Visual Algebra

Lecture 1.3: Group presentations

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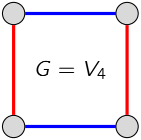
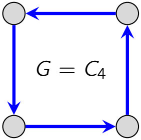
Unlabeled Cayley graphs

Previously, we've labeled the nodes of Cayley graphs with configurations.

If we want to focus on a graph's structure, we can leave the nodes unlabeled.

For example, consider the following two groups of size 4:





The abstract group isomorphic to **Rect** is the Klein 4-group, denoted V_4, named after German mathematician Felix Klein (1849–1925).

Questions

- Are the two groups whose Cayley graphs are shown above isomorphic?
- Can you think of an object whose symmetry group is the group on the right?

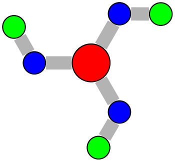
Cyclic groups (preview)

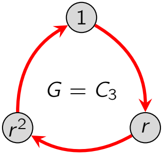
Groups that can be generated by a single action are called cyclic.

These describe shapes that have only rotational symmetry.

The shape of a molecule of boric acid, $B(OH)_3$, is shown below. It should be clear that there are three symmetries:

- the identity action, 1
- 120° counterclockwise rotation, r
- 240° counterclockwise rotation, r^2.





The boric acid molecule is chiral because a mirror reflection is not a symmetry.

Inorganic chemists use group theory to classify molecules and crystals by their symmetries.

The triangle symmetry group $\mathbf{Tri} = \langle r, f \rangle$ contains $C_3 = \langle r \rangle$ as a subset. We say that C_3 is a subgroup of $\mathbf{Tri}$.

Labeling Cayley graphs with actions

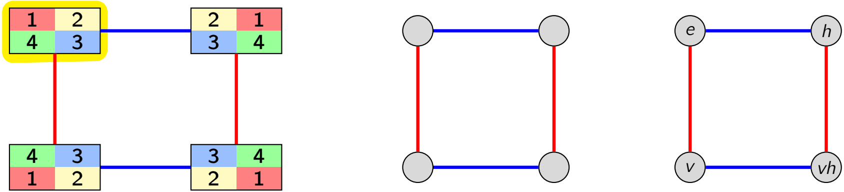
When drawing Cayley graphs, we have done one of two things with the nodes:

1. Label nodes with configurations of an object.
2. Leaving nodes unlabeled.

There is a 3rd choice, since every path represents an action in the group.

3. Label the nodes with actions.

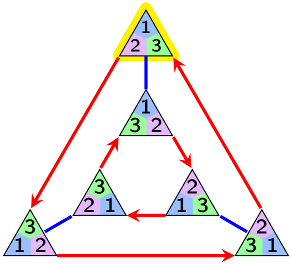
Here is one way to do this for the Klein 4-group, $G = V_4$:

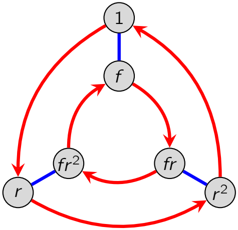


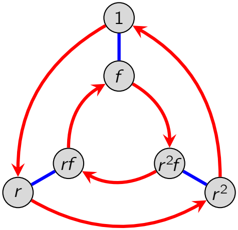
By the "regularity property" of Cayley graphs, it does not matter where we start, or what path we take when labeling.

Labeling Cayley graphs with actions

Here are two canonical ways to label the nodes of the Cayley graph of $\mathbf{Tri} = \langle r, f \rangle$.







Technically, these are right Cayley graphs because we are reading from left-to-right.

In other words, traversing around the graph corresponds to right multiplication.

Remark

Every path corresponds to an action $a \in G$. To compute a path for $ab \in G$:

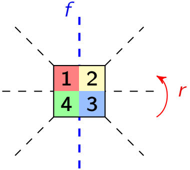
- start at node a (or equivalently, start at the identity node and follow any path for a).
- follow any path corresponding to b.

Another group of size 8

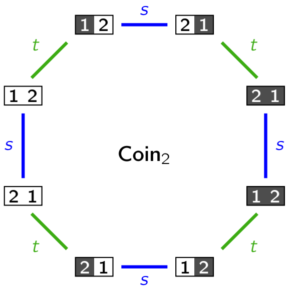
The eight symmetries of a square form a group generated by:

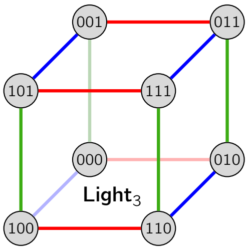
- a 90° counterclockwise rotation r,
- a horizontal flip f.

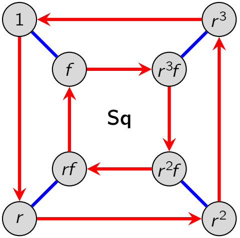
We'll call this group $\mathbf{Sq} = \langle r, f \rangle$.



