# Lecture 2.4: Cayley's theorem

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Math 4120, Modern Algebra

# Overview

We just finished introducing 5 families of groups:

1. cyclic groups
2. abelian groups
3. dihedral groups
4. symmetric groups
5. alternating groups

In this lecture, we will introduce Cayley's theorem, which says that every finite group is isomorphic to a collection of permutations.

# Cayley's theorem

Any set of permutations that forms a group is called a permutation group.

Cayley's theorem says that permutations can be used to construct any finite group.

In other words, every group has the same structure as (we say "*is isomorphic to*") some permutation group.

**Warning!** We are *not* saying that every group is isomorphic to some symmetric group, $S_n$. Rather, every group is isomorphic to a subgroup of a some symmetric group $S_n$ – i.e., a subset of $S_n$ that is *also* a group in its own right.

**Question**

Given a group, how do we associate it with a set of permutations?

# Cayley's theorem; how to construct permutations

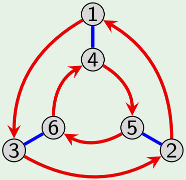
Here is an algorithm given a Cayley diagram with $n$ nodes:

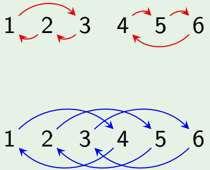
1. number the nodes 1 through $n$,
2. interpret each arrow type in Cayley diagram as a permutation.

The resulting permutations are the generators of the corresponding permutation group.

## Example

Let's try this with $D_3 = \langle r, f \rangle$.




We see that $D_3$ is isomorphic to the subgroup $\langle (132)(456), (14)(25)(36) \rangle$ of $S_6$.

# Cayley's theorem; how to construct permutations

Here is an algorithm given a multiplication table with $n$ elements:

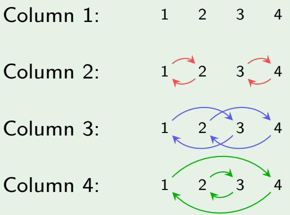
1. replace the table headings with 1 through $n$,
2. make the appropriate replacements throughout the rest of the table,
3. interpret each column as a permutation.

This results in a 1-1 correspondence between the original group elements (not just the generators) and permutations.

## Example

Let's try this with the multiplication table for $V_4 = \langle v, h \rangle$.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 1 | 4 | 3 |
| 3 | 3 | 4 | 1 | 2 |
| 4 | 4 | 3 | 2 | 1 |




We see that $V_4$ is isomorphic to the subgroup $\langle (12)(34), (13)(24) \rangle$ of $S_4$.

# Cayley's theorem

Intuitively, two groups are isomorphic if they have the same structure.

Two groups are *isomorphic* if we can construct Cayley diagrams for each that look identical.

**Cayley's Theorem**

Every finite group is isomorphic to a collection of permutations.

Our algorithms exhibit a 1-1 correspondence between group elements and permutations.

However, we have *not* shown that the corresponding permutations form a group, or that the resulting permutation group has the same structure as the original.

What needs to be shown is that the permutation from the $i^{\text{th}}$ row followed by the permutation from the $j^{\text{th}}$ column, results in the permutation that corresponding to the cell in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of the original table. (See page 85 for a proof.)