpha-\beta}\,\psi_1(x)$, , for $\psi_1(x) \in H^{s^*}_{(\alpha-\beta)}(\mathrm{J})$, a similar analysis leads to

$$u_1 \in H^{\min\{\frac{s^*+(\alpha-\beta)}{2}\,,\,(\alpha-\beta)+\frac{1}{2}-\epsilon\}}(\mathrm{J})\,. \tag{6.9}$$

Combining (6.8) and (6.9) we obtain

$$u \in H^{\min\{\frac{s^*+(\alpha-\beta)}{2}\,,\,\frac{s^*+\beta}{2}\,,(\alpha-\beta)+\frac{1}{2}-\epsilon\,,\,\beta+\frac{1}{2}-\epsilon\}}(\mathrm{I})\,. \tag{6.10}$$

Noting that for $s > -\alpha + 1 + \min\{(\alpha-\beta)\,,\,\beta\}$ that $s^* > \min\{(\alpha-\beta)\,,\,\beta\} + 1$, from (6.10),

$$u \in H^{\min\{(\alpha-\beta)\,,\,\beta\}\,+\,\frac{1}{2}-\epsilon}(\mathrm{I})\,.$$

■

Corresponding to Theorem 5.2 we have the following.

Corollary 6.3 (See Theorem 5.2.) *Assuming the hypothesis of Theorem 5.2 are satisfied, and let* $s^* := \alpha + \min\{s\,,\,\alpha + (\alpha-\beta) + 1\,,\,\alpha + \beta + 1\}$. *Then the unique solution of* (5.17) *satisfies for any* $\epsilon > 0$

$$u \in H^{\min\{\frac{s^*+(\alpha-\beta)}{2},\,\frac{s^*+\beta}{2},\,(\alpha-\beta)+\frac{1}{2}-\epsilon,\,\beta+\frac{1}{2}-\epsilon\}}(\mathrm{I})\,. \tag{6.11}$$

In particular, for $s > -\alpha + 1 + \min\{(\alpha-\beta),\,\beta\}$,

$$u \in H^{\min\{(\alpha-\beta),\,\beta\}\,+\,\frac{1}{2}-\epsilon}(\mathrm{I})\,. \tag{6.12}$$

Proof: Proof follows exactly as that for Corollary 6.2.

■

Corresponding to Theorem 5.3 we have the following.

Corollary 6.4 (See Theorem 5.3.) *Assuming the hypothesis of Theorem 5.3 are satisfied, and let* $s^* := \alpha + \min\{s\,,\,\alpha + (\alpha-\beta) - 1\,,\,\alpha + \beta - 1\}$. *Then the unique solution of* (5.21) *satisfies for any* $\epsilon > 0$

$$u \in H^{\min\{\frac{s^*+(\alpha-\beta)}{2},\,\frac{s^*+\beta}{2},\,(\alpha-\beta)+\frac{1}{2}-\epsilon,\,\beta+\frac{1}{2}-\epsilon\}}(\mathrm{I})\,. \tag{6.13}$$

In particular, for $s > -\alpha + 1 + \min\{(\alpha-\beta),\,\beta\}$,

$$u \in H^{\min\{(\alpha-\beta),\,\beta\}\,+\,\frac{1}{2}-\epsilon}(\mathrm{I})\,. \tag{6.14}$$

Proof: Proof follows exactly as that for Corollary 6.2.

■

Remark: If the regularity of the right hand side function f is further restricted then the regularity of the solution may be improved. For example, if $r = 1/2$ the operator $\mathcal{L}^{\alpha}_{1/2}(\cdot)$ corresponds to the integral fractional Laplacian operator. For this operator Hao and Zhang in [22] showed that for $f \in H^s(\mathrm{I})$ the solution of (5.21) satisfied $u \in H^{\min\{s+\alpha\,,\,\alpha/2\,+\,1/2\,-\,\epsilon\}}(\mathrm{I})$ for any $\epsilon > 0$.

