Visual Algebra

Lecture 1.2: Groups from puzzles

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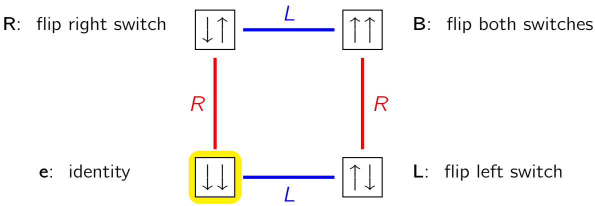
Groups arising from non-symmetry actions

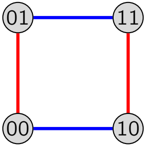
Consider two light switches in the "down" position. Call this our "home state".

Let $\mathbf{Light}_2 = \langle L, R \rangle$ be the group, where

- L: flip left switch
- R: flip right switch

Here is a Cayley graph:





Remark

The Cayley graphs for $\mathbf{Rect} = \{e, v, h, r\}$ and $\mathbf{Light}_2 = \{e, L, R, B\}$ have the same structure. We say they are isomorphic, and write

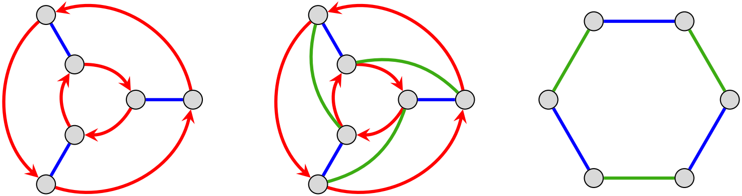
$$\mathbf{Rect} \cong \mathbf{Light}_2 \,.$$

Isomorphic groups

The formal definition of two groups being isomorphic is technical, and involves a structure preserving bijection between them.

If two groups have generating sets that define Cayley graphs of the same structure, they are isomorphic, and we have a "*Yes Certificate*".

If the Cayley graphs are different, then the groups are not necessarily non-isomorphic, as we saw with $\mathbf{Tri} = \langle r, f \rangle = \langle r, f, rf \rangle = \langle f, rf \rangle$.



In other words, a "*No Certificate*" is harder to verify.

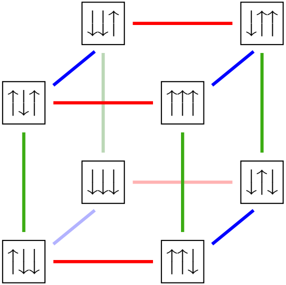
Remark

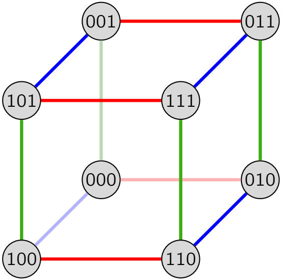
To prove two groups are non-isomorphic, find a property that one has but the other doesn't.

The three light switch group

The "2 *light switch group*" generalizes to the "3 *light switch group*" in the obvious manner:

$$\mathbf{Light}_3 := \langle L, M, R \rangle.$$





Properties of $\mathbf{Light}_3$

- this group is abelian: $ab = ba$, for all $a, b \in \mathbf{Light}_3$
- every action is its own inverse: $a^2 = e$, or $a^{-1} = a$, for all $a \in \mathbf{Light}_3$ (*verify!*)
- any minimal generating set has size 3 (*not immediately obvious*).

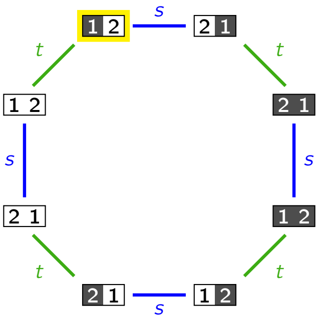
Another group of size 8

Call the following rectangle configuration our *home state*: 

Suppose we are allowed the following operations, or "actions":

- s: swap the two squares
- t: toggle the color of the first square.

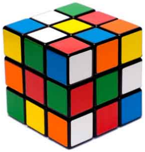
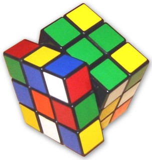
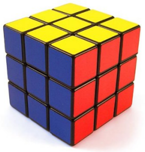
Here is a Cayley graph of this group that we'll call $\textbf{Coin}_2 = \langle s, t \rangle$:



Question: *Are the groups* $\textbf{Coin}_2$ *and* $\textbf{Light}_3$ *isomorphic?*

The Rubik's cube group

One of the most famous groups is the set of actions on the Rubik's Cube.

