Random Boolean Networks

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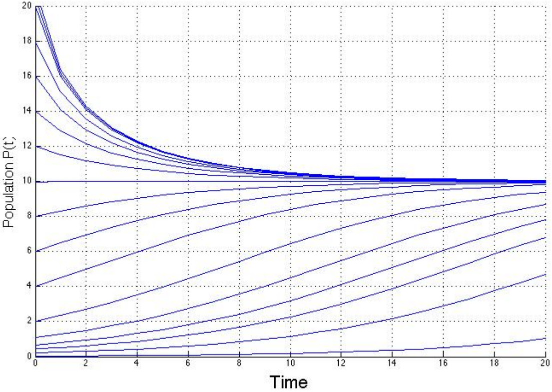
Algebraic Biology

A simple dynamical system map that exhibits chaos

The logistic equation $x_{t+1} = rx_t(1 - x_t)$ is a simple 1D dynamical system map.

In 1976, biologist Robert May showed that it can exhibit chaos.

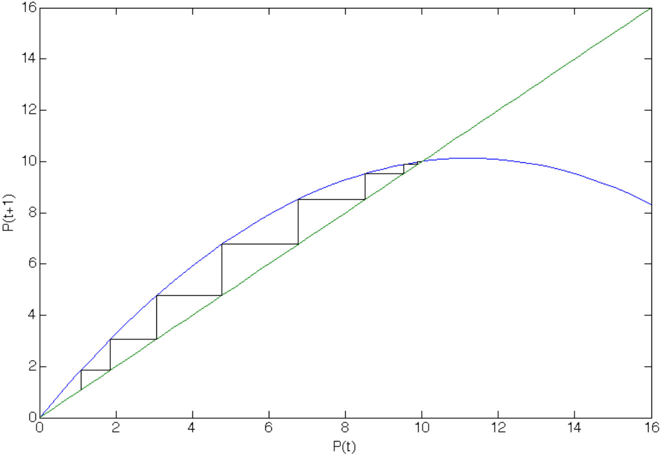
Here are some (interpolated) solutions to $x_{t+1} = rx_t(1 - x_t/M)$ for $r = 1.2$, $M = 125/3$.



Cobwebbing

Given a dynamical system $x_{t+1} = rx_t(1 - x_t/M)$, we can numerically find $x_0, x_1, x_2, \ldots$ by plotting x_{t+1} vs. x_t on the same axes, and then "cobwebbing."

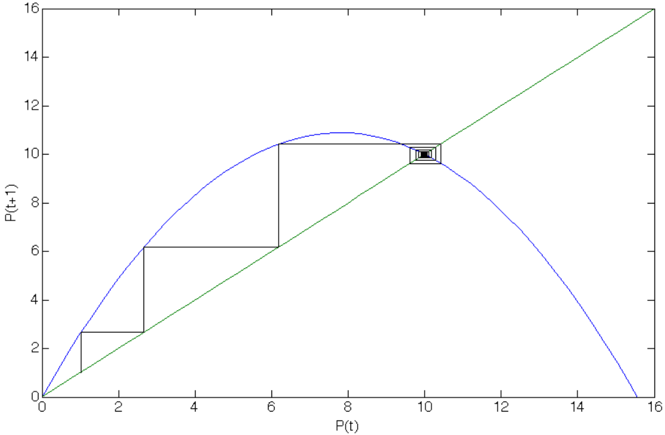
Here is an example, for $r = 1.8$.



Cobwebbing

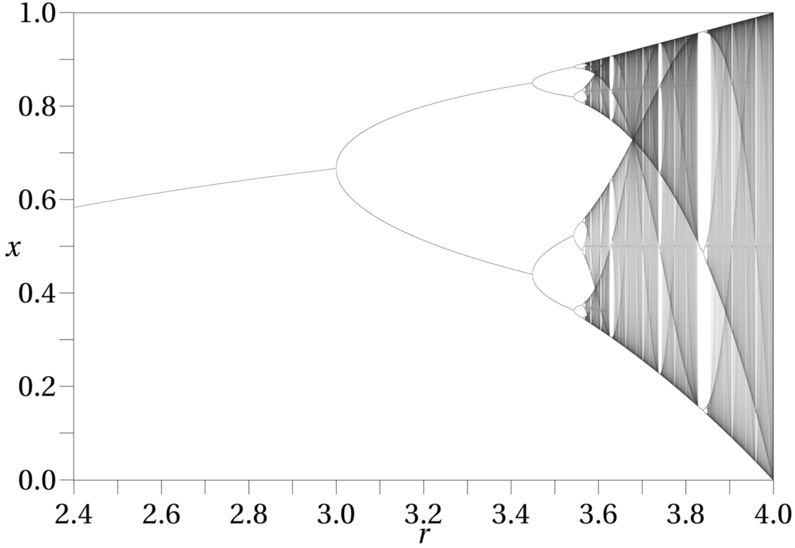
Given a dynamical system $x_{t+1} = rx_t(1 - x_t/M)$, we can numerically find $x_0, x_1, x_2, \ldots$ by plotting x_{t+1} vs. x_t on the same axes, and then "cobwebbing."