7 Proof of Theorem 5.1

An important part of the proof in establishing the regularity of the solution to (1.3) is determining the value of t and σ such that $x^p\psi(x) \in H^t_{(\sigma)}(\mathrm{J})$ for $\psi(x) \in H^s_{(\mu)}(\mathrm{J})$. The general result is given in Theorem 5.1. There are two key terms which arise in the proof of Theorem 5.1. The analysis for one of these terms follows similarly to a term which occurs for the case of s between 0 and 1, discussed in Theorem 7.2. The other of these terms arises for the case of s between 1 and 2, discussed in Theorem 7.3. We begin this section with an embedding theorem which is used in the proofs of the subsequent theorems in this section.

Theorem 7.1 *For $s \geq 0$, $\gamma - s > -1$, then $H^s_{(\gamma)}(\mathrm{J}) \subset L^2_{(\gamma-s)}(\mathrm{J})$.*

Proof: Firstly we consider the case for $0 \leq s \leq 1$.
For $s = 0$ we have that

$$H^s_{(\gamma)}(\mathrm{J}) \;=\; H^0_{(\gamma)}(\mathrm{J}) \;=\; L^2_{(\gamma)}(\mathrm{J}). \tag{7.1}$$

For $s = 1$, consider $\phi \in L^2_{(\gamma-s)}(\mathrm{J}) \;=\; L^2_{(\gamma-1)}(\mathrm{J})$. Then, using Hardy's inequality [8, Lemma 3.2],

$$\begin{aligned}
\|\phi\|^2_{L^2_{(\gamma-1)}} &= \int_{\mathrm{J}} x^{\gamma-1}\big(\phi(x)\big)^2\,dx \\
&\lesssim \int_{\mathrm{J}} x^{\gamma+1}\big(\phi'(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma+1}\big(\phi(x)\big)^2\,dx \\
&\leq \int_{\mathrm{J}} x^{\gamma+1}\big(\phi'(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma}\big(\phi(x)\big)^2\,dx \\
&= \|\phi\|^2_{H^1_{(\gamma)}}\,.
\end{aligned} \tag{7.2}$$

From (7.1) and (7.2) the identity operator $\mathbb{I}$ mapping from $H^s_{(\gamma)}(\mathrm{J}) \longrightarrow L^2_{(\gamma-s)}(\mathrm{J})$, $s = 0, 1$ is a bounded operator.

Additionally, the spaces $L^2_{\sigma}(\mathrm{J})$ are interpolation spaces [34, Lemma 23.1].

Hence, for $\theta \;=\; (1-\theta)0 \,+\, \theta 1$, using the fact that $H^\theta_{(\gamma)}(\mathrm{J})$ and $L^2_{(\theta)}(\mathrm{J})$ are interpolation spaces, it follows that

$$H^\theta_{(\gamma)}(\mathrm{J}) \xrightarrow{\mathbb{I}} L^2_{((1-\theta)\gamma\,+\,\theta(\gamma-1))}(\mathrm{J}) \;=\; L^2_{(\gamma-\theta)}(\mathrm{J})$$

is bounded. Thus, if $u \in H^\theta_{(\gamma)}(\mathrm{J})$ then $u \in L^2_{(\gamma-\theta)}(\mathrm{J})$ with $\|u\|_{L^2_{(\gamma-\theta)}} \;\leq\; C\,\|u\|_{H^\theta_{(\gamma)}}$.

Next, for $1 \leq s \leq 2$, consider $s = 2$. For $\phi \in L^2_{(\gamma-s)}(\mathrm{J}) \;=\; L^2_{(\gamma-2)}(\mathrm{J})$, again using Hardy's inequality (and that $\gamma - s > -1$),

$$\begin{aligned}
\|\phi\|^2_{L^2_{(\gamma-2)}} &= \int_{\mathrm{J}} x^{\gamma-2}\big(\phi(x)\big)^2\,dx \\
&\lesssim \int_{\mathrm{J}} x^{\gamma}\big(\phi'(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma}\big(\phi(x)\big)^2\,dx \\
&\lesssim \int_{\mathrm{J}} x^{\gamma+2}\big(\phi''(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma+2}\big(\phi'(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma}\big(\phi(x)\big)^2\,dx \\
&\leq \int_{\mathrm{J}} x^{\gamma+2}\big(\phi''(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma+1}\big(\phi'(x)\big)^2\,dx \;+\; \int_{\mathrm{J}} x^{\gamma}\big(\phi(x)\big)^2\,dx \\
&= \|\phi\|^2_{H^2_{(\gamma)}}\,.
\end{aligned} \tag{7.3}$$

