Visual Algebra

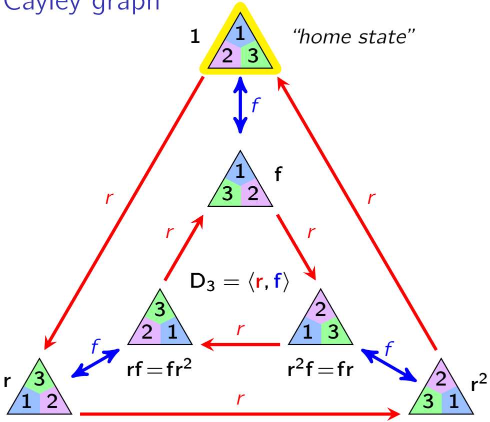
Lecture 0.1: What is *Visual Algebra* all about?

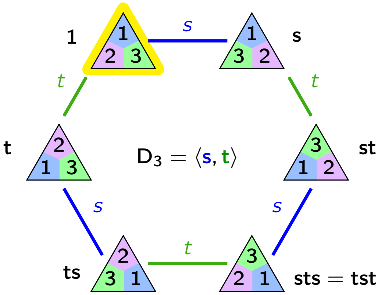
Dr. Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

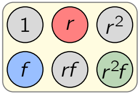
A Cayley graph

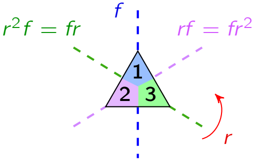
(Chapter 1)





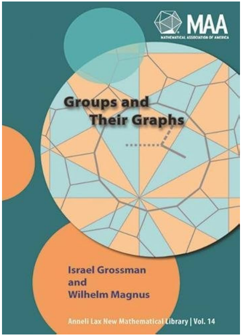
"Group switchboard"

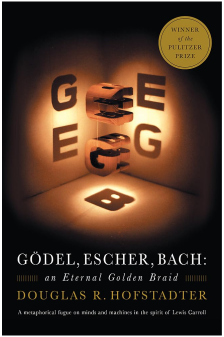


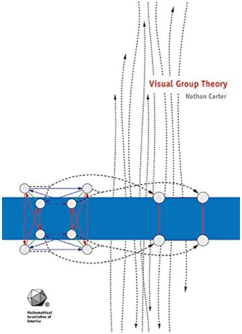


A brief (incomplete) history of Visual Algebra

- **1964** Grossman and Magnus publish *Groups and Their Graphs*.
- **2002** Douglas Hofstadter teaches *Visual Group Theory & Galois Theory* at Indiana.
- **2009** Nathan Carter publishes *Visual Group Theory* (VGT). I meet him at MathFest.
- **2010** I first teach Modern Algebra at Clemson using VGT.
- **2016** I record 43 lectures on VGT.
- **2016** Dana Ernst writes *An Inquiry-Based Approach to Abstract Algebra*.
- **2019** I start writing *Visual Algebra*.
- **2024** I start re-recording a new Visual Algebra YouTube series.







New Visual Algebra recording studio!

Visual Group Theory vs. Visual Algebra YouTube playlists

Visual Group Theory (2016)

- 43 lectures
- 1 semester undergraduate algebra, following Nathan Carter's book (supplemented)

Visual Algebra (2025)

- $\approx$ 100 lectures
- 2+ semesters of undergraduate algebra, following my *Visual Algebra* book
- Or. . . 1+ semester of graduate algebra. (Abridged version of Ch 1–4, coming soon!)

A list of terms original to my book and video series

- A "group switchboard"
- Blue vs. red cosets
- Shoebox diagrams
- Pizza diagrams
- Semiabelian group
- Diquaternion group
- Moderately and fully unnormal
- Moderately and fully uncentral
- G-set posets
- Maximal central ascents & descents
- Chutes and ladders diagram
- The crooked ladder theorem
- (Annotated) subring lattices
- Ideal class group lattices

Visual Algebra topical outline

Table of contents

Visual Algebra courses that I teach

- **Undergraduate Algebra 1**: Chapters 1–5, half of Chapter 8.
- **Undergraduate Algebra 2**: Chapters 8–9, Chapters 6–7, Parts of Chapters 10–11.
- **Graduate Algebra 1**: Chapters 1–9.

