PAlmetto **N**umber **T**heory **S**eries **33**

in honor of Michael Filaseta's 60th birthday

SCHEDULE OF ACTIVITIES

Talks will take place in Brackett 100, 113. Refreshments are available during all breaks in Brackett 111.

Saturday, December 14, 2019

8:00–8:30 [Brackett 111] Coffee and other refreshments

8:30–8:40 [Brackett 100] Welcome from Founding Director, SMSS **Kevin James**

8:40–9:40 [Brackett 100] **Carl Pomerance** (Dartmouth College), *Cyclotomic Polynomials: Problems and Results*

10:00–10:20 [Brackett 100] **Brian Beasley**, (Presbyterian College), *A Brief History of Distribution Results for Powerfree Numbers*

[Brackett 113] **Lea Beneish**, (Emory University), *On Weierstrass mock modular forms and a dimension formula for certain vertex operator algebras*

10:30–10:50 [Brackett 100] **Daniel Baczkowski**, (University of Findlay), *Diophantine equations involving factorials and arithmetic functions*

[Brackett 113] **Holly-Paige Chaos**, (Wake Forest University), *Torsion for CM Elliptic Curves in Degree 2p*

11:00–11:20 [Brackett 100] **Nathan McNew**, (Towson University), *Counting pattern-avoiding integer partitions*

[Brackett 113] **Sunil Chetty**, (College of Saint Benedict and Saint John's University), *On primitive Pythagorean triples of special forms*

11:40–12:40 [Brackett 100] **Jiuya Wang**, (Duke University), *Bounding ℓ-torsion in class groups of elementary abelian extensions*

12:40-2-:00 Lunch

2:00–2:20 [Brackett 100] **Joshua Harrington**, (Cedar Crest College), *Odd Coverings of Subsets of the Integers*

[Brackett 113] **Lydia Eldredge**, (Florida State University), *Small Mahler Measure in Cubic Fields*

2:40–3:40 [Brackett 100] **Kevin Ford**, (UIUC), *Anatomy of random integers, permutations and polynomials*

4:00–4:20 [Brackett 100] **Jesse Thorner**, (University of Florida), *New results on the Chebotarev density theorem*

[Brackett 113] **Jackson Morrow**, (Emory University), *Non-Archimedean entire curves in proper surfaces admitting a dominant morphism to an elliptic curve*

The organizers thank NSF, NSA, and SMSS of Clemson University for their support.

4:30–4:50 [Brackett 100] **Angel Kumchev**, (Towson University), *Two Theorems of Piatetski-Shapiro*

[Brackett 113] **Alexis Newton**, (Wake Forest University), *Sums of Two Sixth Powers*

5:00–5:20 [Brackett 100] **Charlotte Ure**, (University of Virginia), *Brauer Groups of Elliptic Curves*

[Brackett 113] **Cooper O'Kuhn**, (University of Florida), *Mondrian's Problem and Its Relation to Number Theory*

5:30–5:50 [Brackett 100] **Delong Li**, (University of Tennessee - Knoxville), *Denominators of the Weierstrass coefficients of the canonical lifting*

[Brackett 113] **Daozhou Zhu**, (Clemson University), *Distinguishing eigenforms*

Sunday, December 15, 2019

8:00-8:30 [Brackett 111] Coffee and other refreshments

8:30–9:30 [Brackett 100] **Jeffrey Lagarias**, (University of Michigan), *On Toric Orbits in the Affine Sieve*

9:50–10:10 [Brackett 100] **Gang Yu**, (Kent State University), *Imaginary quadratic fields with class groups of* 3*-rank at least* 2

[Brackett 113] **Brandon Alberts**, (University of Connecticut), *Counting Towers of Number Fields*

10:20–10:40 [Brackett 100] **Ognian Trifonov**, (University of South Carolina), *Lattice points close to a plane curve*

[Brackett 113] **Tomer Reiter**, (Emory University), *Isogenies of Elliptic Curves over* $\mathbb{Q}(2^\infty)$

10:50–11:10 [Brackett 100] **Robert Hough**, (Stony Brook), *The local zeta function in enumerating quartic fields*

[Brackett 113] **Biao Wang**, (University at Buffalo), *Möbius function, Liouville function, Ramanujan sum and Chebatorev densities*

11:20–11:40 [Brackett 100] **Jeremy Rouse**, (Wake Forest University), *The density of odd order reductions*

