Visual Algebra

Lecture 1.4: Infinite symmetry groups

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Frieze groups

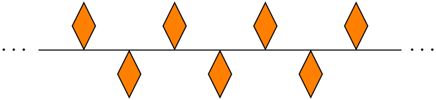
In architecture, a frieze is a long narrow section of a building, often decorated with art.

Figure: A frieze on the Admiralty, in Saint Petersburg.

They were common on ancient Greek, Roman, and Persian buildings.

Sometimes, but not always, such a pattern repeats.

In mathematics, a frieze is a 2-dimensional pattern that repeats in one direction, with a minimal nonzero translational symmetry.

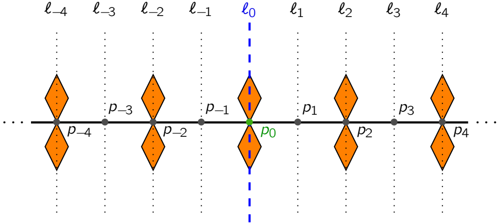


Definition

The symmetry group of a frieze is called a frieze group.

Goal. *Understand and classify the frieze groups.*

Frieze groups



Definition

Let v be the unique vertical reflection. Symmetries come in infinite families. Define

- t: minimal translation to the right
- $g_i := t^i v = v t^i$: glide-reflection
- h_i: horizontal reflection across ℓ_i
- r_i: 180° rotation around p_i

The symmetry group of this frieze consists of the following symmetries:

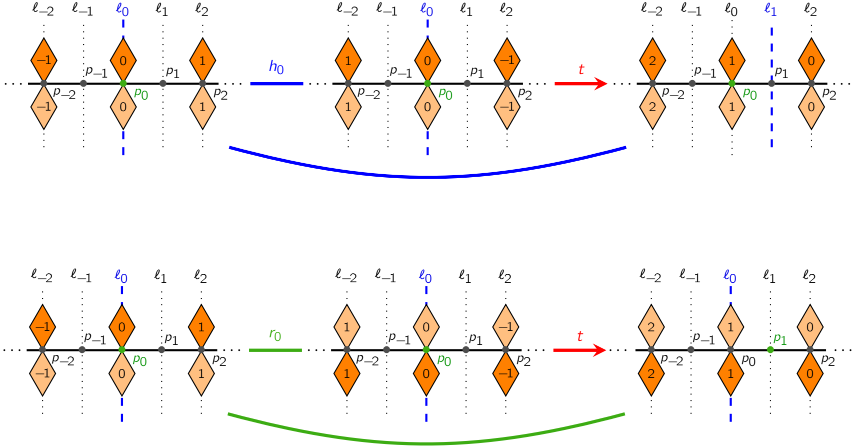
$$\mathbf{Frz}_1 := \{h_i \mid i \in \mathbb{Z}\} \cup \{r_i \mid i \in \mathbb{Z}\} \cup \{t^i \mid i \in \mathbb{Z}\} \cup \{g_i \mid i \in \mathbb{Z}\}.$$

Note that $v = g_0$. Letting $h := h_0$, $r := r_0$, and $g := g_1$, this frieze group is generated by

$$\mathbf{Frz}_1 := \langle t, h, v \rangle = \langle t, h, r \rangle = \langle t, v, r \rangle = \langle g, h, v \rangle = \cdots$$

Frieze groups

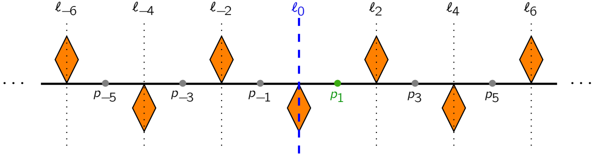
Let's look at how the various reflections and rotations are related:



Similarly, it follows that $h_i t = h_{i+1}$ and $r_i t = r_{i+1}$ for any $i \in \mathbb{Z}$.

A "smaller" frieze group

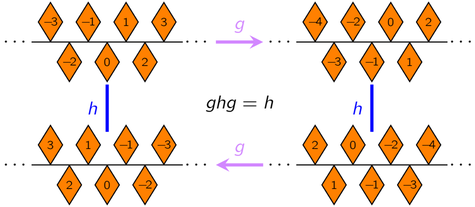
Let's eliminate the vertical symmetry from the previous frieze group.

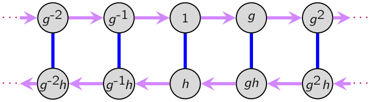


We lose half of the horizontal reflections and rotations in the process. The frieze group is

$$\mathbf{Frz}_2 := \{g^i \mid i \in \mathbb{Z}\} \cup \{h_{2j} \mid j \in \mathbb{Z}\} \cup \{r_{2k+1} \mid k \in \mathbb{Z}\} = \langle g, h \rangle = \langle g, r \rangle,$$

where $h = h_0$, $r = r_1$, $g = g_1 = tv$.

