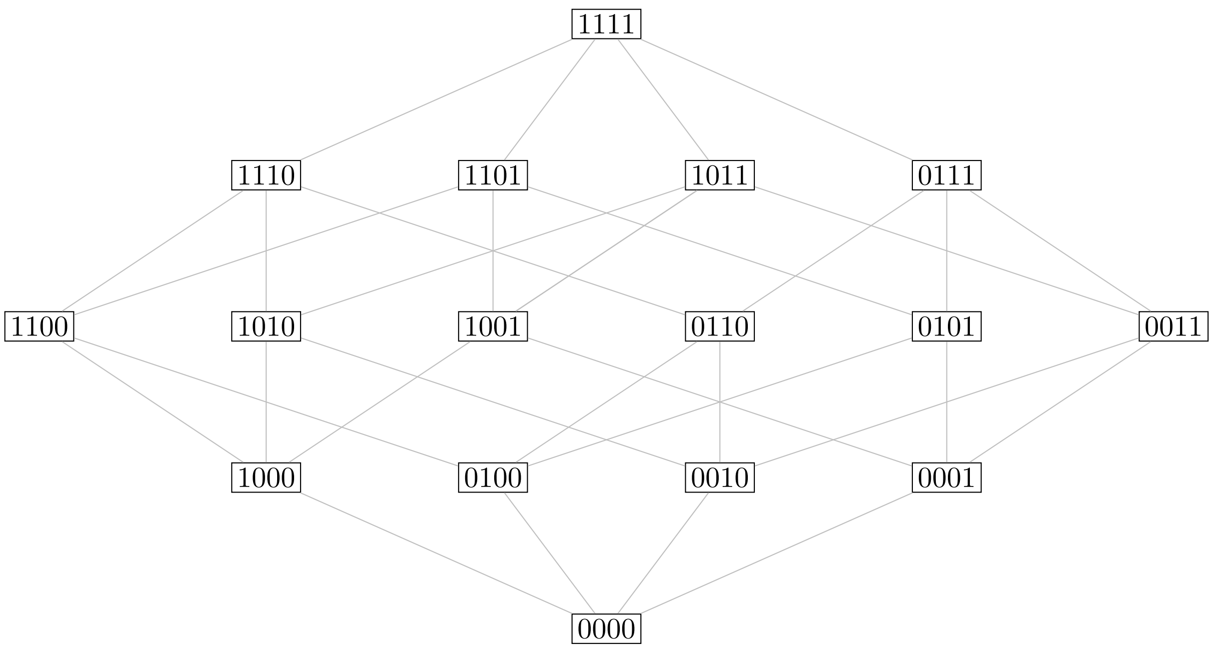
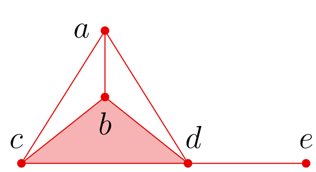
Read. *Mathematical Concepts in Modern Biology*, by Robeva/Hodge. Ch. 3: Inferring the topology of gene regulatory networks: an algebraic approach to reverse engineering, and *Algebraic and Combinatorial Computational Biology*, by Robeva/Hodge. Ch 6: Inferring interactions in molecular networks via primary decompositions of monomial ideals.

Exercises.

1. **Asynchronous automata**. Draw the asynchronous automaton the following Boolean model:
$$(f_1, f_2, f_3, f_4) = (x_2 \wedge \overline{x_3},\ \ \overline{x_1},\ \ x_3 \vee x_4,\ \ x_1 + x_2).$$
Then, partition the nodes into strongly connected components, and draw the acyclic directed graph formed by collapsing the SCCs into single nodes. Find the attractors and classify them by type: fixed point, cyclic attractor, or complex attractor. The Boolean lattice B_4 is shown below.



