# SUMMER 2019 REU: NUMBER THEORY

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## 1. Partition functions in real quadratic fields

A partition of a natural number $n$ is a way of writing $n$ as a sum of a non-increasing sequence of natural numbers. For example, 3 can be partitioned in three distinct ways:

$$3, \quad 2+1, \quad 1+1+1.$$

The number of partitions of $n$ is denoted by the partition function $p(n)$. For example $p(3) = 3$, $p(4) = 5$, $p(5) = 7$ and $p(6) = 11$. In [1] Hardy and Ramanujan observed the following asymptotic formula for the partition function

$$e^{A\sqrt{n}} \ll p(n) \ll e^{B\sqrt{n}}$$

for some suitable positive constants $A$ and $B$.

In the setting of real quadratic fields one can also define partitions and partition functions. For example, let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D$. Let $\gamma = a + b\sqrt{D} \in \mathcal{O}_K^+$ be a totally positive integer, which means that both $a + b\sqrt{D}$ and $a - b\sqrt{D}$ are positive. For instance, $3 + \sqrt{5}$ is totally positive while $2 + \sqrt{5}$ is not. A partition of $\gamma$ is a finite sum

$$\gamma = \sum_{\alpha \in \mathcal{O}_K^+} n_\alpha \alpha$$

with $n_\alpha \in \mathbb{Z}_{\geq 0}$. A partition of $\gamma$ is thus uniquely determined by a sequence of nonnegative integers $\{n_\alpha\}_\alpha$ indexed by totally positive integers $\alpha \in \mathcal{O}_K^+$. The function $p_{\sqrt{D}}(\gamma)$ for $\gamma \in \mathcal{O}_K^+$ is defined to be the total number of such partitions of $\gamma$. For example, $3 \in \mathbb{Q}(\sqrt{5})$ can be partitioned as:

$$3, \quad 2+1, \quad 1+1+1,$$

while 6 can be partitioned as:

$$6, 5+1, 4+2, 3+3, (3+\sqrt{5}) + (3-\sqrt{5}), 4+1+1, 3+2+1, 2+2+2,$$
$$3+1+1+1, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.$$

So $p_{\sqrt{5}}(3) = 3$ and $p_{\sqrt{5}}(6) = 12$. This project aims to develop a fast algorithm to compute the new partition functions $p_{\sqrt{D}}$ and to find an asymptotic formula similar to that of Hardy-Ramanujan.

## 2. Traces of Hecke operators

Modular forms play an important role in modern number theory, for example in Andrew Wiles proof of Fermat's Last Theorem. Modular forms are special holomorphic functions on the upper half complex plane which has nice transformation properties under the action of $\Gamma_0(N)$, where $N$ is a natural number and

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ N|c \right\}$$

There are two quantities associated to each modular form, the weight $k$ and the level $N$. Modular forms of same weight $k$ and level $N$ form a finite dimensional vector space over the field of complex numbers, on which Hecke operators $T_m^{k,N}$ with $m = 1, 2, \cdots$ can be defined. These Hecke operators commute with each other. They preserve a subspace, called the subspace of cuspforms and denoted $S_k(N)$, of the space of modular forms of weight $k$ and level $N$.

Information of the trace $\mathrm{Tr}(T_m^{k,N})$ of each Hecke operator is very important for the study of Hecke operators and cuspforms. The Generalized Lehmer Conjecture states that

$$\mathrm{Tr}(T_m^{k,N}) \neq 0$$

for essentially all $k, N$ and $m$. Note that for $m = 1$ this is trivially true because $T_1^{k,N}$ is just the identity operator. Rouse [2] proved that this holds true for $m = 2$. The main goal of this project is to study traces of Hecke operators and to show that $\mathrm{Tr}(T_3^{k,N}) \neq 0$ for $m = 3$.

## References

[1] G. H. Hardy and S. Ramanujan. Asymptotic Formulaae in Combinatory Analysis, *Proc. London Math. Soc. (2)*, 17:75–115, 1918.

[2] J. Rouse. Vanishing and non-vanishing of traces of Hecke operators, *Trans. Amer. Math. Soc.*, 358, no. 10, 4637–4651, 2006.