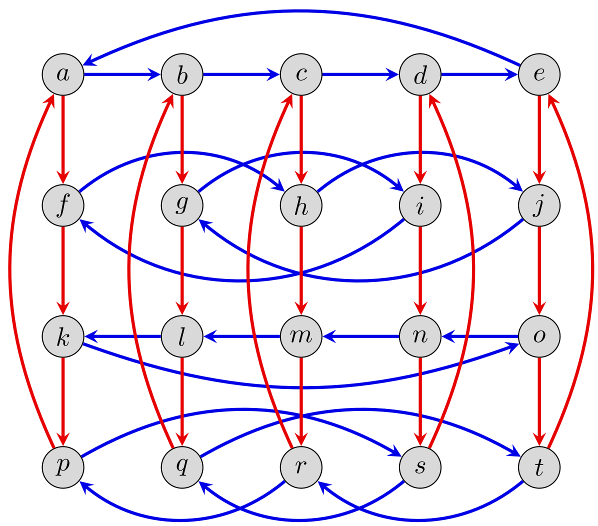
Read Chapter 7 of *Visual Group Theory*, or Chapter 8 of *IBL Abstract Algebra*, and then write up solutions to the following exercises.

1. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Carry out the following steps for both of these subgroups. List the cosets element-wise.

(a) Write G as a disjoint union of the subgroup's left cosets.

(b) Write G as a disjoint union of the subgroup's right cosets.

(c) Compute the normalizer of the subgroup.

(d) Attempt the quotient process, by shrinking the left cosets into individual nodes. The the resulting diagram and determine whether it is a valid Cayley diagram, and if so, which familiar group the quotient is isomorphic to.

(e) Find all conjugates of the subgroup.

2. Let H be a subgroup of an abelian group G. Prove that H and G/H are both abelian.

3. All of the following statements are *false*. For each one, exhibit an explicit counterexample, and justify your reasoning. Assume that each $H_i \trianglelefteq G_i$ for $i = 1, 2$.

(a) If H and G/H is abelian, then G is abelian.

(b) If every proper subgroup H of a group G is cyclic, then G is cyclic.

(c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

(d) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, then $H_1 \cong H_2$.

(e) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.

4. Prove the following "subgroup criterion", which can be very useful when trying to show that a subset is indeed a subgroup: *A nonempty subset H of a group G is a subgroup if and only if $xy^{-1} \in H$ holds for all $x, y \in H$.*

5. Let A be a subset of a group G. The *centralizer* of A, denoted $C_G(A)$, is the set of all elements that commute with everything in A:

$$C_G(A) = \{g \in G \mid ga = ag,\ \forall a \in A\}\,.$$

   If $A = \{a\}$, then we denote the centralizer as $C_G(a)$.

   (a) Prove that $C_G(A)$ is a subgroup of G.

   (b) For D_4, and Q_8, compute the centralizers of each element.

   (c) Compute the centralizers of the following elements in S_4: e, (12), (123), (1234), and (12)(34).

   (d) If A is a subgroup of G, prove that $C_G(A) \trianglelefteq N_G(A)$.

6. Recall that for any group G, conjugacy is an equivalence relation on G, and the *conjugacy class* of an element $x \in G$ is $\mathrm{cl}_G(x) = \{gxg^{-1} \mid g \in G\}$.

   (a) Partition the following groups into conjugacy classes:

   (i) $\mathbb{Z}_4$;
   (ii) D_5;
   (iii) D_8;
   (iv) Q_8;
   (v) S_4;
   (vi) A_4.

   (b) For each of the following elements $\sigma \in S_4$: e, (12), (123), (1234), and (12)(34), compare the size of its centralizer $C_{S_4}(\sigma)$ to the size of its conjugacy class, $\mathrm{cl}_{S_4}(\sigma)$. What do you notice about the product of these two numbers?