Again, using the fact that $H^\theta_{(\gamma)}(\mathrm{J})$ and $L^2_{(\theta)}(\mathrm{J})$ are interpolation spaces, it now follows that for $0 \leq \theta \leq 2$ if $u \in H^\theta_{(\gamma)}(\mathrm{J})$ then $u \in L^2_{(\gamma-\theta)}(\mathrm{J})$ with $\|u\|_{L^2_{(\gamma-\theta)}} \;\leq\; C\,\|u\|_{H^\theta_{(\gamma)}}$.

The argument extends in an obvious manner to arbitrary $s > 2$.

■

Theorem 7.2 *Let $0 \le s < 1$, $\mu > -1$, and $\psi \in H^s_{(\mu)}(\mathrm{J})$. Then $x^p\,\psi \in H^t_{(\sigma)}(\mathrm{J})$ provided*

$$0 \le t \le s\,, \quad \sigma + 2p \ge \mu\,, \quad \sigma + 2p - t > -1\,, \quad \text{and} \quad \sigma + 2p + t \ge \mu + s\,. \tag{7.4}$$

Additionally, when (7.4) is satisfied, there exists $C > 0$ (independent of ψ) such that $\|x^p\,\psi\|_{H^t_{(\sigma)}(\mathrm{J})} \le C\,\|\psi\|_{H^s_{(\mu)}(\mathrm{J})}$.

Proof: Firstly, for $s = t = 0$,

$$\begin{aligned} \|x^p\,\psi\|^2_{H^t_{(\sigma)}} &= \|x^p\,\psi\|^2_{H^0_{(\sigma)}} = \int_{\mathrm{J}} x^\sigma\,\left(x^p\,\psi(x)\right)^2\,dx = \int_{\mathrm{J}} x^{\sigma+2p}\,\left(\psi(x)\right)^2\,dx \\ &\le \int_{\mathrm{J}} x^\mu\,\left(\psi(x)\right)^2\,dx\,,\ \text{provided}\ \ \sigma + 2p \ge \mu\,, \\ &= \|\psi\|^2_{H^0_{(\mu)}} = \|\psi\|^2_{H^s_{(\mu)}}\,. \end{aligned} \tag{7.5}$$

For $0 < s < 1$, in addition to (7.5) we must also consider the semi-norm $|x^p\,\psi|_{H^t_{(\sigma)}(\Lambda)}$.

$$\begin{aligned} |x^p\,\psi|^2_{H^t_{(\sigma)}(\Lambda)} &= \iint_\Lambda x^{\sigma+t}\,\frac{|x^p\,\psi(x) - y^p\,\psi(y)|^2}{|x - y|^{1+2t}}\,dy\,dx \\ &\lesssim \iint_\Lambda x^{\sigma+t}\,y^{2p}\,\frac{|\psi(x) - \psi(y)|^2}{|x - y|^{1+2t}}\,dy\,dx + \iint_\Lambda x^{\sigma+t}\,\psi^2(x)\,\frac{|x^p - y^p|^2}{|x - y|^{1+2t}}\,dy\,dx \\ &:= I_1 + I_2\,. \end{aligned} \tag{7.6}$$

Noting that in Λ, $y < \frac{3}{2}x$, for I_1 we have

$$\begin{aligned} I_1 &\lesssim \iint_\Lambda x^{\sigma+t}\,x^{2p}\,\frac{|\psi(x) - \psi(y)|^2}{|x - y|^{1+2t}}\,dy\,dx \\ &\le \iint_\Lambda x^{\mu+s}\,\frac{|\psi(x) - \psi(y)|^2}{|x - y|^{1+2s}}\,dy\,dx\,, \quad \text{provided}\ \ \sigma + t + 2p \ge \mu + s\,,\ \text{and}\ \ t \le s\,, \\ &= |\psi|^2_{H^s_{(\mu)}(\Lambda)}\,. \end{aligned} \tag{7.7}$$