Here is an example, for $r = 2.8$. What differences do you notice?



Bifurications

The attractors, as a function of r, can be plotted in a bifurication diagram.



Bifurications

The logistic equation is a good example of:

- How chaotic behavior can emerge from a very simple map.
- How tuning a parameter can change a system from ordered to chaotic.

Dynamics of the logistic map

The solutions to $x_{t+1} = rx_t(1 - x_t)$ are non-negative and bounded for $r \in [0, 4]$.

The long-term behavior for various values of r are:

- Fixed point $r^* = \frac{r-1}{r}$ (overdamped) for $1 < r < 2$.
- Fixed point $r^* = \frac{r-1}{r}$ (underdamped) for $2 < r < 3$.
- Size-2 stable attractor for $3 < r < 1 + \sqrt{6} \approx 3.44949$.
- Size-4 stable attractor for $3.44949 < r < 3.54409$.
- $3.56995 < r < 4$ chaos (with a few exception).
- Stable 3-cycle at $r = 1 + \sqrt{8} \approx 3.8284$.
- Stable $2^n p$-cycle for some $r < 4$ for every prime p.

Classical vs. statistical mechanics

Classical mechanics studies the laws of motion of single point masses.

The famous two-body problem was posed and solved by Isaac Newton in 1687.

The three-body problem has no closed form general solution.

In physics, the field of statistical mechanics studies large assemblies of microscopic entities (e.g., atoms, molecules, particles, etc.) using the tools of probability and statistics.

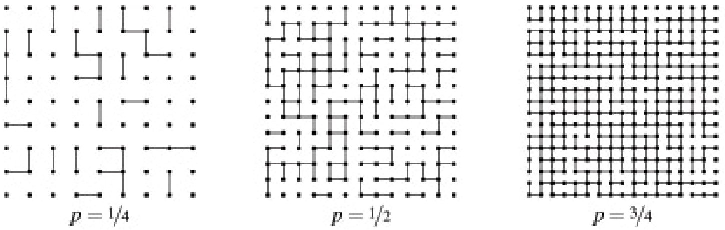
A fundamental concept of stat mech is the statistical ensemble: a large (or infinite) collection of independent copies of the entity. This was introduced by Gibbs in 1902.

Formally, this is a probability distribution over all possible system states.

Lattices and percolation theory

Percolation theory is a topic involving the statistical mechanics of phase transitions.

Imagine a square lattice where each edge is present with some fixed probability p.



- small p: "islands"
  perturbations die out
- $p \approx .5$: ???
- large p: "holes"
  perturbations spread

If this models a blight spreading through a forest, then $p = .5$ is the critical threshold between the impact being finite or infinite.

This is a phase transition.

Statistical mechanics and Boolean networks

In 1969, Stuart Kauffman introduced random Boolean networks (RBNs) as models of gene regulatory networks.

In the Kauffman model, there are N nodes, each one having K randomly chosen inputs.

Each node f_v gets an update function $f_v \colon \mathbb{F}_2^K \to \mathbb{F}_2$, randomly chosen from some distribution.

Kauffman noticed that for $K = 1$, the BNs had small attractors; lots of fixed points. These networks are called stable.

For large K, the BNs had lots of large attractors. These networks are called chaotic.

Networks for $K = 2$ are on the boundary of these phases. They are called critical.

Early evidence seemed to suggest that biological networks shared a number of properties (scaling laws) with these random critical networks.

