Lecture 5.2: Boundary conditions for the heat equation

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4340, Advanced Engineering Mathematics
Last time: Example 1a

The solution to the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = x(1 - x). \]

is \( u(x, t) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{n^3 \pi^3} \sin(n\pi x) e^{-(cn\pi)^2 t}. \)

This time: Example 1b

Solve the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u(0, t) = u(1, t) = 32, \quad u(x, 0) = x(1 - x) + 32. \]
Last time: Example 1a

The solution to the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = x(1 - x). \]

is

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3 \pi^3} \sin(n\pi x) e^{-(cn\pi)^2 t}. \]

This time: Example 1c

Solve the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u(0, t) = 32, \quad u(1, t) = 42, \quad u(x, 0) = x(1 - x) + 32 + 10x. \]
Summary

To solve the initial / boundary value problem

\[ u_t = c^2 u_{xx}, \quad u(0, t) = a, \quad u(L, t) = b, \quad u(x, 0) = h(x), \]

first solve the related **homogeneous problem**, then add this to the **steady-state solution**

\[ u_{ss}(x) = a + \frac{b-a}{L} x. \]
Neumann boundary conditions (type 2)

Example 2

Solve the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u_x(0, t) = u_x(1, t) = 0, \quad u(x, 0) = x(1 - x). \]
Example 2 (cont.)

The general solution to the following BVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u_x(0, t) = u_x(1, t) = 0, \quad u(x, 0) = x(1 - x). \]

is \( u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-(cn\pi)^2 t}. \) Now, we'll solve the remaining IVP.
Mixed boundary conditions

Example 1.5

Solve the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u(0, t) = u_x(1, t) = 0, \quad u(x, 0) = 5 \sin(\pi x/2). \]
Periodic boundary conditions

Example

Solve the following B/IVP for the heat equation:

\[ u_t = c^2 u_{xx}, \quad u(0, t) = u(2\pi, t), \quad u(x, 0) = 2 + \cos x - 3 \sin 2x. \]