Visual Algebra

Lecture 1.1: Groups, symmetry, and Cayley graphs

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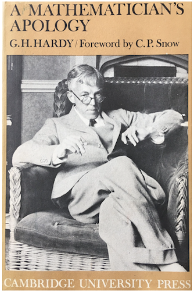
The science of patterns



G.H. Hardy (1877–1947) famously said that "*Mathematics is the Science of Patterns*."

He was also the PhD advisor to the brilliant Srinivasa Ramanujan (1887–1920), the central character in the 2015 film *The Man Who Knew Infinity*.

In his 1940 book *A Mathematician's Apology*, Hardy writes:

> *"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."*

Another theme is that the inherent beauty of mathematics is not unlike elegance found in other forms of art.

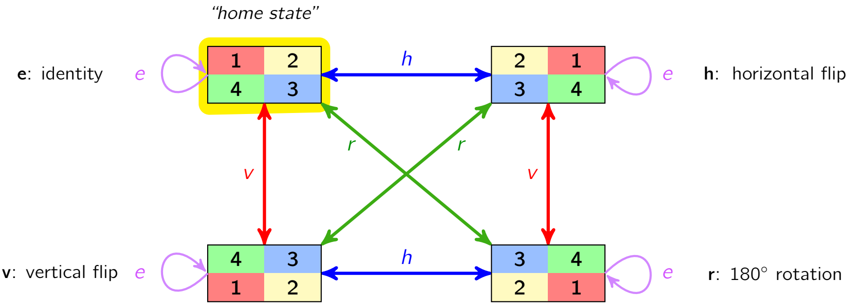
> *"The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way."*

Very few mathematical fields embody visual patterns as well as group theory.

We'll motivate the idea of a group by starting with the symmetries of a rectangle.

Our first group

The four symmetries of a rectangle can be visualalized using a **Cayley graph**, named after British mathematician Arthur Cayley (1821–1895).



The set **Rect** $= \{e, h, v, r\}$ of four symmetries is our first example of a **group**.

Observations?

Groups, informally

A group is a set of actions, satisfying a few mild properties.

Basic properties

- **Closure**: Composing actions in any order is another action.
- **Identity**: There is an identity action e, satisfying
$$ae = a = ea$$
for all actions a. [*Often we use* 1 *instead of* e.]
- **Inverses**: Every action a in has an inverse action b, satisfying
$$ab = e = ba.$$

We call the operation of composing actions multiplication, and write it from left-to-right.

Every group has a generating set, and we use angle brackets to denote this.

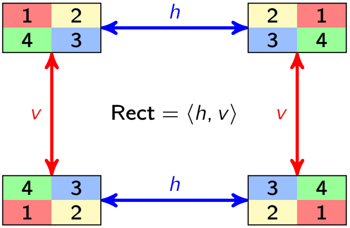
We usually prefer to find a minimal generating set. For example,

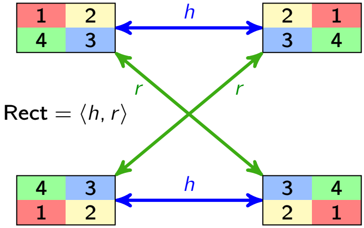
$$\textbf{Rect} = \langle v, h \rangle = \langle v, r \rangle = \langle h, r \rangle = \langle v, h, r \rangle = \langle v, h, e \rangle = \cdots$$

There still something missing from the above definition of a group. (Stay tuned!)

Minimal generating sets

Different minimal generating sets might lead to different Cayley graphs:





Remarks

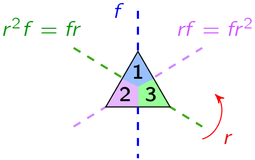
The group **Rect** has some properties that are not always true for other groups:

- it is **abelian**: $ab = ba$ for all $a, b \in$ **Rect**
- every element is its own inverse: $a^{-1} = a$ for all $a \in$ **Rect**
- the Cayley graphs for any two minimal generating sets have the same structure
- all minimal generating sets have the same size.

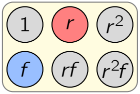
Symmetries of a triangle

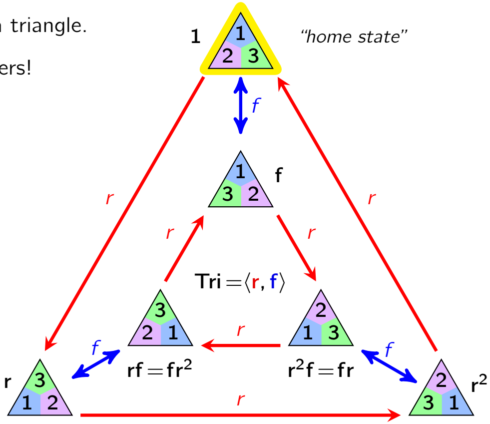
Consider the group of symmetries of a triangle.