Fact

There are 43,252,003,274,489,856,000 distinct configurations of the Rubik's cube.

This toy was invented in 1974 by architect Ernő Rubik (born 1944) of Budapest, Hungary.

His Wikipedia page used to say:

> He is known to be an introvert and hardly accessible person, hard to contact or get for autographs. He typically does not attend speedcubing events. However, he attended the 2007 World Championship in Budapest.[2][3]

The Rubik's cube group

Not impossible . . . just almost impossible.

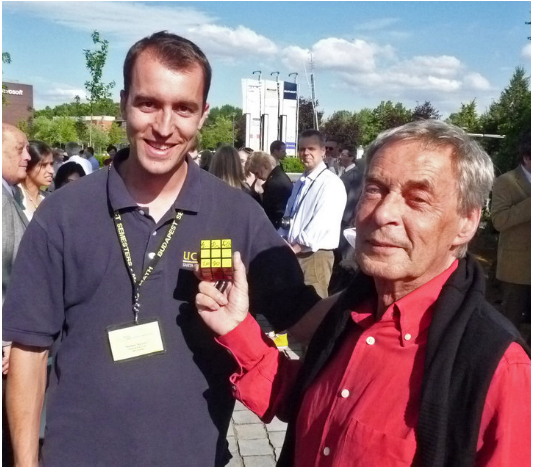


Figure: June 2010, in Budapest, Hungary

The Rubik's cube group

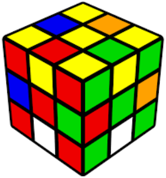
The configurations of the Rubik's cube are different than the actions, but they are in bijective correspondence.

The Rubik's cube group is generated (not minimally!) by 6 actions:

$$\textbf{Rubik} := \langle F, B, R, L, U, D \rangle,$$

where each action is a 90° clockwise turn of one of the six faces:

- **F**: front
- **B**: back
- **R**: right
- **L**: left
- **U**: upper
- **D**: bottom ("down")



In other words, these six actions generate all $\big|\textbf{Rubik}\big| = 43{,}252{,}003{,}274{,}489{,}856{,}000$ actions of the Rubik's cube group.

Theorem (Rokicki, Kociemba, Davidson, Dethrige, 2010)

Every configuration of the Rubik's cube group is at most 20 "moves" from the solved state. Moreover, there are configurations that are exactly 20 moves away.

The Rubik's cube group

Though the Rubik's cube group is generated by 6 actions,

$$\textbf{Rubik} := \langle F,\ B,\ R,\ L,\ U,\ D \rangle,$$

most solution guides also use:

- F', B', R', L', U', D' for 90° counterclockwise turns, and
- $F2, B2, R2, L2, U2, D2$ for 180° turns.

The theorem about "*every configuration is at most* 20 *moves away*" considers this definition for a "*move*."

The following is a standard definition from graph (or network) theory.

Definition

The diameter of a graph is the longest shortest path between any two nodes.

Theorem (Rokicki, Kociemba, Davidson, Dethrige, 2010)

The diameter of the Cayley graph of the Rubik's cube group, with generating set

$$\textbf{Rubik} = \langle F, B, R, L, U, D, F', B', R', L', U', D', F2, B2, R2, L2, U2, D2 \rangle$$

is 20.

The Rubik's cube group

In 2014, Tomas Rokicki and Morley Davidson, with the Ohio Supercomputing Center, solved the Rubik's cube in the "*quarter-turn metric*".

Theorem (Rokicki & Davidson, 2014)

The diameter of the Cayley graph of the Rubik's cube group, with generating set

$$\textbf{Rubik} = \langle F, B, R, L, U, D, F', B', R', L', U', D' \rangle$$

is 26.

In the "*half-turn metric*," there are hundreds of millions of nodes a maximal distance (exactly 20) from the solved state.

In the "*quarter-turn metric*," we only know of three at a maximal distance (exactly 26).

