Lecture 1.5: Multiplication tables

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Math 4120, Modern Algebra

Overview

We are almost ready to introduce the formal definition of a group.

In this lecture, we will introduce one more useful algebraic tool for better understanding groups: multiplication tables.

We will also look more closely at inverses of the actions in a group.

Finally, we will introduce a new group of size 8 called the quaternions which frequently arise in theoretical physics.

Inverses

If g is a generator in a group G, then following the "g-arrow" backwards is an action that we call its inverse, and denoted by g^{-1}.

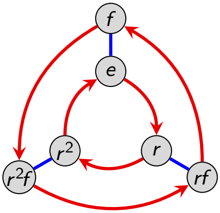
More generally, if g is represented by a path in a Cayley diagram, then g^{-1} is the action achieved by tracing out this path in reverse.

Note that by construction,

$$gg^{-1} = g^{-1}g = e\,,$$

where e is the identity (or "do nothing") action. Sometimes this is denoted by e, 1, or 0.

For example, let's use the following Cayley diagram to compute the inverses of a few actions:



$r^{-1} = \_\_\_\_\_$ because $r\_\_\_\_\_ = e = \_\_\_\_\_r$

$f^{-1} = \_\_\_\_\_$ because $f\_\_\_\_\_ = e = \_\_\_\_\_f$

$(rf)^{-1} = \_\_\_\_\_$ because $(rf)\_\_\_\_\_ = e = \_\_\_\_\_(rf)$

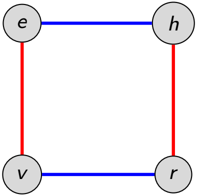
$(r^2f)^{-1} = \_\_\_\_\_$ because $(r^2f)\_\_\_\_\_ = e = \_\_\_\_\_(r^2f)$.

Multiplication tables

Since we can use a Cayley diagram with nodes labeled by actions as a "group calculator," we can create a (group) multiplication table, that shows how every pair of group actions combine.

This is best illustrated by diving in and doing an example. Let's fill out a multiplication table for V_4.

Since order of multiplication can matter, let's stick with the convention that the entry in row g and column h is the element gh (rather than hg).



	e	v	h	r
e	e	v	h	r
v	v	e	r	h
h	h	r	e	v
r	r	h	v	e

Some remarks on the structure of multiplication tables

Comments

- The 1st column and 1st row repeat themselves. (Why?) Sometimes these will be omitted (*Group Explorer* does this).
- Multiplication tables can visually reveal patterns that may be difficult to see otherwise. To help make these patterns more obvious, we can color the cells of the multiplication table, assigning a unique color to each action of the group. Figure 4.7 (page 47) has examples of a few more tables.
- A group is abelian iff its multiplication table is symmetric about the "main diagonal."
- In each row and each column, each group action occurs exactly once. (This will always happen. . . Why?)

Let's state and prove that last comment as as theorem.

A theorem and proof

Theorem

An element cannot appear twice in the same row or column of a multiplictaion table.

Proof

Suppose that in row a, the element g appears in columns b and c. Algebraically, this means

$$ab = g = ac.$$

Multiplying everything on the left by a^{-1} yields

$$a^{-1}ab = a^{-1}g = a^{-1}ac \qquad \Longrightarrow \qquad b = c.$$

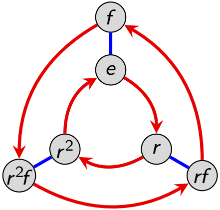
Thus, g (or any element) element cannot appear twice in the same row.

The proof that two elements cannot appear twice in the same column is similar, and will be left as a homework exercise. □

Another example: D_3

Let's fill out a multiplication table for the group D_3; here are several different presentations:

$$\begin{aligned} D_3 &= \langle r, f \mid r^3 = e,\ f^2 = e,\ rf = fr^2 \rangle \\ &= \langle r, f \mid r^3 = e,\ f^2 = e,\ rfr = f \rangle \,. \end{aligned}$$



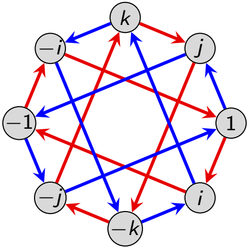
	e	r	r^2	f	rf	r^2f
e	e	r	r^2	f	rf	r^2f
r	r	r^2	e	rf	r^2f	f
r^2	r^2	e	r	r^2f	f	rf
f	f	r^2f	rf	e	r^2	r
rf	rf	f	r^2f	r	e	r^2
r^2f	r^2f	rf	f	r^2	r	e

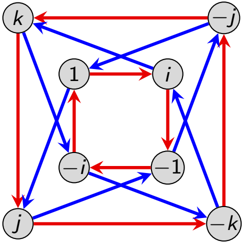
Observations? What patterns do you see?

Just for fun, what group do you get if you remove the "$r^3 = e$" relation from the presentations above? (*Hint*: We've seen it recently!)

Another example: the quaternion group

The following Cayley diagram, laid out two different ways, describes a group of size 8 called the Quaternion group, often denoted $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$.





The "numbers" j and k individually act like $i = \sqrt{-1}$, because $i^2 = j^2 = k^2 = -1$.

Multiplication of $\{\pm i, \pm j, \pm k\}$ works like the cross product of unit vectors in $\mathbb{R}^3$:

$$ij = k, \quad jk = i, \quad ki = j, \qquad\qquad ji = -k, \quad kj = -i, \quad ik = -j\,.$$

Here are two possible presentations for this group:

$$\begin{aligned} Q_4 &= \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle \\ &= \langle i, j \mid i^4 = j^4 = 1,\ iji = j \rangle\,. \end{aligned}$$