2. **Alexander duality**. Consider the simplicial complex Δ over $X = \{a, b, c, d, e\}$ shown below:

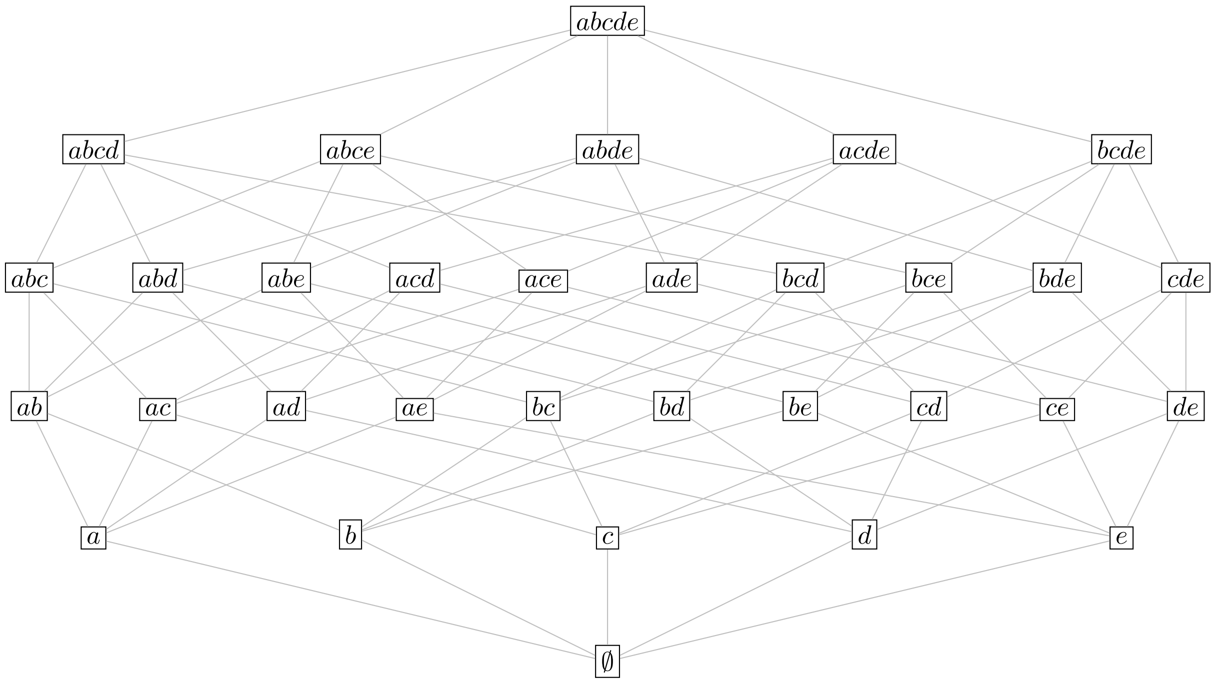


The 5-dimensional Boolean lattice 2^X also appears below.

(i) In the Boolean lattice, color the faces red, and the nonfaces blue.

(ii) Find the maximal faces and minimal generators of the Stanley-Reisner ideal I_{Δ^c}.

(iii) Compute the primary decomposition of I_{Δ^c}.



3. **Reverse-engineering a Boolean model**. Consider a Boolean model $f = (f_1, f_2, f_3)$ whose (synchronous) phase spaces consists of the following:

$$(0,0,1) \xrightarrow{f} (1,0,1) \xrightarrow{f} (1,1,1) \xrightarrow{f} (1,1,0) \xrightarrow{f} (0,1,0) \xrightarrow{f} (0,0,0).$$

In this problem, you will reverse-engineer the wiring diagram and then the models space.

(a) Do the following steps for each $k = 1, 2, 3$.

i. Write down the corresponding set of *data*

$$\mathcal{D}_k := \big\{(\boldsymbol{s}_1, t_{1k}), \dots, (\boldsymbol{s}_5, t_{5k})\big\}$$

that arises from the k^{th} coordinate of this time-series.

ii. Find the monomial ideal $I_{\Delta_k^c}$ of non-disposable sets, and compute its primary decomposition to find the min-sets.

iii. Find the pseudomonomial ideal $J_{\Delta_k^c}$, and compute its primary decomposition to find the signed min-sets.

(b) Find all min-sets and signed min-sets of the original Boolean model.

(c) Use Macaulay2 to compute the vanishing ideal $I(\mathcal{D})$. How big is it?

(d) Find the model space $\mathrm{Mod}(\mathcal{D}) = \prod_{i=1}^{3} \big[f_i + I(\mathcal{D})\big]$. Use Lagrange interpolation to find $f = (f_1, f_2, f_3)$, and Macaulay2 to reduce them modulo $I(\mathcal{D})$.