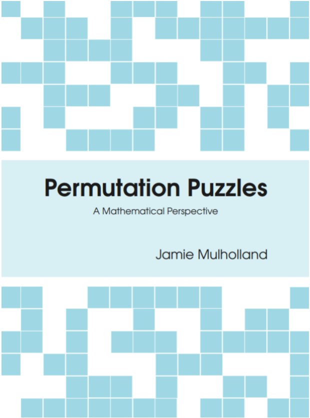
It is conjectured that there are

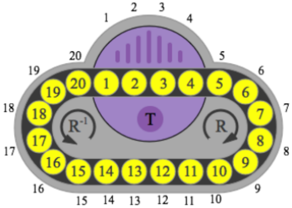
- $\approx$ 36 nodes at a distance of 25
- $\approx$ 150,000 nodes at a distance of 24
- $\approx$ 24 quadrillion (2.4×10^{16}) nodes at a distance of 23.

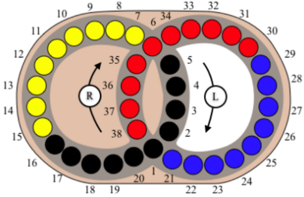
When we study permutation groups, we'll better understand why the quarter-turn metric is more natural than the half-turn metric.

Group theory and puzzles

Mulholland, J. *Permutation Puzzles: a Mathematical Perspective*. Self-published, 2021.





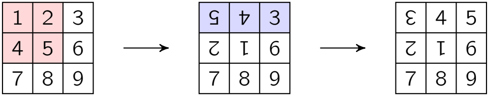


- Course website, *Permutation Puzzles*: `http://www.sfu.ca/~jtmulhol/math302/`
- *Jaap's puzzle page*: `https://www.jaapsch.net/puzzles/`

Spinpossible™ and Spin$_{3\times 3}$

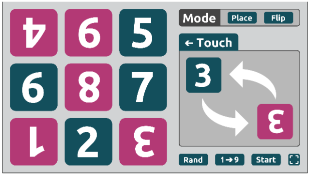
In 2011, Andrew and Alex Sutherland developed a puzzled played on an $n \times n$ grid.

Legal moves consist of rotating rectangular regions by 180°.



Jeff Slye wrote a webapp that explores this:

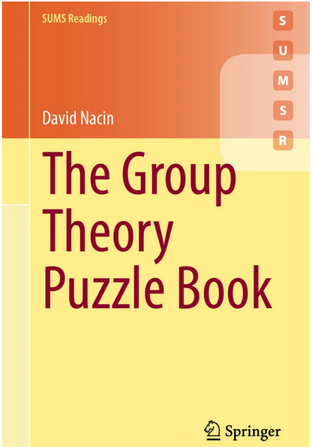
`https://jslyemath.itch.io/spin3x3`



- Sutherland, A., & Sutherland, A. (2011). The mathematics of Spinpossible. Preprint at `arXiv:1110.6645`.
- Ernst, D. C., & Slye, J. (2024). Using the Spin$_{3\times 3}$ Virtual Manipulative to Introduce Group Theory. PRIMUS, 1-12.
- Ernst, D. C. (2016). An inquiry-based approach to abstract algebra. Available online at `danaernst.com`.

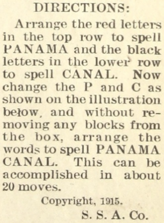
Group theory and puzzles

Nacin, D. (2024). The Group Theory Puzzle Book. Springer. Cham, Switzerland.



Groupoid puzzles

The impossible "Fifteen puzzle" was a craze in the US in the 1870s and 1880s.



DIRECTIONS:

Arrange the red letters in the top row to spell PANAMA and the black letters in the lower row to spell CANAL. Now change the P and C as shown on the illustration below, and without removing any blocks from the box, arrange the words to spell PANAMA CANAL. This can be accomplished in about 20 moves.

Copyright, 1915.
S. S. A. Co.

In an 1879 *American Journal of Mathematics*, paper, Johnson and Story wrote:

> *The "15" puzzle for the last few weeks has been prominently before the American public, and may safely be said to have engaged the attention of nine out of ten persons of both sexes and of all ages and conditions of the community.*

Actions cannot always be composed, so this puzzle's algebraic structure is a "groupoid."

- Jim Storer Puzzles Homepage: `https://www.cs.brandeis.edu/~storer/JimPuzzles/`