Read Chapter 9.1 of *Visual Group Theory* (VGT), or Chapter 13 of *AATA*. Then write up solutions to the following exercises.

1. The commutator subgroup of a group $G$ is the subgroup

$$G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle \,.$$

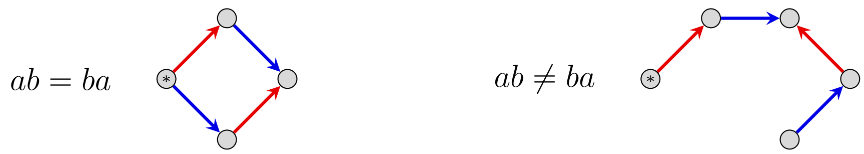
   (a) Show that $G$ is abelian if and only if $G' = \{e\}$.

   (b) Show that $G' \trianglelefteq G$.

   (c) Show that $G'$ is the intersection of all normal subgroups of $G$ that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \trianglelefteq G} N$$

   (d) If we quotient $G$ by $G'$, then we are in essence, "killing" all non-abelian parts of the Cayley diagram, as shown below:



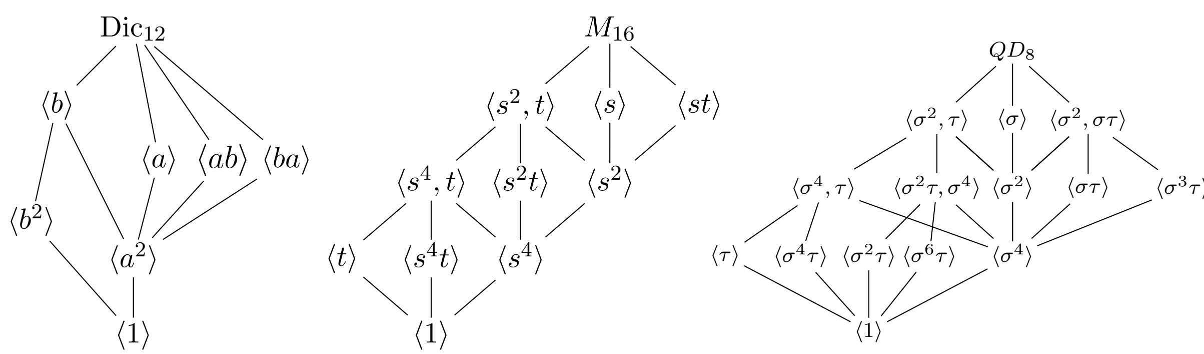

   Prove algebraically that $G/G'$ is indeed abelian.

2. Consider the following nonabelian groups $G$ whose subgroup lattices are shown below.

   (i) The *dicyclic group* $\mathrm{Dic}_{12} = \langle a, b \mid a^4 = b^6 = 1,\ bab = a \rangle$ of order 12.

   (ii) The *modular group* $M_{16} = \langle s, t \mid s^8 = t^2 = 1,\ tst = s^5 \rangle$ of order 16.

   (iii) The *quasidihedral group* $QD_8 = \langle \sigma, \tau \mid \sigma^8 = 1,\ \tau^2 = 1,\ \sigma\tau = \tau\sigma^3 \rangle$ of order 16.




Carry out the following steps for each group $G$.

   (a) On the lattice, label each edge with the corresponding index. Then circle every normal subgroup $N$ and determine which familiar group $G/N$ is isomorphic to. Justify why each $N$ must be normal.

   (b) Find the commutator subgroup $G'$ and the abelianization, $G/G'$.

   (c) Using *Group Explorer*, draw a Cayley diagram with the given generating set.

3. Find the commutator subgroup and abelianization of each of the following groups.

   (a) An abelian group $A$.

   (b) The alternating group $A_n$, for $n \geq 5$. [*Hint*: $A_n$ is a *simple group*, which means its only normal subgroups are $\langle e \rangle$ and $A_n$.]

   (c) The dihedral group $D_n$. [*Hint*: Do the cases of even and odd $n$ separately.]

4. Recall that the automorphism group of $V_4 = \langle h, v \rangle = \{e, h, v, r\}$, where $r = hv$ is

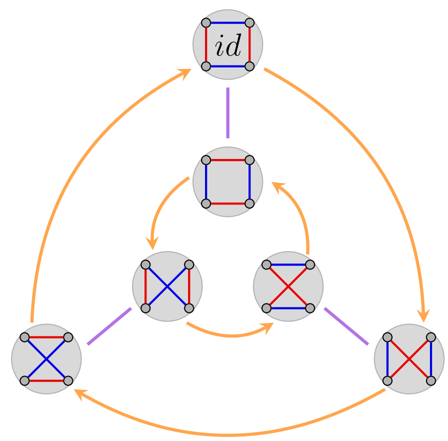
$$\mathrm{Aut}(V_4) = \big\langle \alpha, \beta \mid \alpha^3 = \beta^2 = (\alpha\beta)^2 = id \big\rangle, \quad \text{where} \quad \begin{array}{l} h \overset{\alpha}{\longmapsto} v \\ v \longmapsto r \end{array} \quad \text{and} \quad \begin{array}{l} h \overset{\beta}{\longmapsto} v \\ v \longmapsto h. \end{array}$$

The generating automorphisms are the following permuations of $V_4$:

$$\alpha: \quad e \quad h \quad v \quad r \qquad\qquad \text{and} \qquad\qquad \beta: \quad e \quad h \quad v \quad r$$

The multiplication table and Cayley diagram of $\mathrm{Aut}(V_4) = \langle \alpha, \beta \rangle$, which highlights how automorphisms are "re-wirings", are shown below:

| | $id$ | $\alpha$ | $\alpha^2$ | $\beta$ | $\alpha\beta$ | $\alpha^2\beta$ |
|---|---|---|---|---|---|---|
| $id$ | $id$ | $\alpha$ | $\alpha^2$ | $\beta$ | $\alpha\beta$ | $\alpha^2\beta$ |
| $\alpha$ | $\alpha$ | $\alpha^2$ | $id$ | $\alpha\beta$ | $\alpha^2\beta$ | $\beta$ |
| $\alpha^2$ | $\alpha^2$ | $id$ | $\alpha$ | $\alpha^2\beta$ | $\beta$ | $\alpha\beta$ |
| $\beta$ | $\beta$ | $\alpha^2\beta$ | $\alpha\beta$ | $id$ | $\alpha^2$ | $\alpha$ |
| $\alpha\beta$ | $\alpha\beta$ | $\beta$ | $\alpha^2\beta$ | $\alpha$ | $id$ | $\alpha^2$ |
| $\alpha^2\beta$ | $\alpha^2\beta$ | $\alpha\beta$ | $\beta$ | $\alpha^2$ | $\alpha$ | $id$ |




Repeat the above steps for each of the following groups. Use the Cayley diagram defined by the generating set given. Recall that $\mathrm{Aut}(\mathbb{Z}_n) \cong U_n$.

   (a) $\mathbb{Z}_5 = \langle 1 \rangle$,

   (b) $\mathbb{Z}_6 = \langle 1 \rangle$,

   (c) $\mathbb{Z}_3 \times \mathbb{Z}_2 = \langle (1,0), (0,1) \rangle$.

   (d) $\mathbb{Z}_8 = \langle 1 \rangle$.

5. Let $G$ act on a set $S$. Prove that the stabilizer $\mathrm{Stab}(s)$ is a subgroup of $G$ for every $s \in S$.

6. Suppose the cyclic group $C_5$ acts on a set $S = \{A, B, C, D\}$.

   (a) What are the possible sizes of the orbits?

   (b) What are the possible stabilizer subgroups of each element?

   (c) Draw the action diagram.

   Fully explain your reasoning for each part.