Visual Group Theory vs. Visual Algebra YouTube series

New topics in *Visual Algebra* (by Chapter)

- Semidirect products (2,4)
- Central products (4)
- Cayley graphs on polytopes (2)
- Free & transitive G-sets (5)
- Fixators (5)
- Orbit counting theorem (5)
- Inner and outer automorphisms (5)
- Action equivalence vs. equivariance (5)
- Groups extensions (6)
- Simplicity of A_n (6)
- Short exact sequences (6)
- Composition series; Jordan-Hölder (6)
- Composition series (6)
- Nilpotent groups (6)
- University properties (7)
- Basic category theory (7)
- Products and coproducts (7)
- Free groups & free products (7)
- Group presentations formalized (7)
- Zorn's lemma (8)
- Primary ideals (8)
- Nil and Jacobian radials (8)
- Rings & fields of fractions (8)
- Ideal class group (9)
- Sunzi remainder theorem (9)
- Hilbert's basis theorem (9)
- Cyclotomic polynomials (10)
- Separable field extensions (10)
- Transitive groups (11)
- Galois theory proofs (11)
- Symmetric polynomials (11)
- Galois group mod p reduction (11)

Also new in Visual Algebra: https://beta.lmfdb.org/Groups/Abstract/

⌂ → Groups → Abstract

Abstract groups

· Login
Citation · Feedback · Hide Menu

The database currently contains 544,831 groups from many different sources, the largest of which is S_{47} of order 47!. In addition, it contains 275,379,753 of their subgroups and 39,933,457 of their irreducible complex characters. You can browse further statistics.

Learn more

- Source and acknowledgements
- Completeness of the data
- Reliability of the data
- Abstract group labeling

Browse

By order: 1-64 65-127 128 129-255 256 257-383 384 385-511 513-1000 1001-1500 1501-2000 2001-

By nilpotency class: 1 2 3 4 5 6 7 8 9 (and not nilpotent)

By property: abelian nonabelian solvable nonsolvable simple perfect rational

Some interesting groups or a random group

Search for subgroups or complex characters

Search

Advanced search options

Order	3	e.g. 4, or a range like 3..5	Exponent	2, 3, 7	e.g. 2, or list of integers like 2, 3, 7
Automorphism group	4.2	e.g. 4.2	Nilpotency class	3	e.g. 4, or a range like 3..5
Automorphism group order	3	e.g. 4, or a range like 3..5	Commutator	4.2, 8	e.g. 4 or 4.2 (order or label)
Center	4.2, 8	e.g. 4 or 4.2 (order or label)	Abelianization	4.2, 8	e.g. 4 or 4.2 (order or label)
Central quotient	4.2, 8	e.g. 4 or 4.2 (order or label)			
Abelian			Direct product		
Cyclic			Semidirect product		
Nilpotent			Perfect		
Simple			Solvable		
Transitive degree	3	e.g. 4, or a range like 3..5	Permutation degree	3	e.g. 4, or a range like 3..5
Number of subgroups	3	e.g. 4, or a range like 3..5	Number of normal subgroups	3	e.g. 4, or a range like 3..5
Number of conjugacy classes	3	e.g. 4, or a range like 3..5			
Order statistics	1^1, 2^3, 3^2	e.g. 1^1, 2^3, 3^2			
Results to display	50				

Display: List of groups Random group

Visual Algebra myths

Myth #1

A *Visual Algebra* course is less rigorous than a traditional algebra course.

Myth #2

A *Visual Algebra* course is easier than a traditional algebra course.

Myth #3

A *Visual Algebra* course just amounts to "supplementing" existing material with visuals.

Myth #4

The visual pedagogy is primarily targeted to students not going to grad school.

Myth #5

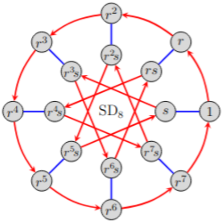
Students who don't learn visually won't enjoy or do well in a *Visual Algebra* course.

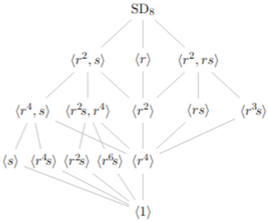
Myth #6

Teaching a *Visual Algebra* class would just be too difficult.