For a nice survey of a stat mech view of Boolean networks:

- Drossel, B. (2008). Random Boolean networks. *Reviews of nonlinear dynamics and complexity*, 69-110.

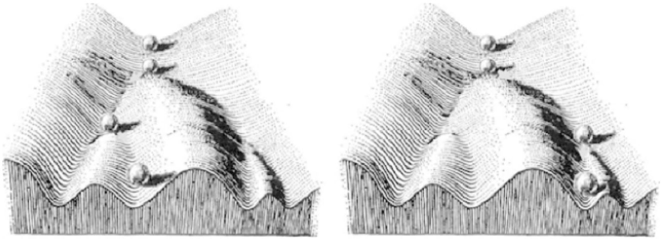
Canalization

In 1942, geneticist C.H. Waddington developed the concept of a epigenetic landscape

He also introduced canalization, a measure of evolutionary robustness.

It quantifies how a population can produce the same phenotype, despite changes to its environment or genotype.

He described these as "canals" in the epigenetic landscapes.



- Waddington, C. H. (1942). Canalization of development and the inheritance of acquired characters. *Nature* **150**(3811), 563–565.
- Waddington, C. H. (1957). The strategy of the genes. London: George Allen & Unwin.

Canalization

Canalization was quantified in Boolean functions by Stuart Kauffman in 1993.

Definition

A Boolean function $f\colon \mathbb{F}_2^n \to \mathbb{F}_2$ is canalizing if has some (non-fictitious) input x_i for which

$$f(x_1,\dots,x_n) = \begin{cases} b & x_i = a \\ g(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) & x_i = \overline{a}. \end{cases}$$

Canalizing Boolean functions have the following form

$$f(x_1,\dots,x_n) = y_i \lozenge g(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n),$$

where $y_i \in \{x_i, \overline{x_i}\}$ and $\lozenge \in \{\wedge, \vee\}$.

If the function g is canalizing, and so on (iterate n times), then f is nested canalizing.

In the early 2000s, Kauffman et al. showed that random Boolean networks built with canalizing (and nested canalizing) functions are stable.

- Karlsson F, Hörnquist M (2007). Order and chaos in Boolean gene networks depends on the mean fraction of canalizing functions. *Physica A* **384**:747–575.
- Kauffman, S., Peterson, C., Samuelsson, B., & Troein, C. (2004). Genetic networks with canalyzing Boolean rules are always stable. *Proc. Natl. Acad. Sci.*, **101**(49), 17102–17107.

Ensembles of random Boolean networks

A random Boolean network consists of N nodes, each with K randomly chosen inputs.

An ensemble of networks is a large number of these RBNs generated in this fashion.

Note that *all* possible topologies occurs, but with different probabilities.

For example, with $N = 2$ and $K = 1$, there are three possible topologies:

$p = .25$ $\quad$ $p = .25$ $\quad$ $p = .5$

A given node v is the input of each node with probability K/N.

Proposition

In the thermodynamic limit $N \to \infty$, the number of out-nodes follows a Poisson distribution

$$P(\text{out-degree is } d) = \frac{K^d}{d!} e^{-K}.$$

Creating random Boolean networks

A random Boolean network on N nodes, each having K inputs is an NK-network.

The first step is to pick a network topology, via some distribution.

Then, the functions $f_i \colon \mathbb{F}_2^K \to \mathbb{F}_2$ are sampled via some distribution.

One example is to pick random Boolean functions with bias p.

This means that the truth table is a length-2^K vector of i.i.d. Bernoulli random variables.

x	0	0	1	1	0	0	1	1
y	0	1	0	1	0	1	0	1
z	0	0	0	0	1	1	1	1
$f(x,y,z)$	a_{000}	a_{010}	a_{100}	a_{110}	a_{001}	a_{011}	a_{101}	a_{111}

The weight of a Boolean function $f(x_1, \dots, x_K)$ is the number of 1s in its truth table.

Each function occurs with probability

$$\Pr(f) = p^w (1-p)^{M-w}, \qquad \text{where } f \text{ has weight } w, \text{ and } M = 2^K.$$

Note that a uniform distribution is the special case of $p = \frac{1}{2}$.