[Brackett 113] **Lee Troupe**, (Mercer University), *Divisor sums representable as the sum of two squares*

12:00–1:00 [Brackett 100] **Soumya Sankar**, (University of Wisconsin), *Proportion of ordinary curves in some families*

1:00 **END OF CONFERENCE**

Abstracts

Brandon Alberts, University of Connecticut, *Counting Towers of Number Fields*

Fix a number field K and a finite transitive subgroup $G \leq S_n$. Malle's conjecture proposes asymptotics for counting the number of G-extensions of number fields F/K with discriminant bounded above by X. A recent and fruitful approach to this problem introduced by Lemke Oliver, Wang, and Wood is to count inductively. Fix a normal subgroup $T \trianglelefteq G$. Step one: for each G/T-extension L/K, first count the number of towers of fields $F/L/K$ with $\mathrm{Gal}(F/L) \cong T$ and $\mathrm{Gal}(F/K) \cong G$ with discriminant bounded above by X. Step two: sum over all choices for the G/T-extension L/K. In this talk we discuss the close relationship between step one of this method and the first Galois cohomology group. This approach suggests a refinement of Malle's conjecture which gives new insight into the problem. We give the solution to step one when T is an abelian normal subgroup of G, and convert this into nontrivial lower bounds for Malle's conjecture whenever G has an abelian normal subgroup.

Daniel Baczkowski, University of Findlay, *Diophantine equations involving factorials and arithmetic functions*

At the turn of the millennium, Florian Luca established that for a fixed rational number r there are only a finite number of positive integers n and m for which $f(n!) = r{\cdot}m!$ where $f \in \{d, \phi, \sigma\}$. Here d, ϕ, and σ denote the number of divisors, Euler's totient, and sum of divisors functions, respectively. The speaker along with Filaseta, Luca, and Trifonov generalized these results. We will describe these generalizations and highlight the main ideas utilized in the proof of the more difficult case of the sum of divisors σ.

Brian Beasley, Presbyterian College, *A Brief History of Distribution Results for Powerfree Numbers*

To examine the distribution of gaps between k-free numbers, Paul Erdős posed the problem of establishing an asymptotic formula for the sum of the powers of the lengths of such gaps. Michael Filaseta contributed to the ongoing work on the original problem and noted that Erdős' question may be generalized to consider gaps between positive integers m for which $f(m)$ is k-free, where f is an irreducible polynomial satisfying certain conditions. This talk, representing joint work with Filaseta in the 1990s, will present some results in the general case. It will also note the more recent research of others in this context under the assumption of the *abc*-conjecture.

Lea Beneish, Emory University, *On Weierstrass mock modular forms and a dimension formula for certain vertex operator algebras*

In joint work with Michael Mertens, using techniques from the theory of mock modular forms and harmonic Maass forms, especially Weierstrass mock modular forms, we establish several dimension formulas for certain holomorphic, strongly rational vertex operator algebras, complementing previous work by van Ekeren, Ḿoller, and Scheithauer.

Holly-Paige Chaos, Wake Forest University, *Torsion for CM Elliptic Curves in Degree 2p*

Let E be an elliptic curve defined over a number field F. By the Mordell-Weil theorem we know that the points of E with coordinates in F can be given the structure of a finitely generated abelian group. We will focus on the subgroups of points with finite order. For a given prime p and an elliptic curve E defined over a number field of degree $2p$, we would like to know exactly what torsion subgroups arise. Before discussing recent progress on this query, specifically in the case of elliptic curves with complex multiplication (CM), I will provide a brief overview on elliptic curves as well as outline some significant classical results.

Sunil Chetty, College of Saint Benedict and Saint John's University, *On primitive Pythagorean triples of special forms*

We explore how to generate families of primitive Pythagorean triples, parameterized by the difference of hypotenuse and longer leg or by the difference in the two legs. Ideally, we aim for a recursive process to generate primitive triples in a family. The former case has been well-studied by various individuals, seemingly without special attention to primitive triples, while the latter seems to be less well-studied. (This is research conducted with an undergraduate student)

Lydia Eldredge, Florida State University, *Small Mahler Measure in Cubic Fields*

The Mahler measure of a polynomial with integer coefficients, is the product, taken over all roots whose modulus is greater than 1 and the absolute value of its leading coefficient. We'll talk about the problem of determining the smallest Mahler measure of a primitive element in a cubic field. This smallest value is known to depend on the discriminant of the field by work of Silverman and Ruppert. We'll discuss an algorithm to determine this minimum value, our numerical findings, and bounds for this minimum for some families of cubic fields.

