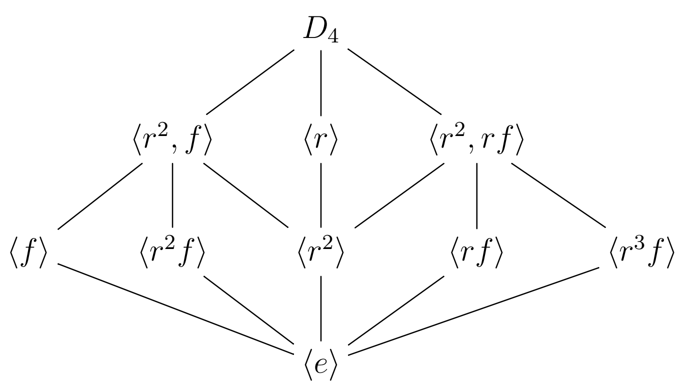
Read Chapter 6 of *Visual Group Theory*, or Chapters 7.3, 8.1 of *IBL Abstract Algebra*, and then write up solutions to the following exercises.

1. Draw the subgroup lattice of the alternating group $A_4 = \langle (1\,2\,3), (1\,2)(3\,4) \rangle$. Then carry out the following steps for two of its subgroups, $H = \langle (1\,2\,3) \rangle$ and $K = \langle (1\,2)(3\,4) \rangle$. When writing a coset, list all of its elements.
   (a) Write A_4 as a disjoint union of the subgroup's left cosets.
   (b) Write A_4 as a disjoint union of the subgroup's right cosets.
   (c) Find all conjugates of the subgroup, and determine whether it is normal in A_4.

2. The *center* of a group G is the set
$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}.$$
   (a) Prove that $Z(G)$ is a subgroup of G, and that it is normal in G.
   (b) Compute the center of the following groups: C_6, D_4, D_5, Q_8, A_4, S_4, and $D_4 \times Q_8$.

3. The subgroup lattice of D_4 is shown below:



For each of the 10 subgroups of D_4, find all of its conjugates, and determine whether it is normal in D_4. Fully justify your answers. [*Hint*: You can do this problem without actually computing xHx^{-1} for any subgroup $H \le D_4$.]

4. Consider a chain of subgroups $K \le H \le G$.
   (a) Prove or disprove: If $K \trianglelefteq H \trianglelefteq G$, then $K \trianglelefteq G$.
   (b) Prove or disprove: If $K \trianglelefteq G$, then $K \trianglelefteq H$.

5. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets
$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \mid h \in H\}.$$
Prove that if $H \trianglelefteq G$, then $aHbH = abH$.

6. Prove that $A \times \{e_B\}$ is a normal subgroup of $A \times B$, where e_B is the identity element of B. That is, show first that it is a subgroup, and then that it is normal.