Sampling of Boolean functions

There are $2^{2^1} = 4$ one-variable Boolean functions:

x	constant		inv't	
0	1	0	0	1
1	1	0	1	0

There are $2^{2^2} = 16$ two-variable Boolean functions:

x_1x_2	constant		canalizing, 1 input				canalizing, 2 inputs								inv't	
00	1	0	0	1	0	1	1	0	0	0	0	1	1	1	1	0
01	1	0	0	1	1	0	0	1	0	0	1	0	1	1	0	1
10	1	0	1	0	0	1	0	0	1	0	1	1	0	1	0	1
11	1	0	1	0	1	0	0	0	0	1	1	1	1	0	1	0

When creating a random Boolean network, the functions must be sampled using some distribution. Examples include:

- Uniform distribution
- Biased functions
- Weighted classes
- Only canalizing functions
- Only threshold functions
- All functions are the same

Update function distributions

Definition

In an ensemble of networks, an update function probability distribution is:

- permutation invariant if $\Pr(f) = \Pr(g)$ whenever f and g have the same weight
- inversion invariant if $\Pr(f) = \Pr(\overline{f})$, where $\overline{f}$ is the inverted rule.

A common permutation-invariant example is the biased probability distribution:

$$\Pr(f) = p^w(1-p)^{M-w}, \qquad \text{where } f \text{ has weight } w, \text{ and } M = 2^K.$$

x_1x_2	constant		canalizing, 1 input				canalizing, 2 inputs								inv't	
00	1	0	0	1	0	1	1	0	0	0	0	1	1	1	1	0
01	1	0	0	1	1	0	0	1	0	0	1	0	1	1	0	1
10	1	0	1	0	0	1	0	0	1	0	1	1	0	1	0	1
11	1	0	1	0	1	0	0	0	0	1	1	1	1	0	1	0
prob.	p^4	q^4	p^2q^2				pq^3				p^3q				p^2q^2	
$p = \frac{1}{3}$	$\frac{1}{81}$	$\frac{16}{81}$	$\frac{4}{81}$				$\frac{8}{81}$				$\frac{2}{81}$				$\frac{4}{81}$	

RBNs with inversion-invariant distributions

Proposition

An ensemble of RBNs with an inversion-invariant distribution has an average of one fixed point per network.

Proof

Pick any $x \in \mathbb{F}_2^n$, and let $y = f(x)$.

Since $\Pr(f) = \Pr(\overline{f})$, the probability that $y_i = x_i$ is $1/2$, for each i.

Therefore, the probability that $f(x) = y$ is $(1/2)^N = 2^{-N}$.

Thus, each of the 2^N states is a fixed points in the proportion 2^{-N} of all networks. □

RBNs with biased distributions

A random Boolean function f with bias p is a length-2^K string of iid Bernoulli random variables.

Proposition

An ensemble of RBNs with a biased distribution

$$\Pr(f) = p^w (1-p)^{M-w}, \qquad \text{where } f \text{ has weight } w, \text{ and } M = 2^K.$$

has an average of one fixed point per network.

Proof

Since biased $\Rightarrow$ permutation-invariant, for any $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{F}_2^N$, the transitions $f(\mathrm{x}) = \mathrm{z}$ and $f(\mathrm{y}) = \mathrm{z}$ are equally likely.

Consider a state $\mathrm{z} \in \mathbb{F}_2^N$ with Hamming weight $w \in \{0, \dots, N\}$.

The average number of states x such that $f(\mathrm{x}) = \mathrm{z}$ is $2^N p^w (1-p)^{N-w}$.

Also, *every* state leads to z equally often, so this is a fixed point in the proportion $p^w (1-p)^{N-w}$ of networks.

The mean number of fixed points is the sum of this over all states, which is 1. □

Counting length-L cycles (Samuelsson & Troein, 2003)

Consider an L-cycle $\mathrm{x}_1 \to \mathrm{x}_2 \to \cdots \to \mathrm{x}_L \to \mathrm{x}_1$ in a $K = 2$ network.

Fix a coordinate $i \in \{1, \dots, N\}$. There are 2^L possible sequences $x_{1i} \to x_{2i} \to \cdots \to x_{Li}$.

Let $n_j \in \{0, \dots, N\}$ be # nodes with sequence $j \in \{0, \dots, m = 2^L - 1\}$ on a length-L cycle.

Let $(P_L)^j_{\ell,k}$ be the probability that a node with input sequences ℓ and k generates output sequence j. (*Depends on the probability distribution of update functions*.)