Kevin Ford, UIUC, *Anatomy of random integers, permutations and polynomials*

We discuss various problems about the multiplicative structure of random integers, permutations and polynomials over a finite field. In particular, we describe probabilistic models which connect the structure of all three objects: integers, permutations and polynomials..

Joshua Harrington, Cedar Crest College, *Odd Coverings of Subsets of the Integers*

Let S be a set of integers. A covering system of S is a finite collection of congruences such that every integer in the set satisfies at least one of the congruences in the collection. An odd covering of S is a covering system such that all moduli are distinct, odd, and greater than 1. Filaseta and Harvey recently investigated the existence of odd coverings of certain subsets of the integers. In this talk we extend this investigation and present a connection between these coverings and coverings recently studied by the presenter.

Robert Hough, Stony Brook, *The local zeta function in enumerating quartic fields*

An exact formula is obtained for the Fourier transform of the local condition of maximality modulo primes $p > 3$ in the prehomogeneous vector space $2 \otimes \mathrm{Sym}^2(\mathbb{Z}_p^3)$ parametrizing quartic fields, thus solving the local 'quartic case' in enumerating quartic fields. This extends earlier works of Taniguchi-Thorne which treat the cubic case, and the mod p condition in the quartic case.

Angel Kumchev, Towson University, *Two Theorems of Piatetski-Shapiro*

Two of Piatetski-Shapiro's early papers deal with questions about the Diophantine properties of fractional powers of integers. One of these introduced what is now known as Piatetski-Shapiro primes. The other introduced the study of the Waring-Goldbach problem for fractional exponents. In this talk, I will review some of the history of both problems and then will present some recent hybrid results. This is joint work with Zhivko Petrov (Sofia University).

Jeffrey Lagarias, University of Michigan, *On Toric Orbits in the Affine Sieve*

The affine sieve of Bourgain, Gamburd and Sarnak extends the Brun sieve to allow one to prove the existence of infinitely many integers having a bounded number of prime factors for all suitable polynomial functions of integer orbits of thin discrete groups L in a linear algebraic group G having the property that the Zariski closure of L is Levi-semisimple. The affine sieve results do not apply to such groups whose Zariski closure has a nontrivial homomorphic image that is an algebraic torus. We consider heuristics for orbits in algebraic tori where (conjecturally) there are integer orbits in which the conclusion fails: these orbits have some polynomial function statistic where only finitely many solutions have any fixed number of prime factors. We present and analyze a probabilistic model, refining a model of Salehi Golsefidy and Sarnak, which makes quantitative predictions on growth of the number of prime factors along such orbits, and compare it to familiar data: Fibonacci and Lucas numbers. (This is joint work with Alex Kontorovich (Rutgers)).

Delong Li, University of Tennessee - Knoxville, *Denominators of the Weierstrass coefficients of the canonical lifting*

Given an ordinary elliptic curve $E/k : y_0^2 = x_0^3 + a_0x_0 + b_0$ in characteristic $p \geq 5$, the canonical lifting $\mathbf{E}$ over the ring of Witt vectors is given by $\mathbf{E}/W(k) : y^2 = x^3 + ax + b$, where $a = (a_0, A_1, A_2, \ldots)$ and $b = (b_0, B_1, B_2, \ldots)$ are functions of a_0 and b_0. Finotti has proved that these functions A_i and B_i can be taken to be rational functions on a_0 and b_0 and raised questions about their denominators. In this talk we will find all the possible factors of the denominators, and give an upper bound for each factor, in the case when these functions are obtained from formulas for the j-invariant of the canonical lifting. We will also show some computations done with Magma. This is joint work with L. Finotti.

Nathan McNew, Towson University, *Counting pattern-avoiding integer partitions*

A partition α contains another partition (or pattern) μ if the Ferrers board for μ is attainable from α under removal of rows and columns. We say α avoids μ if it does not contain μ. We count the number of partitions of n avoiding a fixed pattern μ, in terms of generating functions and their asymptotic growth rates.