If $p = 0$ then $I_2 = 0$. To bound I_2 for $p \neq 0$ we introduce the change of variable: $y = \frac{1}{z}x$, where $\frac{2}{3} < z < \frac{3}{2}$. With this change of variable, we have

$$dy = \frac{-1}{z^2}x\,dz\,, \quad |x - y| = x\,\frac{1}{z}|z - 1|\,, \quad \text{and for } I_2$$

$$I_2 = \int_0^{1/2} x^{\sigma+2p-t}\psi^2(x)\,dx\,\int_{2/3}^{3/2} z^{-1+2t-2p}\,|1 - z|^{-1-2t}|z^p - 1|^2\,dz \tag{7.8}$$

$$\lesssim \|\psi\|^2_{L^2_{(\sigma+2p-t)}} \cdot \int_{2/3}^{3/2} |1 - z|^{-1-2t}|z^p - 1|^2\,dz\,. \tag{7.9}$$

Using Hardy's inequality [8, Lemma 3.2], we bound the integral in (7.9) as follows.

$$\begin{aligned}
\int_{2/3}^{3/2} |1-z|^{-1-2t}|z^p-1|^2\,dz &= \int_{2/3}^{1}(1-z)^{-1-2t}(z^p-1)^2\,dz \;+\; \int_{1}^{3/2}(z-1)^{-1-2t}(z^p-1)^2\,dz \\
&\lesssim \int_{2/3}^{1}(1-z)^{-1-2t+2}(z^{p-1})^2\,dz \;+\; \int_{1}^{3/2}(z-1)^{-1-2t+2}(z^{p-1})^2\,dz \\
&\lesssim 1\,, \qquad \text{provided } t<1\,. && (7.10)
\end{aligned}$$

From Theorem 7.1, we have

$$\begin{aligned}
\|\psi\|_{L^2_{(\sigma+2p-t)}} &\lesssim \|\psi\|_{H^t_{(\sigma+2p)}}, \qquad \text{provided } \sigma+2p-t>-1\,, \\
&\le \|\psi\|_{H^s_{(\mu)}}, \qquad \text{provided } \sigma+2p\ge\mu\,,\ \text{ and } t\le s\,. && (7.11)
\end{aligned}$$

Finally, combining (7.5)-(7.11) we obtain the stated results.

■

Theorem 7.3 *Let* $1\le s<2$, $\mu>-1$, *and* $\psi\in H^s_{(\mu)}(\mathrm{J})$. *Then* $x^p\,\psi\in H^t_{(\sigma)}(\mathrm{J})$ *provided*

$$0\le t\le s\,,\quad \sigma+2p\ge\mu\,,\quad \sigma+2p-t>-1\,,\quad \text{and}\quad \sigma+2p+t\ge\mu+s\,. \tag{7.12}$$

Additionally, when (7.12) *is satisfied, there exists* $C>0$ *(independent of* ψ*) such that* $\|x^p\,\psi\|_{H^t_{(\sigma)}(\mathrm{J})}\le C\,\|\psi\|_{H^s_{(\mu)}(\mathrm{J})}$.

Proof: For $0\le t<1$ Theorem 7.2 applies. We assume that $t\ge1$. Hence, of interest is

$$\|x^p\,\psi\|_{L^2_{(\sigma)}},\qquad \|D\,(x^p\,\psi)\,\|_{L^2_{(\sigma+1)}},\qquad |D\,(x^p\,\psi)\,|_{H^{t-1}_{(\sigma+1)}(\Lambda)}\,.$$

From Theorem 7.2,

$$\|x^p\,\psi\|_{L^2_{(\sigma)}}\lesssim\|\psi\|_{H^0_{(\mu)}}\,,\quad \text{provided}\quad \sigma+2p\ge\mu\,. \tag{7.13}$$

For $D\,(x^p\,\psi)$ we have

$$\begin{aligned}
D\,(x^p\,\psi) &= x^p\,\psi' + p\,x^{p-1}\,\psi \quad\text{and}\\
(D\,(x^p\,\psi))^2 &\lesssim \left(x^p\,\psi'\right)^2 + \left(x^{p-1}\,\psi\right)^2\,.
\end{aligned}$$