The mean number of length-L cycles in a $K = 2$ network is:

$$\langle C_L \rangle_N = \frac{1}{L} \sum_{\{n_j\}} \frac{N!}{n_0! \cdots n_m!} \prod_j \left(\sum_{\ell,k} \frac{n_\ell n_k}{N^2} (P_L)^j_{\ell,k} \right)^{n_j} .$$

- $\frac{1}{L}$: any of the L states on the cycle could be the starting point
- $\frac{N!}{n_0! \cdots n_m!}$: # ways to divide N nodes into groups of sizes $n_0, \dots, n_m$.
- $\sum_{\{n_j\}}$: over all ways to chose $n_0, \dots, n_m$ so that $n_0 + \cdots + n_m = N$
- Product: probability that each node z with a sequence j is connected to nodes x, y, w/ sequences ℓ & k and has a update function s.t. $f(\mathrm{x}) = f(\mathrm{y}) = \mathrm{z}$.

Annealed approximation

A mean field theory (MFT) is a stat mech concept to reduce high-dimensional stochastic model to a simple "averaged" one.

One example: Derrida and Pomeau's annealed approximation of random Boolean networks.

RBN assumptions

- The network is infinitely large (fluctuations of global quantities are negligible).
- The inputs of each node can be reset each time-step ("annealed").

Quantities of interest for RBNs (see Drossel, 2008)

- Time-evolution of weight (proportion of 1s).
- Time-evolution of the (Hamming) distance between states of identical networks.
- Statistics of small perturbations.

Important theme

Quantify and understand the three phases of RBNs: chaotic, stable, and critical, and how they depend on parameters.

The time evolution of the proportion of 1s and 0s

Pick a random $x \in \mathbb{F}_2^N$, and consider the update function $f_i \colon \mathbb{F}_2^K \to \mathbb{F}_2$ at node i.

Let p_m be the probability that if m inputs are 1, the output is 1:

$$p_m = \Pr\left[f_i(x) = 1 \mid m \text{ inputs are } 1\right]$$

Let's compute p_m for the biased rules and weighted rules.

m	x_1x_2	constant		canalizing, 1 input				canalizing, 2 inputs								inv't	
0	00	1	0	0	1	0	1	1	0	0	0	0	1	1	1	1	0
1	01	1	0	0	1	1	0	0	1	0	0	1	0	1	1	0	1
1	10	1	0	1	0	0	1	0	0	1	0	1	1	0	1	0	1
2	11	1	0	1	0	1	0	0	0	0	1	1	1	1	0	1	0
prob.		p^4	q^4	p^2q^2				pq^3				p^3q				p^2q^2	
p_m		p		p				p				p				p	

m	x_1x_2	constant		canalizing, 1 input				canalizing, 2 inputs								inv't	
0	00	1	0	0	1	0	1	1	0	0	0	0	1	1	1	1	0
1	01	1	0	0	1	1	0	0	1	0	0	1	0	1	1	0	1
1	10	1	0	1	0	0	1	0	0	1	0	1	1	0	1	0	1
2	11	1	0	1	0	1	0	0	0	0	1	1	1	1	0	1	0
prob.		α		β				γ								δ	
p_m		$\frac{1}{2}$		$\frac{1}{2}$				$\frac{1}{2}$								$\frac{1}{2}$	

The time evolution of the proportion of 1s and 0s

Consider a threshold function with threshold $T \in \mathbb{Z}$ and signs $c_{ij} \in \{-1, 1\}$:

$$f_i(x_1, \ldots, x_K) = \begin{cases} 1 & \text{if } \sum_{j=1}^{K} c_{ij}(2x_j - 1) \geq T \\ 0 & \text{otherwise.} \end{cases}$$

Here are examples, for various values of the threshold T:

1	x_1	0	0	1	1	0	0	1	1
1	x_2	0	1	0	1	0	1	0	1
-1	x_3	0	0	0	0	1	1	1	1
c_{ij}	$\sum c_{ij}(2x_j - 1)$	-1	1	1	3	-3	-1	-1	1
$T = -1$	$f_i(x, y, z)$	1	1	1	1	0	1	1	1
$T = 1$	$f_i(x, y, z)$	0	1	1	1	0	0	0	1
$T = 3$	$f_i(x, y, z)$	0	0	0	1	0	0	0	0