Jackson Morrow, Emory University, *Non-Archimedean entire curves in proper surfaces admitting a dominant morphism to an elliptic curve*

The conjectures of Green-Griffiths-Lang predict the precise interplay between different notions of hyperbolicity: Brody hyperbolic, arithmetically hyperbolic, Kobayashi hyperbolic, algebraically hyperbolic, and groupless. In his thesis, W. Cherry defined a notion of non-Archimedean hyperbolicity; however, his definition does not seem to be the "correct" version, as it does not mirror complex hyperbolicity. In recent work, A. Javanpeykar and A. Vezzani introduced a new non-Archimedean notion of hyperbolicity, which fixed this issue and also stated a non-Archimedean version of the Green-Griffiths-Lang conjecture.
In this talk, I will discuss complex and non-Archimedean notions of hyperbolicity and a proof of the non-Archimedean Green-Griffiths-Lang conjecture for surfaces admitting a dominant morphism to an elliptic curve.

Alexis Newton, Wake Forest University, *Sums of Two Sixth Powers*

What is the smallest positive integer that is a sum of two rational sixth powers, but not a sum of two integer sixth powers? Consider some positive integer m and look for a solution to $a^6 + b^6 \equiv 0 \mod m^6$. If (x, y) is a nonzero solution to this equivalence, then $(\frac{x}{m})^6 + (\frac{y}{m})^6 \in \mathbb{Z}$. Then using a lattice reduction algorithm, we can scale x and y by some c to ensure cy and cx are "small $\mod 5^6$". This gives the possible solution 164,634,913. In this talk, we will discuss the tools used to determine whether 164,634,913 is the smallest positive integer that is a sum of two rational sixth powers, but not a sum of two integer sixth powers. These methods include local solvability testing, elliptic curve maps, and the Mordell-Weil Sieve, which is a method for showing a curve C has no rational points.

Cooper O'Kuhn, University of Florida, *Mondrian's Problem and Its Relation to Number Theory*

Though seemingly geometric in nature, Mondrian's Problem, a problem involving integer-sided rectangles tessellating an integer-sided square, appears to have strong connections to important topics in Number Theory such as Erdos' Multiplication Table and Sieve Theory. In this talk, we discuss how advancements in these areas have lead to progress towards understanding Mondrian's Problem.

Carl Pomerance, Dartmouth College, *Cyclotomic Polynomials: Problems and Results*

Perhaps the prettiest family of polynomials, the cyclotomics are a fertile source of interesting problems. For example, what can be said about the largest coefficient? In the other direction, which cyclotomic polynomials have all coefficients in $\{0, 1, -1\}$? What can be said about prime values, or the largest prime factor of a value? This talk will survey many of these problems, including a new one on the difference of two cyclotomic polynomials that has a connection to the prime k-tuples conjecture.

Tomer Reiter, Emory University, *Isogenies of Elliptic Curves over* $\mathbb{Q}(2^\infty)$

Let $\mathbb{Q}(2^\infty)$ be the compositum of all quadratic extensions of $\mathbb{Q}$. Torsion subgroups of rational elliptic curves base changed to $\mathbb{Q}(2^\infty)$ were classified by Laska, Lorenz and Fujita. Recently Daniels, Lozano-Robledo, Najman, and Sutherland classified torsion subgroups of rational elliptic curves base changed to $\mathbb{Q}(3^\infty)$, the compositum of all cubic extensions of $\mathbb{Q}$. We classify all cyclic isogenies of prime power degree of rational elliptic curves base changed to $\mathbb{Q}(2^\infty)$, and give a list of possible degrees of cyclic isogenies for all but finitely many elliptic curves over $\mathbb{Q}(2^\infty)$.

Jeremy Rouse, Wake Forest University, *The density of odd order reductions*

Suppose that $E/\mathbb{Q}$ is an elliptic curve and $\alpha \in E(\mathbb{Q})$ is a point of infinite order. What is the density of primes p for which $\alpha \in E(\mathbb{F}_p)$ has odd order? Answering this questions turns out to be equivalent to describing the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the preimages of α under multiplication by 2^n. I will discuss joint work with David Zureick-Brown determining the image of the ordinary 2-adic representation, and more recent work describing the possible images of the arboreal 2-adic representation. As an application, if E is a non-CM elliptic curve and $\alpha \in E(\mathbb{Q})$ has the property that $\alpha + T$ is not twice a point over $\mathbb{Q}$ for any 2-power rational torsion point T, then the density of primes p for which $\alpha \in E(\mathbb{F}_p)$ has odd order is between $1/224$ and $1969/2688$.