Then, using Hardy's inequality [8, Lemma 3.2] (using $\sigma+2p-t>-1$, i.e., $\sigma+2p>0$),

$$\begin{aligned}
\|D\,(x^p\,\psi)\,\|^2_{L^2_{(\sigma+1)}} &\lesssim \int_0^{3/4} x^{\sigma+1}\left(x^p\,\psi'(x)\right)^2\,dx \;+\; \int_0^{3/4} x^{\sigma+1}\left(x^{p-1}\,\psi(x)\right)^2\,dx \\
&= \int_0^{3/4} x^{\sigma+2p+1}\left(\psi'(x)\right)^2\,dx \;+\; \int_0^{3/4} x^{\sigma+2p-1}\left(\psi(x)\right)^2\,dx \\
&\lesssim \int_0^{3/4} x^{\sigma+2p+1}\left(\psi'(x)\right)^2\,dx \\
&\qquad + \int_0^{3/4} x^{\sigma+2p+1}\left(\psi'(x)\right)^2\,dx \;+\; \int_0^{3/4} x^{\sigma+2p+1}\left(\psi(x)\right)^2\,dx \\
&\lesssim \|\psi\|^2_{H^1_{(\mu)}}\,,\quad \text{provided}\quad \sigma+2p\ge\mu\,. && (7.14)
\end{aligned}$$

Equation (7.14), together with (7.13) establishes the stated result for $t = s = 1$.

Next, for $|D\,(x^p\,\psi)\,|_{H^{t-1}_{(\sigma+1)}(\Lambda)}$ we have

$$
\begin{aligned}
|D\,(x^p\,\psi)\,|^2_{H^{t-1}_{(\sigma+1)}(\Lambda)} \;&=\; \iint_\Lambda x^{\sigma+t}\,\frac{|D\,(x^p\,\psi(x))\;-\;D\,(y^p\,\psi(y))\,|^2}{|x\;-\;y|^{1+2(t-1)}}\,dy\,dx \\
&\lesssim\; \iint_\Lambda x^{\sigma+t}\,\frac{|x^{p-1}\,\psi(x)\;-\;y^{p-1}\,\psi(y)|^2}{|x\;-\;y|^{1+2(t-1)}}\,dy\,dx\;+\;\iint_\Lambda x^{\sigma+t}\,\frac{|x^p\,\psi'(x)\;-\;y^p\,\psi'(y)|^2}{|x\;-\;y|^{1+2(t-1)}}\,dy\,dx \\
&:=\; I_3\;+\;I_4\,.
\end{aligned}
\tag{7.15}
$$

For I_3, proceeding as in (7.6),

$$
\begin{aligned}
I_3 \lesssim\; &\iint_\Lambda x^{\sigma+t}\,y^{2(p-1)}\,\frac{|\psi(x)\;-\;\psi(y)|^2}{|x\;-\;y|^{1+2(t-1)}}\,dy\,dx\;+\;\iint_\Lambda x^{\sigma+t}\,\psi^2(x)\,\frac{|x^{p-1}\;-\;y^{p-1}|^2}{|x\;-\;y|^{1+2(t-1)}}\,dy\,dx \\
:=\;& I_{3,1}\;+\;I_{3,2}\,.
\end{aligned}
$$

For $I_{3,1}$, using $y \,<\, \frac{3}{2}x$, and the introducing the change of variable $y \,=\, \frac{1}{z}x$, where $\frac{2}{3} < z < \frac{3}{2}$ we obtain

$$
\begin{aligned}
I_{3,1} &\lesssim \int_{x=0}^{1/2} x^{\sigma+t+2(p-1)} \int_{z=2/3}^{3/2} \frac{z^{1+2(t-1)-2(p-1)}}{(x\,|z-1|)^{1+2(t-1)}}\,|\psi(x)\;-\;\psi(\frac{1}{z}x)|^2\,z^{-2}\,x\,dz\,dx \\
&\lesssim \int_{x=0}^{1/2} x^{\sigma+2p-t}\Bigg(\int_{z=2/3}^{1}(1-z)^{1-2t}\,(\psi(x)\;-\;\psi(\frac{1}{z}x))^2\,dz \\
&\qquad\qquad +\int_{z=1}^{3/2}(z-1)^{1-2t}\,(\psi(x)\;-\;\psi(\frac{1}{z}x))^2\,dz\Bigg)dx \\
&\lesssim \int_{x=0}^{1/2} x^{\sigma+2p-t}\Bigg(\int_{z=2/3}^{1}(1-z)^{3-2t}\,z^{-4}\,x^2\,(\psi'(\frac{1}{z}x))^2\,dz \\
&\qquad\qquad +\int_{z=1}^{3/2}(z-1)^{3-2t}\,z^{-4}\,x^2\,(\psi'(\frac{1}{z}x))^2\,dz\Bigg)dx \\
&\qquad \text{(using Hardy's inequality)} \\
&\lesssim \int_{x=0}^{1/2} x^{\sigma+2p-t+2}\int_{z=2/3}^{3/2}|1-z|^{3-2t}\,(\psi'(\frac{1}{z}x))^2\,dz\,dx\,.
\end{aligned}
\tag{7.16}
$$

