Lecture 3.7: Conjugacy classes

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Math 4120, Modern Algebra

Overview

Recall that for $H \leq G$, the conjugate subgroup of H by a fixed $g \in G$ is

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\} .$$

Additionally, H is normal iff $gHg^{-1} = H$ for all $g \in G$.

We can also fix the element we are conjugating. Given $x \in G$, we may ask:

"*which elements can be written as gxg^{-1} for some $g \in G$?*"

The set of all such elements in G is called the conjugacy class of x, denoted $\mathrm{cl}_G(x)$. Formally, this is the set

$$\mathrm{cl}_G(x) = \{gxg^{-1} \mid g \in G\} .$$

Remarks

- In any group, $\mathrm{cl}_G(e) = \{e\}$, because $geg^{-1} = e$ for any $g \in G$.
- If x and g commute, then $gxg^{-1} = x$. Thus, when computing $\mathrm{cl}_G(x)$, we only need to check gxg^{-1} for those $g \in G$ that *do not commute* with x.
- Moreover, $\mathrm{cl}_G(x) = \{x\}$ iff x commutes with everything in G. (Why?)

Conjugacy classes

Lemma

Conjugacy is an equivalence relation.

Proof

- *Reflexive*: $x = exe^{-1}$.
- *Symmetric*: $x = gyg^{-1} \Rightarrow y = g^{-1}xg$.
- *Transitive*: $x = gyg^{-1}$ and $y = hzh^{-1} \Rightarrow x = (gh)z(gh)^{-1}$. □

Since conjugacy is an equivalence relation, it partitions the group G into equivalence classes (conjugacy classes).

Let's compute the conjugacy classes in D_4. We'll start by finding $\text{cl}_{D_4}(r)$. Note that we only need to compute grg^{-1} for those g that *do not* commute with r:

$$frf^{-1} = r^3, \qquad (rf)r(rf)^{-1} = r^3, \qquad (r^2f)r(r^2f)^{-1} = r^3, \qquad (r^3f)r(r^3f)^{-1} = r^3.$$

Therefore, the conjugacy class of r is $\text{cl}_{D_4}(r) = \{r, r^3\}$.

Since conjugacy is an equivalence relation, $\text{cl}_{D_4}(r^3) = \text{cl}_{D_4}(r) = \{r, r^3\}$.

Conjugacy classes in D_4

To compute $\text{cl}_{D_4}(f)$, we don't need to check e, r^2, f, or r^2f, since these all commute with f:

$$rfr^{-1} = r^2f, \qquad r^3f(r^3)^{-1} = r^2f, \qquad (rf)f(rf)^{-1} = r^2f, \qquad (r^3f)f(r^3f)^{-1} = r^2f\,.$$

Therefore, $\text{cl}_{D_4}(f) = \{f, r^2f\}$.

What is $\text{cl}_{D_4}(rf)$? Note that it has size greater than 1 because rf does not commute with everything in D_4.

It also *cannot* contain elements from the other conjugacy classes. The only element left is r^3f, so $\text{cl}_{D_4}(rf) = \{rf, r^3f\}$.

The "Class Equation", visually:
Partition of D_4 by its
conjugacy classes

<table>
<tr><td>e</td><td rowspan="2">r
r^3</td><td>f r^2f</td></tr>
<tr><td>r^2</td><td>rf r^3f</td></tr>
</table>

We can write $D_4 = \underbrace{\{e\} \cup \{r^2\}}_{\text{these commute with everything in } D_4} \cup \{r, r^3\} \cup \{f, r^2f\} \cup \{r, r^3f\}$.

The class equation

Definition

The center of G is the set $Z(G) = \{z \in G \mid gz = zg, \ \forall g \in G\}$.

Observation

$\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.

Proof

Suppose x is in its own conjugacy class. This means that

$$\mathrm{cl}_G(x) = \{x\} \iff gxg^{-1} = x, \ \forall g \in G \iff gx = xg, \ \forall g \in G \iff x \in Z(G).$$

□

The Class Equation

For any finite group G,

$$|G| = |Z(G)| + \sum |\,\mathrm{cl}_G(x_i)|$$

where the sum is taken over distinct conjugacy classes of size greater than 1.

More on conjugacy classes

Proposition

Every normal subgroup is the union of conjugacy classes.

Proof

Suppose $n \in N \lhd G$. Then $gng^{-1} \in gNg^{-1} = N$, thus if $n \in N$, its entire conjugacy class $\mathrm{cl}_G(n)$ is contained in N as well. □

Proposition

Conjugate elements have the same order.

