Lecture 5.2: The orbit-stabilizer theorem

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120, Modern Algebra

Orbits, stabilizers, and fixed points

Suppose G acts on a set S. Pick a configuration $s \in S$. We can ask two questions about it:

(i) What other states (in S) are reachable from s? (We call this the orbit of s.)

(ii) What group elements (in G) fix s? (We call this the stabilizer of s.)

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(i) The orbit of $s \in S$ is the set

$$\mathrm{Orb}(s) = \{s.\phi(g) \mid g \in G\}\,.$$

(ii) The stabilizer of s in G is

$$\mathrm{Stab}(s) = \{g \in G \mid s.\phi(g) = s\}\,.$$

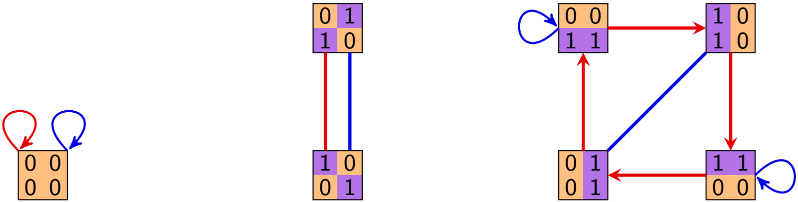
(iii) The fixed points of the action are the orbits of size 1:

$$\mathrm{Fix}(\phi) = \{s \in S \mid s.\phi(g) = s \text{ for all } g \in G\}\,.$$

Note that the orbits of ϕ are the connected components in the action diagram.

Orbits, stabilizers, and fixed points

Let's revisit our running example:



The orbits are the 3 connected components. There is only one fixed point of ϕ. The stabilizers are:

$$\text{Stab}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)=D_4,\qquad \text{Stab}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)=\{e,r^2,rf,r^3f\},\qquad \text{Stab}\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)=\{e,f\},$$

$$\text{Stab}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)=\{e,r^2,rf,r^3f\},\qquad \text{Stab}\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)=\{e,r^2f\},$$

$$\text{Stab}\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)=\{e,f\},$$

$$\text{Stab}\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)=\{e,r^2f\}.$$

Observations?

Orbits and stabilizers

Proposition

For any $s \in S$, the set $\text{Stab}(s)$ is a subgroup of G.

Proof (outline)

To show $\text{Stab}(s)$ is a group, we need to show three things:

(i) *Contains the identity*. That is, $s.\phi(e) = s$.

(ii) *Inverses exist*. That is, if $s.\phi(g) = s$, then $s.\phi(g^{-1}) = s$.

(iii) *Closure*. That is, if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh) = s$.

You'll do this on the homework.

Remark

The kernel of the action ϕ is the set of all group elements that fix everything in S:

$$\text{Ker}\,\phi = \{g \in G \mid \phi(g) = e\} = \{g \in G \mid s.\phi(g) = s \text{ for all } s \in S\}.$$

Notice that

$$\text{Ker}\,\phi = \bigcap_{s \in S} \text{Stab}(s).$$

The Orbit-Stabilizer Theorem

The following result is another one of the central results of group theory.

Orbit-Stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and any $s \in S$,

$$|\,\mathrm{Orb}(s)| \cdot |\,\mathrm{Stab}(s)| = |G|\,.$$

Proof

Since $\mathrm{Stab}(s) \leq G$, Lagrange's theorem tells us that

$$\underbrace{[G : \mathrm{Stab}(s)]}_{\text{number of cosets}} \cdot \underbrace{|\,\mathrm{Stab}(s)|}_{\text{size of subgroup}} = |G|.$$

Thus, it suffices to show that $|\,\mathrm{Orb}(s)| = [G : \mathrm{Stab}(s)]$.

Goal: Exhibit a bijection between elements of $\mathrm{Orb}(s)$, and right cosets of $\mathrm{Stab}(s)$.

That is, *two elements in G send s to the same place iff they're in the same coset*.

Proof (cont.)

Let's look at our previous example to get some intuition for why this should be true.

We are seeking a bijection between $\mathrm{Orb}(s)$, and the right cosets of $\mathrm{Stab}(s)$.

That is, two elements in G send s to the same place iff they're in the same coset.

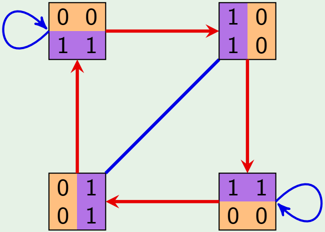
Let $s = \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}$

Then $\mathrm{Stab}(s) = \langle f \rangle$.

Partition of D_4 by the right cosets of H :

$G = D_4$ and $H = \langle f \rangle$

e	r	r^2	r^3
f	fr	fr^2	fr^3
H	Hr	Hr^2	Hr^3



Note that $s.\phi(g) = s.\phi(k)$ iff g and k are in the same right coset of H in G.

The Orbit-Stabilizer Theorem: $|\text{Orb}(s)| \cdot |\text{Stab}(s)| = |G|$

Proof (cont.)

Throughout, let $H = \text{Stab}(s)$.

"$\Rightarrow$" *If two elements send s to the same place, then they are in the same coset.*

Suppose $g, k \in G$ both send s to the same element of S. This means:

$$\begin{aligned}
s.\phi(g) = s.\phi(k) \quad &\Longrightarrow \quad s.\phi(g)\phi(k)^{-1} = s \\
&\Longrightarrow \quad s.\phi(g)\phi(k^{-1}) = s \\
&\Longrightarrow \quad s.\phi(gk^{-1}) = s \qquad \text{(i.e., } gk^{-1} \text{ stabilizes } s\text{)} \\
&\Longrightarrow \quad gk^{-1} \in H \qquad \text{(recall that } H = \text{Stab}(s)\text{)} \\
&\Longrightarrow \quad Hgk^{-1} = H \\
&\Longrightarrow \quad Hg = Hk
\end{aligned}$$

"$\Leftarrow$" *If two elements are in the same coset, then they send s to the same place.*

Take two elements $g, k \in G$ in the same right coset of H. This means $Hg = Hk$.

This is the last line of the proof of the forward direction, above. We can change each $\Longrightarrow$ into $\Longleftrightarrow$, and thus conclude that $s.\phi(g) = s.\phi(k)$. $\square$

If we have instead, a left group action, the proof carries through but using left cosets.