Next, letting $w = \frac{1}{z}x$, then $dw = \frac{1}{z}\,dx$ and

$$\begin{aligned}
I_{3,1} &\lesssim \int_{z=2/3}^{3/2} |1-z|^{3-2t} \int_{w=0}^{1/2z} (z\,w)^{\sigma+2p-t+2}\,(\psi'(w))^2\,z\,dw\,dz \\
&\lesssim \int_{2/3}^{3/2} |1-z|^{3-2t}\,dz \int_{0}^{3/4} w^{\sigma+2p-t+2}\,(\psi'(w))^2\,dw \\
&\lesssim \|\psi'\|^2_{L^2_{(\sigma+2p-t+2)}}\,, \text{ as } t<2\,, \\
&\lesssim \|\psi'\|^2_{H^{t-1}_{(\sigma+2p+1)}}\,, \text{ (using Theorem 7.1) provided } \sigma+2p-t+2>-1\,, \\
&\lesssim \|\psi'\|^2_{H^{s-1}_{(\mu+1)}}\,, \text{ provided } \sigma+2p\geq\mu\,, \text{ and } t\leq s\,, \\
&\lesssim \|\psi\|^2_{H^s_{(\mu)}}\,. \qquad (7.17)
\end{aligned}$$

If $p=1$ then $I_{3,2}=0$. For $I_{3,2}$ with $p\neq 1$, proceeding as in the approach used to obtain the bound for I_2, (7.8) - (7.10),

$$\begin{aligned}
I_{3,2} &\lesssim \int_{0}^{1/2} x^{(\sigma+1)-(t-1)+2(p-1)}(\psi(x))^2\,dx \int_{2/3}^{3/2} z^{-1+2(t-1)-2(p-1)}\,|1-z|^{-1-2(t-1)}\,|z^{p-1}-1|^2\,dz \\
&\lesssim \int_{x=0}^{1/2} x^{\sigma+2p-t}(\psi(x))^2\,dx\,, \text{ provided } (t-1)<1\,, \text{ i.e., } t<2\,, \\
&\lesssim \int_{0}^{1/2} x^{\sigma+2p-t+2}(\psi'(x))^2\,dx + \int_{0}^{1/2} x^{\sigma+2p-t+2}(\psi(x))^2\,dx\,, \\
&\qquad \text{(using Hardy's inequality) provided } \sigma+2p-t>-1\,, \\
&\lesssim \|\psi'\|^2_{L^2_{(\sigma+2p-t+2)}} + \|\psi\|^2_{L^2_{(\sigma+2p-t+2)}} \\
&\lesssim \|\psi'\|^2_{H^{t-1}_{(\sigma+2p+1)}} + \|\psi\|^2_{H^t_{(\sigma+2p+2)}}\,, \text{ (using Theorem 7.1) provided } \sigma+2p-t+2>-1\,, \\
&\lesssim \|\psi'\|^2_{H^{s-1}_{(\mu+1)}} + \|\psi\|^2_{H^s_{(\mu+2)}}\,, \text{ provided } \sigma+2p\geq\mu \text{ and } t\leq s, \\
&\lesssim \|\psi\|^2_{H^s_{(\mu)}}\,. \qquad (7.18)
\end{aligned}$$