Question: *Do any of these groups have the same structure? (Are they "isomorphic"?)*







Can you find a property that one group (not graph!) has that the others do not?

Group presentations

Thus far, we've described a group by its generators.

$G = \langle r, f \rangle$ means "*G is generated by r and f*."

However, this doesn't tell us *how* they generate.

Definition

A group presentation for G is a description of the group as

$$G = \Big\langle \text{generators} \;\Big|\; \text{relations} \Big\rangle .$$

The vertical bar can be thought of as meaning "subject to".

Even for a fixed set of generators, a group presentation is not unique.

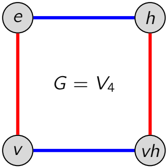
Key idea

A presentation is just an algebraic way to encode a Cayley graph.

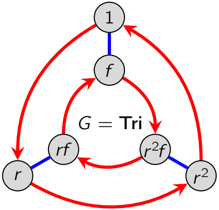
But, it doesn't necessarily tell us *which* Cayley graph!

Group presentations

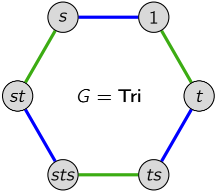
Here are some example of presentations:



$\langle v, h \mid v^2 = h^2 = e,\ vh = hv \rangle$



$\langle r, f \mid r^3 = f^2 = 1,\ rf = fr^2 \rangle$



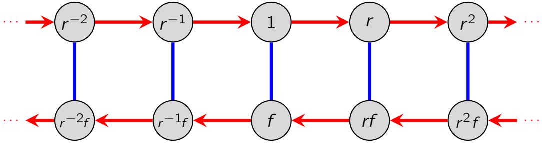
$\langle s, t \mid s^2 = t^2 = 1,\ sts = tst \rangle$

The relation $r^3 = 1$ is redundant in the second presentation:

$$rf = fr^2 \quad \Rightarrow \quad f(rf) = r^2 \quad \Rightarrow \quad (frf)^2 = r^4 \quad \Rightarrow \quad fr^2f = r^4 \quad \Rightarrow \quad (fr^2)f = r^4$$

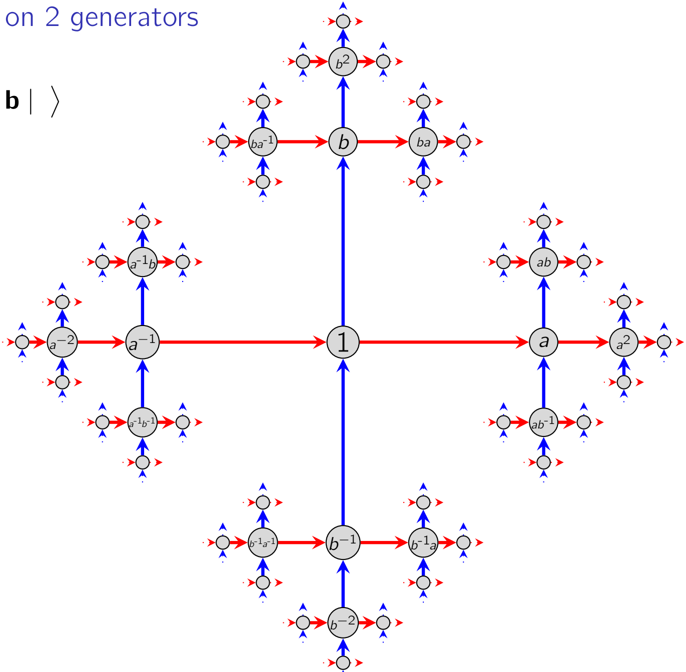
$$\Rightarrow \quad (rf)f = r^4 \quad \Rightarrow \quad r = r^4 \quad \Rightarrow \quad \color{red}{1 = r^3}.$$

But removing $r^3 = 1$ from **Tri** $= \langle r, f \mid r^3 = f^2 = 1,\ rf = fr^{-1} \rangle$ yields an infinite group.



