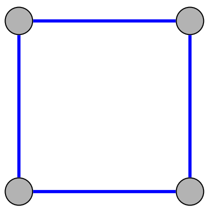
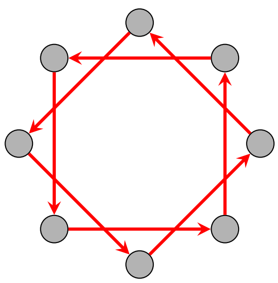
MthSc 412: Abstract Algebra (Fall 2010)
Midterm 1

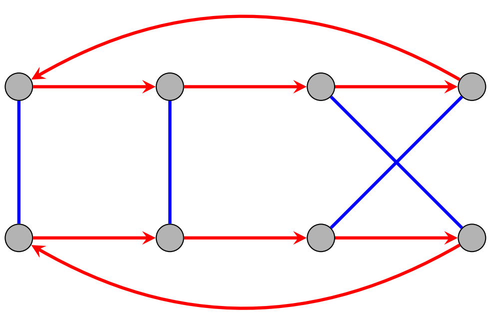
NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (2 points each) Determine whether each of the following diagrams are Cayley diagrams. If the answer is "no," briefly explain why.

(a) 

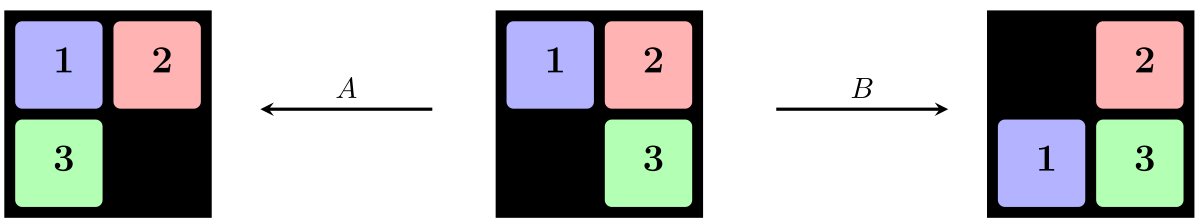
(b) 

(c) 

2. (4 points) Complete the following table so that it represents the multiplication table for a group.

$*$	e	a	b	c
e				
a		c		
b				
c				

3. (10 points) Consider the following toy that was proposed in lecture. There are three squares labeled 1, 2, and 3, on a 2×2 grid, with one empty square (denoted with an X), so the three squares can slide around. This empty square can be filled in using one of two actions: (A) Fill in the empty square in the clockwise direction, and (B) Fill in the empty square in the counterclockwise direction. These two actions generate a group. The actions of the generators are shown below.



(a) How many actions does this group have? What is the size of the smallest generating set? Justify your assertions.

(b) Draw an abstract Cayley diagram for this group using a minimal generating set (you can, but do not need to label the nodes with configurations or elements). What familiar group is this isomorphic to?

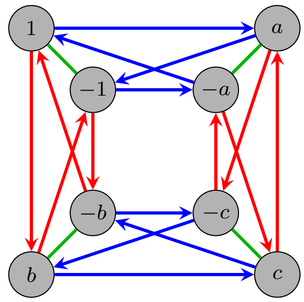
4. (12 points) In this problem, we will prove one of the questions from the class discussion board.

   (a) Prove that a node of a Cayley diagram cannot have two *incoming* edges of the same generator (i.e., edges of the same color). *Hint: Suppose there were two nodes x and y, with an edge for generator g leading from x to z, and from y to z. What can you conclude?*

(b) Use the result from Part (a) (even if you could not prove it) to show that in the Cayley diagram of a finite group, if we start at any node x and traverse edges corresponding with a fixed action (i.e., edges of the same color), we eventually wind up back at x.

(c) By Part (b), we know that Cayley diagrams are built from cycles. Show that in a Cayley diagram, all cycles corresponding with the same generator have the same length. You may use the result we proved in class about self-loops provided you can properly state it.

5. (10 points) Consider the following Cayley diagram of a group G with generators a, b, and -1.



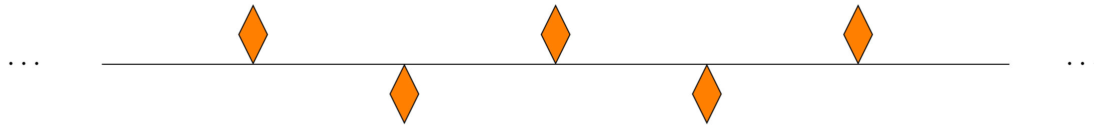
(a) Create the multiplication table for G from this Cayley diagram.

$*$	1	-1	a	b	c	$-a$	$-b$	$-c$
1								
-1								
a								
b								
c								
$-a$								
$-b$								
$-c$								

(b) Using your table, find the inverse of $-c$.

(c) Is this a *minimal* set of generators for G? How can you determine this from only the structure of the Cayley graph?

6. (8 points) Consider the following frieze pattern.



   (a) Describe a minimal set of generators for the group of symmetries of this infinite repeating pattern. (By minimal, I mean that you cannot eliminate any single generator from your list and still generate the whole group.)

   (b) Draw the *Cayley diagram of actions* (i.e., pick a node e, and label the other nodes with actions) of the group of symmetries of this frieze pattern using the generating set you described above. (It will be infinite, just draw enough of it to make the repeating pattern clear. Be sure to distinguish edges corresponding to different generators.)