With I_4, proceeding as in (7.6),

$$\begin{aligned}
I_4 &\lesssim \iint_\Lambda x^{\sigma+t}\,y^{2p}\,\frac{|\psi'(x)-\psi'(y)|^2}{|x-y|^{1+2(t-1)}}\,dy\,dx + \iint_\Lambda x^{\sigma+t}\,(\psi'(x))^2\,\frac{|x^p-y^p|^2}{|x-y|^{1+2(t-1)}}\,dy\,dx \\
&\lesssim |\psi'|^2_{H^{s-1}_{(\mu+1)}(\Lambda)} \quad \text{(provided } \sigma+2p+t\geq\mu+s\,, \text{ and } t\leq s\text{)} \\
&\quad + \|\psi'\|^2_{H^{s-1}_{(\mu+1)}} \quad \text{(provided } \sigma+2p-t+2>-1\,, \sigma+2p\geq\mu\,, \text{ and } 1<t\leq s\text{)} \\
&\lesssim \|\psi\|^2_{H^s_{(\mu)}}\,. \qquad (7.19)
\end{aligned}$$

Combining (7.13) - (7.19) we obtain the stated result.

■

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1

The proof is an induction argument, using Theorem 7.3 as the initial step.

For $t = s = n$ we have,

$$
\begin{aligned}
\|D^t(x^p\,\psi(x))\|^2_{L^2_{(\sigma+t)}} &\lesssim \sum_{j=0}^{n} \|x^{p-j}\,D^{n-j}\psi(x)\|^2_{L^2_{(\sigma+n)}} \\
&= \sum_{j=0}^{n} \int_0^{3/4} x^{\sigma+n}\,\left(x^{p-j}\,D^{n-j}\psi(x)\right)^2\,dx \\
&= \sum_{j=0}^{n} \int_0^{3/4} x^{\sigma+n+2(p-j)}\,\left(D^{n-j}\psi(x)\right)^2\,dx \\
&\quad \text{(applying Hardy's inequality } j \text{ times, using } \sigma + 2p - n > -1) \\
&\lesssim \sum_{j=0}^{n}\sum_{k=0}^{j} \int_0^{3/4} x^{\sigma+2p+n}\,\left(D^{n-j+k}\psi(x)\right)^2\,dx \\
&\lesssim \sum_{j=0}^{n} \|D^j\psi\|^2_{L^2_{(\sigma+2p+n)}} \\
&\lesssim \sum_{j=0}^{n} \|D^j\psi\|^2_{L^2_{(\mu+n)}} \quad (\text{using } \sigma + 2p \geq \mu) \\
&\lesssim \sum_{j=0}^{n} \|D^j\psi\|^2_{L^2_{(\mu+j)}} \lesssim \|\psi\|^2_{H^n_{(\mu)}} = \|\psi\|^2_{H^s_{(\mu)}}\,.
\end{aligned}
\tag{7.20}
$$

Equation (7.20), together with the induction assumption establishes the result for $t = s = n$.

For $n < t, s < n+1$ we also need to consider $|D^n(x^p\,\psi(x))|_{H^{t-n}_{(\sigma+n)}(\Lambda)}$.

$$
\begin{aligned}
|D^n(x^p\,\psi(x))|^2_{H^{t-n}_{(\sigma+t)}(\Lambda)} &= \sum_{j=0}^{n} \iint_\Lambda x^{\sigma+t} \frac{|x^{p-j}\,D^{n-j}\psi(x) - y^{p-j}\,D^{n-j}\psi(y)|^2}{|x-y|^{1+2(t-n)}}\,dy\,dx \\
&\lesssim \sum_{j=0}^{n} \left(\iint_\Lambda x^{\sigma+t}\,y^{2(p-j)}\,\frac{|D^{n-j}\psi(x) - D^{n-j}\psi(y)|^2}{|x-y|^{1+2(t-n)}}\,dy\,dx \right. \\
&\qquad \left. + \iint_\Lambda x^{\sigma+t}\,\left(D^{n-j}\psi(x)\right)^2\,\frac{|x^{p-j} - y^{p-j}|^2}{|x-y|^{1+2(t-n)}}\,dy\,dx \right) \\
&= \iint_\Lambda x^{\sigma+t}\,y^{2p}\,\frac{|D^n\psi(x) - D^n\psi(y)|^2}{|x-y|^{1+2(t-n)}}\,dy\,dx \\
&\quad + \sum_{j=1}^{n} \iint_\Lambda x^{\sigma+t}\,y^{2(p-j)}\,\frac{|D^{n-j}\psi(x) - D^{n-j}\psi(y)|^2}{|x-y|^{1+2(t-n)}}\,dy\,dx \\
&\quad + \sum_{j=0}^{n} \iint_\Lambda x^{\sigma+t}\,\left(D^{n-j}\psi(x)\right)^2\,\frac{|x^{p-j} - y^{p-j}|^2}{|x-y|^{1+2(t-n)}}\,dy\,dx\,.
\end{aligned}
\tag{7.21}
$$