Creative Visual Algebra homework and exam problems

2. (24 pts) Answer the following questions about the *semidihedral group* $G = \mathrm{SD}_8$, whose Cayley grpah and subgroup lattice are shown below.





(a) The subgroup $\langle r^2\rangle$ is isomorphic to ____________, and $G/\langle r^2\rangle \cong$____________.

(b) The subgroup $\langle r^4\rangle$ is isomorphic to ____________, and $G/\langle r^4\rangle \cong$____________.

(c) G has three order-8 subgroups: $\langle r^2, s\rangle \cong$_____, $\langle r\rangle \cong$_____, and $\langle r^2, rs\rangle \cong$_____.

(d) Find all ways that G can be written as a direct or semidirect product of two of its proper subgroups.

(e) Partition the subgroups into conjugacy classes by circling them on the lattice. Every subgroup should be contained in some circle.

(f) The normalizer of $H = \langle s\rangle$ is $N_G(H) =$__________, and $N_G(H)/H \cong$__________.

(g) The center of this group is $Z(G) =$______________, and its inner automorphism group is $\mathrm{Inn}(G) \cong$________________.

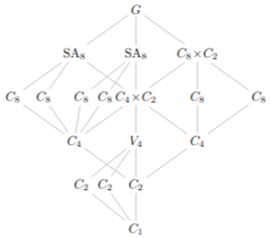
(h) The commutator subgroup is $G' =$_____, and the abelianization is $G/G' \cong$_____.

(i) The generator $s \in G$ is conjugate to exactly _______ element(s) of G, and thus its centralizer is $C_G(s) =$______________________.

(j) Is G a simple group? Justify your answer in a single sentence.

(k) Exactly _____ of the 15 subgroups of G have normalizer equal to G, and _____ of them have normalizer $\langle 1\rangle$.

3. (28 points) Consider the group of order 32 whose subgroup lattice appears below.



(a) $G/(C_4 \times C_2) \cong$ ____________, and $G/C_8 \cong$ ____________ (when it is defined).

(b) The quotients of G by its three order-4 subgroups, reading from left-to-right, are $G/C_4 \cong$ ____________, $G/V_4 \cong$ ____________, and $G/C_4 \cong$ ____________.

(c) The commutator subgroup is $G' =$ _____ and the abelianization is $G/G' \cong$ _____.

(d) The center of G trivially must be contained in the center of all of its subgroups. Recall that the center of SA_8 has order 4, and $\mathrm{SA}_8/Z(\mathrm{SA}_8) \cong V_4$. Circle this group on the subgroup lattice.

(e) Consider the descending central series $G = L_0 \trianglerighteq L_1 \trianglerighteq L_2 \trianglerighteq \cdots$. Determine which order-4 subgroup L_1 is, with justification, and mark this on the subgroup lattice.

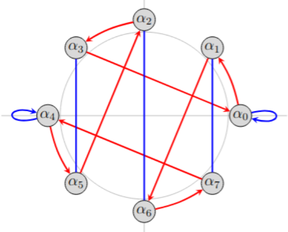
(f) It's now possible to determine the ascending and descending central series, by inspection. Mark these on the subgroup lattice, with justification. You may cite basic properties that we proved, such as (i) p-groups are nilpotent, (ii) the ascending and descending central series have the same length, and (iii) $L_k \leq Z_{n-k}$ for all $k \geq 0$.

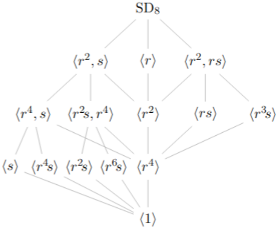
(g) For each non-normal subgroup H, circle its conjugacy class, $\mathrm{cl}_G(H)$.

(h) What is the inner automorphism group, $\mathrm{Inn}(G)$, isomorphic to, and why?

(i) Is G the semidirect product of any nontrivial proper subgroups? Why or why not?

4. (22 pts) The Galois group $G = \mathrm{Gal}(x^8 - 2)$ acts on the set $S = \{\alpha_0, \dots, \alpha_7\}$ of roots of $x^8 - 2$, where $\alpha_i = \sqrt[8]{2}e^{2\pi i/8}$. An action graph is shown below.