Soumya Sankar, University of Wisconsin, *Proportion of ordinary curves in some families*

A curve over a field of characteristic p is called ordinary if the p-torsion of its Jacobian is the largest possible. One can ask, what is the probability that a curve over a finite field is ordinary? I will talk about perspectives on this question and answer it in the case of families such as Artin-Schreier and superelliptic curves.

Jesse Thorner, University of Florida, *New results on the Chebotarev density theorem*

(Joint with Asif Zaman) I will describe improvements in the uniformity of the Chebotarev density theorem. This consists of both pointwise results (for a given Galois extension L/K of number fields) and average results (for $L/\mathbb{Q}$ as L varies in a family of number fields, each of which is Galois over $\mathbb{Q}$). Our pointwise results are highly uniform even if Landau–Siegel zeros exist, giving a uniform improvement over the seminal work of Lagarias and Odlyzko. Our average results require no unproven hypotheses toward the strong Artin conjecture, giving a uniform improvement over a recent breakthrough of Pierce, Turnage-Butterbaugh, and Wood.

Ognian Trifonov, University of South Carolina, *Lattice points close to a plane curve*

We establish new upper bounds for the number of lattice points close to a plane convex curve of bounded curvature.

Lee Troupe, Mercer University, *Divisor sums representable as the sum of two squares*

How often does the sum of the proper divisors of a number n, denoted $s(n)$, possess some arithmetically interesting property? In 2014, Pollack showed that the number of $n \leq x$ such that $s(n)$ is prime is $O(x/\log x)$, and in 2015 he showed that the set of natural numbers n for which $s(n)$ is a palindrome has asymptotic density zero. This talk concerns a new result in the same vein: The number of $n \leq x$ such that $s(n)$ can be written as a sum of two squares has order of magnitude $x/\sqrt{\log x}$, which coincides (up to constants) with the number of $n \leq x$ which can be written as a sum of two squares.

Charlotte Ure, University of Virginia, *Brauer Groups of Elliptic Curves*

The Brauer group of an elliptic curve is an important invariant associated to the curve. Elements in this group are equivalence classes of central simple algebras over the function field. The Brauer group can detect arithmetic and algebraic properties of the underlying curve. In particular, Manin showed that certain elements in the Brauer obstruct the existence of rational points on the elliptic curve. In this talk, I will discuss how we can achieve an explicit description of the prime torsion of these groups over any base field.

Biao Wang, University at Buffalo, *Möbius function, Liouville function, Ramanujan sum and Chebatorev densities*

For the Möbius function μ, it is well-known that the prime number theorem is equivalent to $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$. In 1977, Alladi showed a formula on the restricted sum of $\frac{\mu(n)}{n}$. In 2017, Dawsey generalized Alladi's result to the setting of Chebotarev densities for finite Galois extensions of $\mathbb{Q}$. In this talk, we will introduce the analogues of their formulas with respect to Liouville function and Ramanujan sum, which are closely related to the Möbius function.

Jiuya Wang, Duke University, *Bounding ℓ-torsion in class groups of elementary abelian extensions*

Elementary abelian groups are finite groups in the form of $(\mathbb{Z}/p\mathbb{Z})^r$ for a prime number p. We prove a non-trivial bound on the ℓ-torsion in class groups of non-cyclic elementary abelian extensions. The bound we obtain is pointwise and unconditional. It also breaks the GRH bound when r is large enough.

Gang Yu, Kent State University, *Imaginary quadratic fields with class groups of* 3*-rank at least* 2

We construct a family of imaginary quadratic fields whose class group has 3-rank at least 2. We show that, for every large X, there are $\gg X^{1/2-\epsilon}$ such fields with discriminants $-D$ satisfying $D \leq X$.

Daozhou Zhu, Clemson University, *Distinguishing eigenforms*

Let $f = \sum_{n=1}^{\infty} a_n(f)q^n$ and $g = \sum_{n=1}^{\infty} a_n(g)q^n$ be two normalized Hecke eigenforms of level one. Assuming the irreducibility of characteristic polynomials of Hecke operators T_2, Trevor Vilardi and Hui Xue proved that $f = g$ iff $a_2(f) = a_2(g)$. In this talk, we will show that $f = g$ iff $a_3(f) = a_3(g)$, assuming the irreducibility of characteristic polynomials of T_3.