The first term in (7.21) is bounded in a similar manner to I_1 in Theorem 7.2, with $\sigma \to \sigma + n$,

$t \to t - n$, to obtain

$$\iint_\Lambda x^{\sigma+t}\, y^{2p}\, \frac{|D^n\psi(x) \;-\; D^n\psi(y)|^2}{|x \;-\; y|^{1\,+\,2(t-n)}}\, dy\, dx \;\lesssim\; |\psi|^2_{H^s_{(\mu)}(\Lambda)}\,, \;\text{ provided }\; \sigma + 2p + t \ge \mu + s\,,\; \text{ and }\; t \le s\,. \tag{7.22}$$

For the second term in (7.21) the terms in the summation are bounded in a similar manner to $I_{3,1}$ in Theorem 7.3, $\sigma \to \sigma + n - 1$, $t \to t - n + 1$, $p \to p + 1 - j$, to obtain

$$\begin{aligned}
&\iint_\Lambda x^{\sigma+t}\, y^{2(p-j)}\, \frac{|D^{n-j}\psi(x) \;-\; D^{n-j}\psi(y)|^2}{|x \;-\; y|^{1\,+\,2(t-n)}}\, dy\, dx \\
&\quad\lesssim\; \|D^{n-j+1}\psi\|^2_{L^2_{(\sigma\,+\,2p\,+n-j+1-(t-n+j-1))}} \\
&\quad\lesssim\; \|D^{n-j+1}\psi\|^2_{H^{t-(n-j+1)}_{(\sigma\,+\,2p\,+n-j+1)}}\,, \;\;\text{provided } \sigma \,+\, 2p + n - j + 1 - (t - n + j - 1) \;>\; -1 \\
&\quad\lesssim\; \|D^{n-j+1}\psi\|^2_{H^{s-(n-j+1)}_{(\mu+n-j+1)}}\,, \;\;\text{provided } \sigma \,+\, 2p \;\ge\; \mu \;\text{ and } t \le s, \\
&\quad\lesssim\; \|\psi\|^2_{H^s_{(\mu)}}\,.
\end{aligned} \tag{7.23}$$

For the third term in (7.21) the terms in the summation are bounded in a similar manner to I_2 in Theorem 7.2, $\sigma \to \sigma + n$, $t \to t - n$, $p \to p - j$, to obtain

$$\begin{aligned}
&\iint_\Lambda x^{\sigma+t}\, \left(D^{n-j}\psi(x)\right)^2\, \frac{|x^{p-j} \;-\; y^{p-j}|^2}{|x \;-\; y|^{1\,+\,2(t-n)}}\, dy\, dx \\
&\quad\lesssim\; \|D^{n-j}\psi\|^2_{L^2_{(\sigma\,+\,2(p-j)\,+n-(t-n))}} \;=\; \|D^{n-j}\psi\|^2_{L^2_{(\sigma\,+\,2p\,+n-j-(t-n+j))}}\,, \;\;\text{provided } t \;<\; n+1, \\
&\quad\lesssim\; \|D^{n-j}\psi\|^2_{H^{t-(n-j)}_{(\sigma\,+\,2p\,+n-j)}}\,, \;\;\text{provided } \sigma \,+\, 2p + n - j - (t - n + j) \;>\; -1 \\
&\quad\lesssim\; \|D^{n-j}\psi\|^2_{H^{s-(n-j)}_{(\mu+n-j)}}\,, \;\;\text{provided } \sigma \,+\, 2p \;\ge\; \mu \;\text{ and } t \le s, \\
&\quad\lesssim\; \|\psi\|^2_{H^s_{(\mu)}}\,.
\end{aligned} \tag{7.24}$$

Combining (7.20) - (7.24) the stated result follows.

■

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