If $c_{ij} = 1$ and -1 with equal prob., then $c_{ij}(2x_j - 1) = +1$ and -1 with equal prob., so

$$\begin{aligned} p_m &= \Pr\left[f_i(x) = 1 \mid m \text{ inputs are } 1\right] \\ &= \Pr\left[\text{sum of } K \text{ random vars from } \{-1, 1\} \text{ is } \geq -T\right] \\ &= \left(\frac{1}{2}\right)^K \sum_{\ell \geq (K-T)/2} \binom{K}{\ell}, \qquad \text{where } \ell = \#1\text{s}. \end{aligned}$$

The time evolution of the proportion of 1s and 0s

For $x \in \mathbb{F}_2^N$, let $b_t = w(x)/N$, i.e., the proportion of nodes of x in state 1.

Within the annealed approximation, let's compute b_{t+1} as a function of b_t.

Now, let's update node i via $f_i \colon \mathbb{F}_2^K \to \mathbb{F}_2$. Since the K inputs are reassigned at each step,

$$\Pr\big[m \text{ inputs are } 1\big] = b_t^m (1-b_t)^{K-m} =$$

This is just the proportion of nodes that have m inputs in state 1.

If $p_m = \Pr\big[f_i(x) = 1 \mid m \text{ inputs are } 1\big]$, then the proportion of nodes in state 1 in the next time step is

$$\begin{aligned} b_{t+1} &= \sum_{m=0}^{K} \binom{K}{m} p_m b_t^m (1-b_t)^{K-m} \\ &= p_m \sum_{m=0}^{K} \binom{K}{m} b_t^m (1-b_t)^{K-m} = p_m \qquad \text{(if } p_m \text{ doesn't depend on } m\text{).} \end{aligned}$$

Note that p_m is independent of m for biased, weighted, and threshold functions.

In this case, b_t reaches its stationary value p_m after just one time step.

The time evolution of the proportion of 1s and 0s

Recall that $f\colon \mathbb{F}_2^n \to \mathbb{F}_2$ is canalizing if has some (non-fictitious) input x_i for which

$$f(x_1,\dots,x_n)=\begin{cases} b & x_i=a \\ g(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) & x_i=\overline{a}. \end{cases}$$

RBNs can be built with canalizing functions as follows (Moreira & Amaral, 2005):

- Choose one input $j\in\{1,\dots,K\}$ at random to be canalizing.
- Let $\eta=$ probability that the canalizing input is $x_j=1$.
- Let $r=$ probability that the canalized output is $b=1$.
- Pick the function g with bias p.

The proportion of 1s in the next time step is

$$b_{t+1}=\underbrace{b_t\eta r+(1-b_t)(1-\eta)r}_{\text{prob. } x_i=a \text{ and } b=1}+\underbrace{b_t(1-\eta)p+(1-b_t)\eta p}_{\text{prob. } x_i=\overline{a} \text{ and } g=1}=r+\eta(p-r)+b_t(p-r)(1-2\eta).$$

This 1D dynamical system map has a unique fixed point (set $b_{t+1}=b_t=b^*$)

$$b^*=\frac{r+\eta(p-r)}{1-(p-r)(1-2\eta)}.$$

This fixed point is stable. (Set $b_t=b^*+h_t$, $b_{t+1}=b^*+h_{t+1}$; and $|h_{t+1}/h_t|<1$).

The time evolution of the proportion of 1s and 0s

For some RBNs, the one-dimensional map $b_{t+1} = F(b_t)$ has an unstable fixed point, have periodic oscillations, or is chaotic.

For example, if $K = 2$ and the functions are all $f(x_1, x_2) = \overline{x_1 \wedge x_2}$ ("NAND"), then

$$b_{t+1} = 1 - b_t^2 \qquad \Longrightarrow \qquad b^* = \frac{-1+\sqrt{5}}{2}, \qquad |h_{t+1}/h_t| = (1-\sqrt{5}) > 1.$$

This system oscillates between 0 and 1. (Period 2 attractor.)

For general K, the function $f(x_1, \ldots, x_K) = \begin{cases} 1 & x_1 = \cdots = x_K \\ 0 & \text{otherwise} \end{cases}$ leads to the map

$$b_{t+1} = b_t^K + (1-b_t)^K,$$

which is chaotic for $K \geq 5$.