This group is **nonabelian** – order matters!



"Group switchboard"





Equivalences of actions like the following are called relations:

$$r^3 = 1, \qquad f^2 = 1, \qquad rf = fr^2, \qquad fr = r^2 f.$$

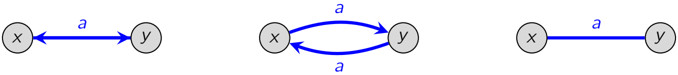
The Cayley graph makes it easy to find the inverse of each action:

$$1^{-1} = 1, \qquad r^{-1} = r^2, \qquad (r^2)^{-1} = r, \qquad f^{-1} = f, \qquad (rf)^{-1} = rf, \qquad (r^2f)^{-1} = r^2 f.$$

Cayley graph structure

Any action with $a^2 = 1$ is its own inverse: $a^{-1} = a$.

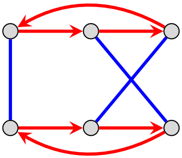
When this happens, we will use undirected arrows instead of bi-directed or double arrows:

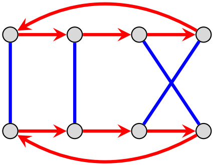


Four of the six actions in $\mathbf{Tri} = \langle r, f \rangle$ are their own inverse.

Cayley graphs must have a certain regularity: if $rf = fr^2$ holds from one node, it must hold from *every* node.

Do either of the following graphs have this regularity property?





A different generating set for the triangle symmetry group

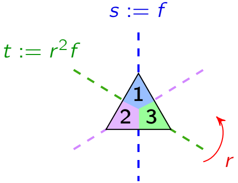
Recall the triangle symmetry group

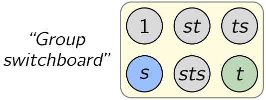
$$\mathbf{Tri} = \{\underbrace{1,\ r,\ r^2}_{\text{rotations}},\ \underbrace{f,\ rf,\ r^2f}_{\text{reflections}}\}.$$

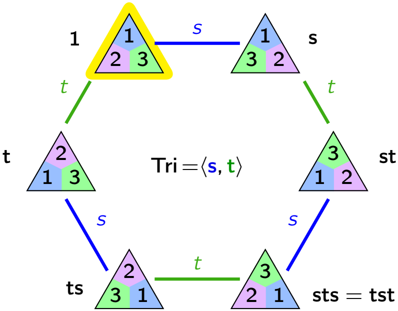
Notice that the composition of two reflections is a 120° rotation:

$$(rf)\cdot f = rf^2 = r\cdot 1 = r, \qquad f\cdot rf = f\cdot fr^2 = 1\cdot r^2 = r^2.$$

Let's see a Cayley graph corresponding to $\mathbf{Tri} = \langle s, t\rangle$, where $s := f$ and $t := r^2f = fr$.





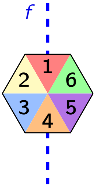


Minimal vs. minimum generating sets

There is a subtle but important difference between minimal and minimum.

For example, consider the group of symmetries of a hexagon:

$$\mathbf{Hex} = \{ \underbrace{1, r, r^2, r^3, r^4, r^5}_{\textit{rotations}}, \underbrace{f, rf, r^2f, r^3f, r^4f, r^5f}_{\textit{reflections}} \}.$$



$\langle r, f \rangle$ is a minimum (and hence minimal) generating set

$\langle r^2, r^3, f \rangle$ is a minimal (but not minimum) generating set

A generating set S is:

- minimal if removing any element makes it no longer generate.
- minimum if it is minimal and $|S| \leq |T|$ for all other generating sets.

For any generating set S of G,

$$\text{minimum} \Rightarrow \text{minimal}, \qquad \text{but} \qquad \text{minimal} \not\Rightarrow \text{minimum}.$$

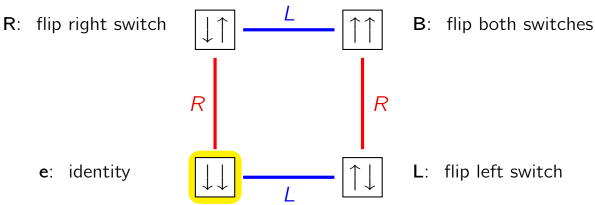
Groups arising from non-symmetry actions

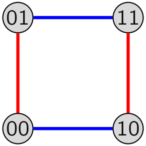
Consider two light switches in the "down" position. Call this our "home state".

Let $\mathbf{Light}_2 = \langle L, R \rangle$ be the group, where

- L: flip left switch
- R: flip right switch

Here is a Cayley graph:





Remark

The Cayley graphs for $\mathbf{Rect} = \{e, v, h, r\}$ and $\mathbf{Light}_2 = \{e, L, R, B\}$ have the same structure. We say they are isomorphic, and write

$$\mathbf{Rect} \cong \mathbf{Light}_2 \,.$$