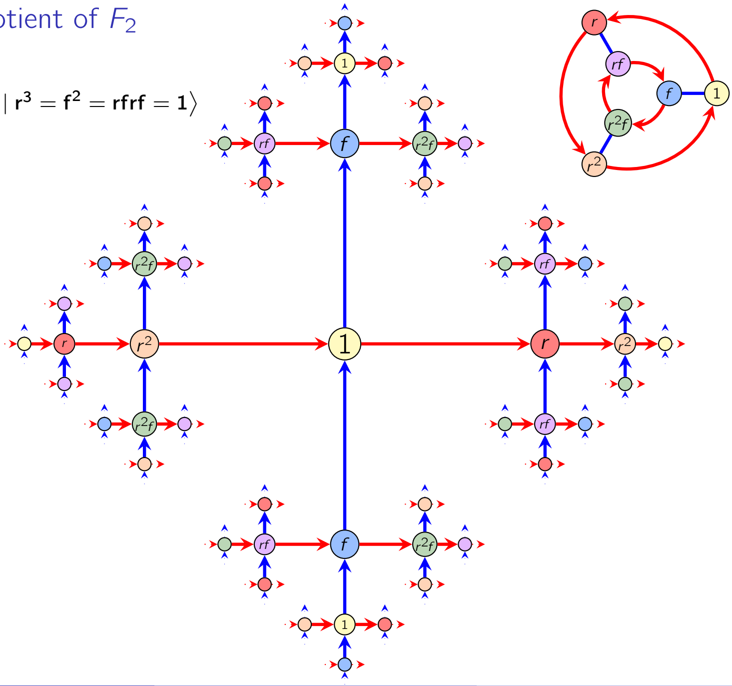
The free group on 2 generators

$$\mathbf{F_2} = \langle \mathbf{a}, \mathbf{b} \mid \ \rangle$$

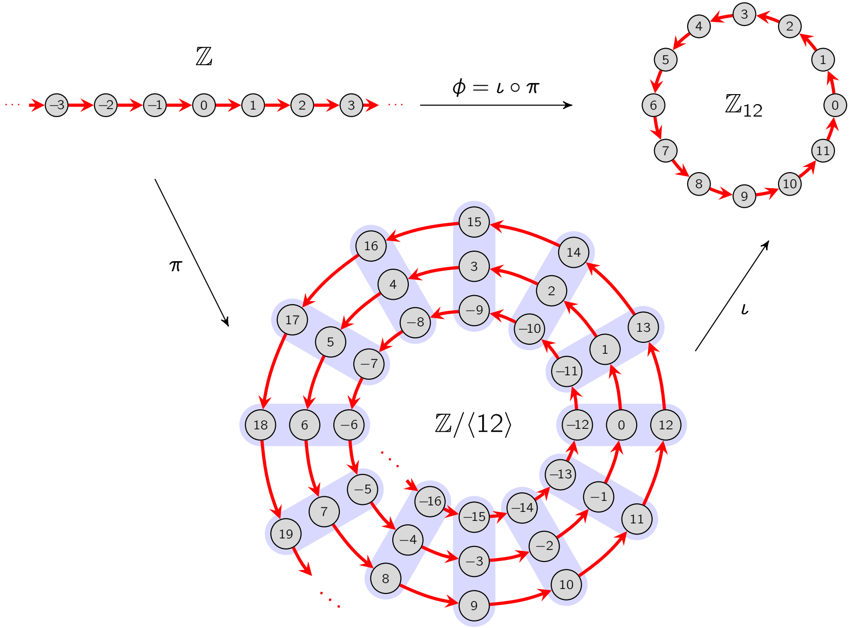


Tri as a quotient of F_2

$$\mathbf{Tri} = \langle \mathbf{r}, \mathbf{f} \mid \mathbf{r}^3 = \mathbf{f}^2 = \mathbf{rfrf} = \mathbf{1} \rangle$$



Constructing $\mathbb{Z}_{12} = \{0, 1, \dots, 11\}$ from the free group on 1 generator



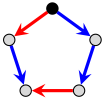
Group presentations

The word problem

Given a presentation $G = \langle g_1, \dots, g_n \mid r_1 = e, \dots, r_m = e \rangle$, is $G = \{e\}$?

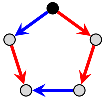
Exercise

Show that $G = \langle a, b \mid ab = b^2 a,\ ba = a^2 b \rangle$ is the trivial group.



$aba^{-1}b^{-2} = 1$

and



$bab^{-1}a^{-2} = 1$

$\Longrightarrow$

$G = \langle a, b, \mid a = b = 1 \rangle = \langle 1 \rangle$

An even harder problem is the isomorphism problem: Given G_1 and G_2, is $G_1 \cong G_2$?

Question

Given a group presentation that "looks like" a large group, *how can we be absolutely sure?*

Unsolvability of the word problem

Theorem

The word problem is unsolvable, even for finitely presented groups.

4-dimensional sphere problem

Given a 4-dimensional surface, determine whether it is homeomorphic to the 4-sphere.

Every surface S has a group $\pi_1(S)$ called the fundamental group of all "looped paths."

Four dimensions is big enough that for *any* G, we can build a surface for which $\pi_1(S) \cong G$.

Theorem

The 4-dimensional sphere problem is unsolvable.

Summary of the proof

Suppose there exists a solution, and let G be a group.

1. Build a surface S such that $\pi_1(S) \cong G$.
2. Determine whether S is a 4-sphere (all loops on a sphere are trivial).
3. This solves the word problem for G. (Contradiction)