(a) This action has ____________ orbit(s), and (is)(is not) [$\longleftarrow$ *circle one*] transitive.

(b) The orbit containing α_0 has size ____________, and $\mathrm{stab}(\alpha_0) =$ ____________.

(c) The orbit containing α_1 has size ____________, and $\mathrm{stab}(\alpha_1) =$ ____________.

(d) $\mathrm{stab}(\alpha_2) =$ ____________________ and $\mathrm{stab}(\alpha_3) =$ ____________________.

(e) The automorphism group of S, as a G-set, is isomorphic to ________________.
[*Tip:* At this point, go back and make sure that your answers to the previous parts of this problem agree with what you have for #2(f,g)!]

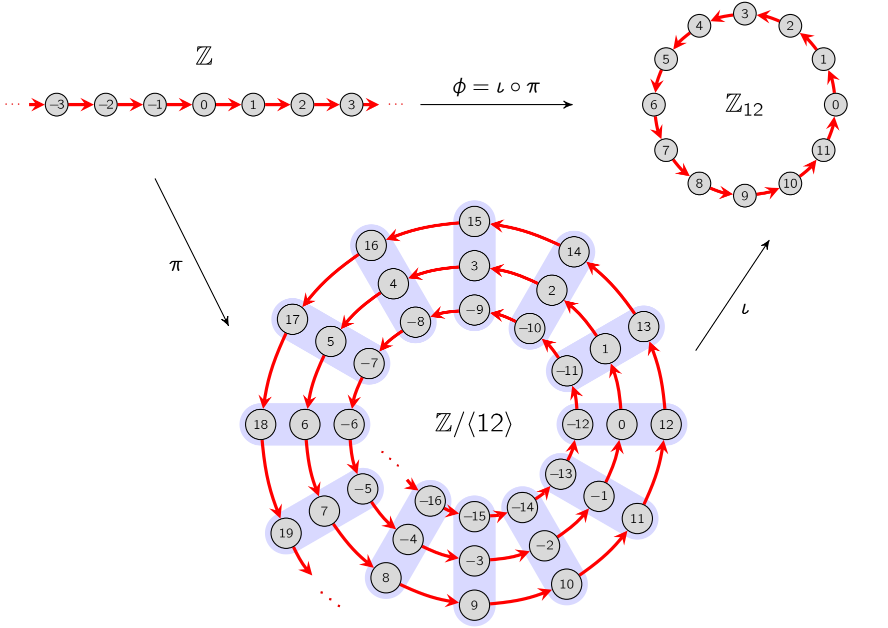
(f) By inspection, we can compute the *fixators* of the following: $\mathrm{fix}(1) =$ __________,
$\mathrm{fix}(s) =$ ____________, $\mathrm{fix}(rs) =$ ____________, and $\mathrm{fix}(r^2 s) =$ ____________.

(g) For this action, $\mathrm{Ker}(\phi) =$ ________________, and $\mathrm{Fix}(\phi) =$ ________________.

(h) The subgroup $H = \langle s\rangle$ of G has order ____________, and index ____________.
It has ________________ right coset(s), and ________________ left coset(s).

(i) Let $G = SD_8$ act on the right cosets of $H = \langle s\rangle$ by right multiplication. Draw the action graph. Use colors, or solid vs. dashed lines to distinguish generator edges.

Constructing $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$ as a quotient (Chapter 4)



Where to learn more

- Subscribe to my YouTube channel!
- Follow @VisualAlgebra on BlueSky and Twitter/X.
- Visual Algebra webpage for slides, HW, exams:

  http://www.math.clemson.edu/~macaule/visualalgebra.html

- Read my articles!
  - Macauley, M. (2024). Dihedralizing the quaternions. *Amer. Math. Monthly*, **131**(4), 294–308.
  - Macauley, M. (2025). Cayley tables and lattices of finite rings. *Math. Mag.*, In press.
- Nathan Carter's *Visual Group Theory* book.
- Dana Ernst's inquiry based learning visual algebra book: http://danaernst.com/

Future to do list (as of December 2024)

- Finalize *Visual Algebra* and publish it.
- Finish recording *Visual Algebra* and *Graduate Visual Algebra* playlists.
- Put LaTeX files for my slides on GitHub.

Feel free to get in touch!

THANK YOU!!!