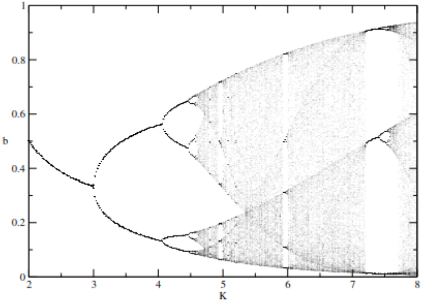
- Moreira, A.A., & Amaral, L.A.N. (2005). Canalizing Kauffman networks: Nonergodicity and its effect on their critical behavior. *Phys. Rev. Lett.*, **94**(21), 218702.
- Andrecut, M. & Ali, M.K. (2001). Chaos in a simple Boolean network. *Intl. J. Mod. Phys. B*, **15**(01), 17-23.

The time evolution of the proportion of 1s and 0s

Considers K as a continuous parameter, the 1D dynamical system map

$$b_{t+1} = b_t^K + (1 - b_k)^K$$

leads to the following bifuritication diagram (see Drossel, 2008):



Assuming the annealed approximation results apply to the original NK-networks,

- almost all states with a value of b_0 undergo the same trajectory b_t with time.
- this trajectory is the same for almost all networks
- the map $b_t \mapsto b_{t+1}$ is the same initially as it is when all states have reach an attractor.

Boolean derivatives, activities, and sensitivities

Define the Boolean j^{th} partial derivative as

$$\frac{\partial f(x)}{\partial x_j} = f(x+e_j) - f(x) = \begin{cases} 1 & \text{toggling the } j^{\text{th}} \text{ bit flips the output} \\ 0 & \text{otherwise.} \end{cases}$$

The activity of x_j in f is the "*probability that toggling the j^{th} bit flips the output*:"

$$\alpha_j^f := \frac{1}{2^K} \sum_{x \in \mathbb{F}_2^K} \frac{\partial f(x)}{\partial x_j} = E\left[\tfrac{\partial f(x)}{\partial x_j}\right].$$

The sensitivity of f at x is the "*number of Hamming neighbors on which f is different*:"

$$s^f(x) = \big|\{i \in [1,\dots,K] \mid f(x+e_i) \neq f(x)\}\big| = \sum_{i=1}^{K} \frac{\partial f(x)}{\partial x_i}.$$

The average sensitivity of f, over all $x \in \mathbb{F}_2^K$ is

$$s^f := E\left[s^f(x)\right] = \frac{1}{2^K} \sum_{x \in \mathbb{F}_2^K} s^f(x) = \frac{1}{2^K} \sum_{x \in \mathbb{F}_2^K} \sum_{i=1}^{K} \frac{\partial f(x)}{\partial x_i} = \sum_{i=1}^{K} \Big(\frac{1}{2^K} \sum_{x \in \mathbb{F}_2^K} \frac{\partial f(x)}{\partial x_i} \Big) = \sum_{i=1}^{K} \alpha_i^f.$$

Activity and sensitivity

Consider a random Boolean function f with bias p, as a length-2^K vector (truth table).

For any $x \in \mathbb{F}_2^K$, the probability that $f(x) \neq f(x+e_i)$ is $2p(1-p)$, and so all activities are

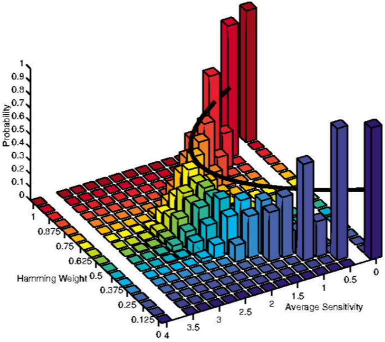
$$E\left[\alpha_1^f\right] = \cdots = \left[\alpha_K^f\right] = 2p(1-p).$$

Thus, the (expected) average sensitivity is

$$E\left[s^f\right] = \sum_{i=1}^{K} E\left[\alpha_i^f\right] = \sum_{i=1}^{K} 2p(1-p) = 2Kp(1-p).$$

Here is a histogram of Boolean functions on $K = 4$ variables.

The black line is the curve $2Kp(1-p)$.



Lyapunov exponents

The Lyapunov exponent of a dynamical system measure the divergence rate of infinitesimally close trajectories.

Consider an equilibrium position x_0 in the phase space, and a perturbation $x_0' = x_0 + \epsilon_0$.