Proof

Consider x and $y = gxg^{-1}$.

If $x^n = e$, then $(gxg^{-1})^n = (gxg^{-1})(gxg^{-1}) \cdots (gxg^{-1}) = gx^n g^{-1} = geg^{-1} = e$. Therefore, $|x| \geq |gxg^{-1}|$.

Conversely, if $(gxg^{-1})^n = e$, then $gx^n g^{-1} = e$, and it must follow that $x^n = e$. Therefore, $|x| \leq |gxg^{-1}|$. □

Conjugacy classes in D_6

Let's determine the conjugacy classes of $D_6 = \langle r, f \mid r^6 = e,\ f^2 = e,\ r^i f = fr^{-i} \rangle$.

The center of D_6 is $Z(D_6) = \{e, r^3\}$; these are the *only* elements in size-1 conjugacy classes.

The only two elements of order 6 are r and r^5; so we must have $\mathrm{cl}_{D_6}(r) = \{r, r^5\}$.

The only two elements of order 3 are r^2 and r^4; so we must have $\mathrm{cl}_{D_6}(r^2) = \{r^2, r^4\}$.

Let's compute the conjugacy class of a reflection $r^i f$. We need to consider two cases; conjugating by r^j and by $r^j f$:

- $r^j (r^i f) r^{-j} = r^j r^i r^j f = r^{i+2j} f$
- $(r^j f)(r^i f)(r^j f)^{-1} = (r^j f)(r^i f) f r^{-j} = r^j f r^{i-j} = r^j r^{j-i} f = r^{2j-i} f$.

Thus, $r^i f$ and $r^k f$ are conjugate iff i and k are both even, or both odd.

The Class Equation, visually:
Partition of D_6 by its
conjugacy classes

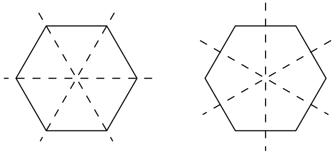
e	r	r^2	f	$r^2 f$	$r^4 f$
r^3	r^5	r^4	rf	$r^3 f$	$r^5 f$

Conjugacy "preserves structure"

Think back to linear algebra. Two matrices A and B are *similar* (=conjugate) if $A = PBP^{-1}$.

Conjugate matrices have the same eigenvalues, eigenvectors, and determinant. In fact, they represent the *same linear map*, but under a change of basis.

If n is even, then there are two "types" of reflections of an n-gon: the axis goes through two corners, or it bisects a pair of sides.



Notice how in D_n, conjugate reflections have the same "type." Do you have a guess of what the conjugacy classes of reflections are in D_n when n is odd?

Also, conjugate rotations in D_n had the same rotating angle, but in the opposite direction (e.g., r^k and r^{n-k}).

Next, we will look at conjugacy classes in the symmetric group S_n. We will see that conjugate permutations have "the same structure."

Cycle type and conjugacy

Definition

Two elements in S_n have the same cycle type if when written as a product of disjoint cycles, there are the same number of length-k cycles for each k.

We can write the cycle type of a permutation $\sigma \in S_n$ as a list $c_1, c_2, \ldots, c_n$, where c_i is the number of cycles of length i in σ.

Here is an example of some elements in S_9 and their cycle types.

- (1 8) (5) (2 3) (4 9 6 7) has cycle type 1,2,0,1.
- (1 8 4 2 3 4 9 6 7) has cycle type 0,0,0,0,0,0,0,0,1.
- $e = (1)(2)(3)(4)(5)(6)(7)(8)(9)$ has cycle type 9.

Theorem

Two elements $g, h \in S_n$ are conjugate if and only if they have the same cycle type.

Big idea

Conjugate permutations have the same structure. Such permutations are *the same up to renumbering*.

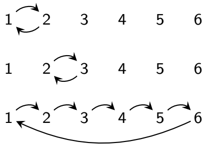
An example

Consider the following permutations in $G = S_6$:

$g = (1\ 2)$

$h = (2\ 3)$

$r = (1\ 2\ 3\ 4\ 5\ 6)$



Since g and h have the same cycle type, they are conjugate:

$$(1\ 2\ 3\ 4\ 5\ 6)\,(2\ 3)\,(1\ 6\ 5\ 4\ 3\ 2) = (1\ 2)\,.$$

Here is a visual interpretation of $g = rhr^{-1}$: