Lecture 1.1: What is a group?

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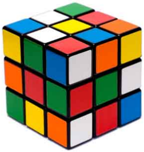
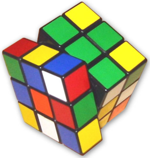
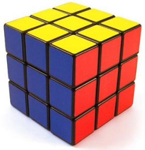
Math 4120, Modern Algebra

A famous toy

Our introduction to group theory will begin by discussing the famous Rubik's Cube.







It was invented in 1974 by Ernő Rubik of Budapest, Hungary.

Ernő Rubik is a Hungarian inventor, sculptor and professor of architecture.

According to his Wikipedia entry:

> *He is known to be a very introverted and hardly accessible person, almost impossible to contact or get for autographs.*

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Not impossible . . . just almost impossible.

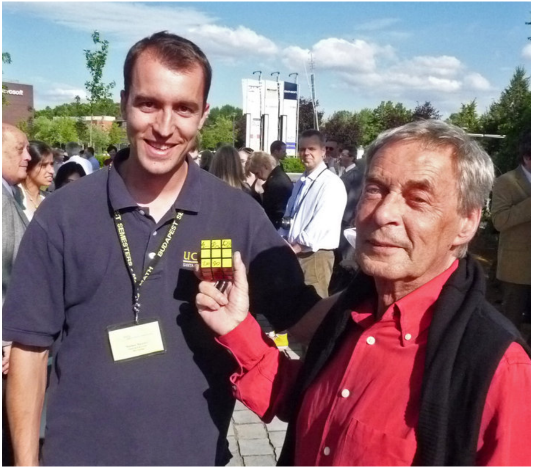

Figure: June 2010, in Budapest, Hungary

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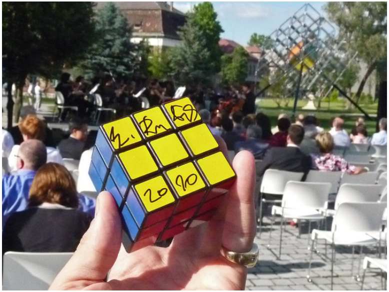
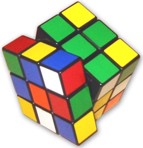
- The cube comes out of the box in the solved position:



- But then we can scramble it up by consecutively rotating one of its 6 faces:



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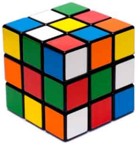
- The result might look something like this:



- The goal is to return the cube to its original solved position, again by consecutively rotating one of the 6 faces.

Since Rubik's Cube does not seem to require any skill with numbers to solve it, you may be inclined to think that this puzzle is not mathematical.

Big idea

Group theory is not primarily about numbers, but rather about patterns and symmetry; something the Rubik's Cube possesses in abundance.

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Let's explore the Rubik's Cube in more detail.

In particular, let's identify some key features that will be recurring themes in our study of patterns and symmetry.

First, some questions to ponder:

- How did we scramble up the cube in the first place? How do we go about unscrambling the cube?
- In particular, what actions, or moves, do we *need* in order to scramble and unscramble the cube? (There are many correct answers.)
- How is Rubik's Cube different from checkers?
- How is Rubik's Cube different from poker?

Four key observations

Observation 1

There is a predefined list of moves that never changes.

Observation 2

Every move is reversible.

Observation 3

Every move is deterministic.

Observation 4

Moves can be combined in any sequence.

In this setting a *move* is a twist of one of the six faces, by 0°, 90°, 180°, or 270°.

We could add more to our list, but as we shall see, these 4 observations are sufficient to describe the aspects of the mathematical objects that we wish to study.

What does group theory have to do with this?

Group theory studies the mathematical consequences of these 4 observations, which in turn will help us answer interesting questions about symmetrical objects.

Group theory arises everywhere! In puzzles, visual arts, music, nature, the physical and life sciences, computer science, cryptography, and of course, all throughout mathematics.

Group theory is one of the most beautiful subjects in all of mathematics!

Instead of considering our 4 observations as descriptions of Rubik's Cube, let's rephrase them as rules (axioms) that will define the boundaries of our objects of study.

Advantages of our endeavor:

1. We make it clear what it is we want to explore.
2. It helps us speak the same language, so that we know we are discussing the same ideas and common themes, though they may appear in vastly different settings.
3. The rules provide the groundwork for making logical deductions, so that we can discover new facts (many of which are surprising!).

Rules of a group

Our rules:

Rule 1

There is a predefined list of actions that never changes.

Rule 2

Every action is reversible.

Rule 3

Every action is deterministic.

Rule 4

Any sequence of consecutive actions is also an action.

Rules of a group

What changes were made in the rephrasing?

Comments

- We swapped the word *move* for *action*.
- The (usually short) list of actions required by Rule 1 is our set of building blocks; called the generators.
- Rule 4 tells us that any sequence of the generators is also an action.

Finally, here is our unofficial definition of a group. (We'll make things a bit more rigorous later.)

Definition (informal)

A group is a set of actions satisfying Rules 1–4.

Observations about the "Rubik's Cube group"

Frequently, two sequences of moves will be "indistinguishable." We will say that two such moves are *the same*. For example, rotating a face (by 90°) once has the same effect as rotating it five times.

Fact

There are 43,252,003,274,489,856,000 distinct configurations of the Rubik's cube.

While there are infinitely many possible sequences of moves, starting from the solved position, there are 43,252,003,274,489,856,000 "truly distinct" moves.

All 4.3×10^{19} moves are generated by just 6 moves: a 90° clockwise twist of one of the 6 faces.

Let's call these generators a, b, c, d, e, and f. Every word over the alphabet $\{a, b, c, d, e, f\}$ describes a unique configuration of the cube (starting from the solved position).

Summary of the big ideas

Loosing speaking a group is a **set of actions** satisfying some mild properties: deterministic, reversibility, and closure.

A generating set for a group is a **subcollection of actions** that together can produce all actions in the group – like a spanning set in a vector space.

Usually, a generating set is *much smaller* than the whole group.

Given a generating set, the individual actions are called generators.

The set of all possible ways to scramble a Rubik's cube is an example of a group. Two actions are the same if they have the *same "net effect"*, e.g., twisting a face 1 time vs. twisting a face 5 times.

Note that the group is the set of actions one can perform, *not* the set of configurations of the cube. However, there is a bijection between these two sets.

The Rubik's cube group has 4.3×10^{19} actions but we can find a generating set of size 6.