After time t, this perturbation is $\epsilon_t \approx \epsilon_0 e^{t\lambda(x_0)}$, i.e., it

- grows if $\lambda > 0$; the dynamical system is chaotic
- shrinks if $\lambda < 0$; the dynamical system is stable.

In higher dimensions, the Jacobian matrix must be considered. This divergence will depend on the eigenvalues. The largest is the maximum Lyapunov exponent.

In random NK-networks, the Lyapunov exponent is the logarithm of average sensitivity:

$$\lambda = \ln E\left[s^f\right] = \ln\left[2Kp(1-p)\right].$$

- Luque, B., & Solé, R. V. (2000). Lyapunov exponents in random Boolean networks. *Physica A* **284**(1-4), 33-45.

Lyapunov exponents and sensitivity

The normalized Hamming distance between states $x, y \in \mathbb{F}_2^N$ is $h_t = \frac{1}{N}\sum_{i=1}^{N}\big[(x(t) - y(t)\big]$.

If N is large and $h_t \approx 0$ is small, then

$$h_{t+1} \approx e^{\lambda} h_t,$$

where $e^{\lambda} = E(s^f)$, the average sensitivity.

The average sensitivity for commonly used functions are:

- **biased functions**: $2Kp(1-p)$
- **weighted classes,** $K = 1$: $1 - \delta$
- **weighted classes,** $K = 2$: $\alpha + 2\beta + \gamma = 1 + \beta - \delta$
- **canalizing functions**: $r(1-p) + (1-r)p + (K-1)(\eta(1-b_t) + (1-\eta)b_t)2p(1-p)$
- **threshold functions**: $K\left(\frac{1}{2}\right)^{K-1}\binom{K-1}{\lfloor (K-T)/2 \rfloor}$.

Derrida plots

Start with two randomly chosen states $x(t)$ and $y(t)$ in $\mathbb{F}_2^N$.

The normalized Hamming distance between them is $\rho(t) = \frac{1}{N}\sum_{i=1}^{N}\big[(x(t) - y(t)\big]$.

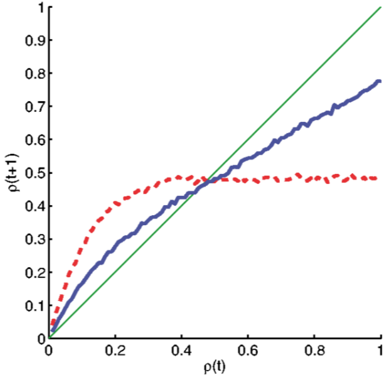
The Derrida plot of a RBN is the graph of $\rho(t+1)$ over $\rho(t)$.

This measures how perturbations grow or shrink over time.

Derrida plots are typically generated experimentally, by randomly sampling many pairs of vectors.

For $\rho(t) \approx 0$, the RBN is:

- chaotic if the curve lies above $y = x$,
- ordered if the curve lies below $y = x$,
- critical if the curve lies on $y = x$.



Criticality

In networks that are ordered:

- small perturbations die out,
- most attractors tend to be small, with fixed points.

In networks that are chaotic:

- small perturbations spread throughout the network,
- there tends to be large attractors (finite proportion of state space).

In networks that are critical:

- the growth of small perturbations follows a power law, $n(s) \sim s^{-3/2}$,
- the dynamics exhibits robustness and evolvability, of living systems.

- Aldana M, Balleza E, Kauffman S, Resendis O (2007) Robustness and evolvability in gene regulatory networks. *J. Theor. Biol.* 245:433–448.
- Drossel, B. (2008). Random Boolean networks. *Reviews of nonlinear dynamics and complexity*, 69-110.

Criticality in biological networks

In the early 2000s, it was hypothesized that many real-world biological networks lie near the critical phase.

- Nykter M, Price ND, Aldana M, Ramsey SA, Kauffman SA, Hood LE, Yli-Harja O, Shmulevich I (2008) Gene expression dynamics in the macrophage exhibit criticality. *Proc. Natl. Acad. Sci.* **105**(6): 1897—1900.
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- Mora, T., & Bialek, W. (2011). Are biological systems poised at criticality?. *J. Stat. Phys.* **144** 268–302.

In 2018, a group of scientists studied 67 networks from the Cell Collective database, and showed that they all exhibited near-critical behavior.

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