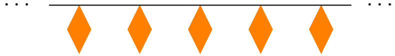


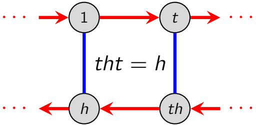


$$\mathbf{Frz}_2 = \langle g, h \mid h^2 = 1,\ ghg = h \rangle$$

Other friezes generated by two symmetries

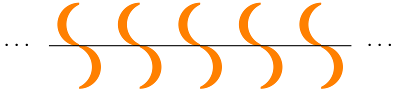
Frieze 3: eliminate the vertical flip and all rotations

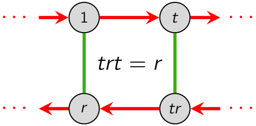




$$\mathbf{Frz}_3 = \{t^i \mid i \in \mathbb{Z}\} \cup \{h_j \mid j \in \mathbb{Z}\} = \langle t, h \mid h^2 = 1,\ tht = h \rangle$$

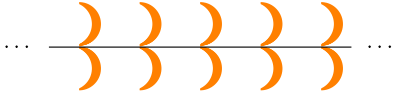
Frieze 4: eliminate the vertical flip and all horizontal flips

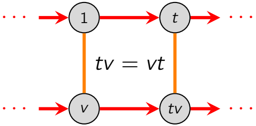




$$\mathbf{Frz}_4 = \{t^i \mid i \in \mathbb{Z}\} \cup \{r_j \mid j \in \mathbb{Z}\} = \langle t, r \mid r^2 = 1,\ trt = r \rangle$$

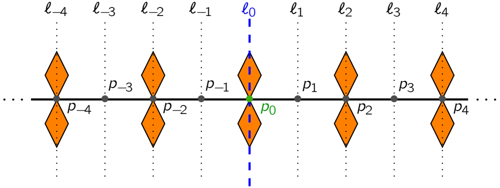
Frieze 5: eliminate all horizontal flips and rotations





$$\mathbf{Frz}_5 = \{t^i \mid i \in \mathbb{Z}\} \cup \{g_j \mid j \in \mathbb{Z}\} = \langle t, v \mid v^2 = 1,\ tv = vt \rangle$$

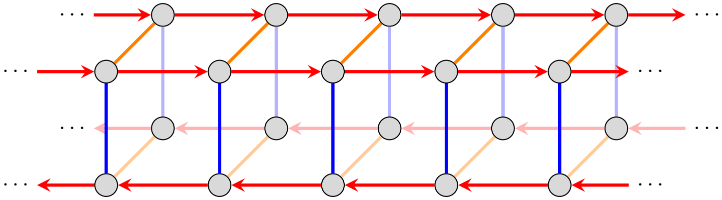
A Cayley graph of our first frieze group



A presentation for this frieze group is

$$\mathbf{Frz}_1 = \langle t, h, v \mid h^2 = v^2 = 1,\ hv = vh,\ tv = vt,\ tht = h \rangle.$$

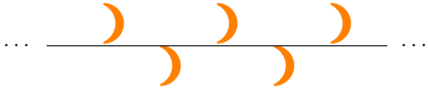
We can make a Cayley graph by piecing together the "tiles" on the previous slide:

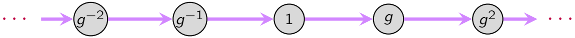


Classification of frieze groups

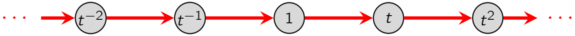
Since frieze groups are infinite, each one must contain a translation.

Frieze 6





Frieze 7



The frieze groups are $\mathbf{Frz}_6 = \langle g \mid \quad \rangle \cong \mathbf{Frz}_7 = \langle t \mid \quad \rangle$.

Theorem

There are 7 different frieze groups, but only 4 up to isomorphism.

Wallpaper and crystal groups

A frieze is a pattern than repeats in one dimension.

A next natural step is to look at discrete patterns that repeat in higher dimensions.

- a 2-dimensional repeating pattern is a wallpaper.
- a 3-dimensional repeating pattern is a crystal. The branch of mathematical chemistry that studies crystals is called crystallography.

In two dimensions, patterns can have 2-fold, 3-fold, 4-fold, or 6-fold symmetry.

Patterns can also have reflective symmetry, or be "chiral."

Symmetry groups of wallpapers are called wallpaper groups.

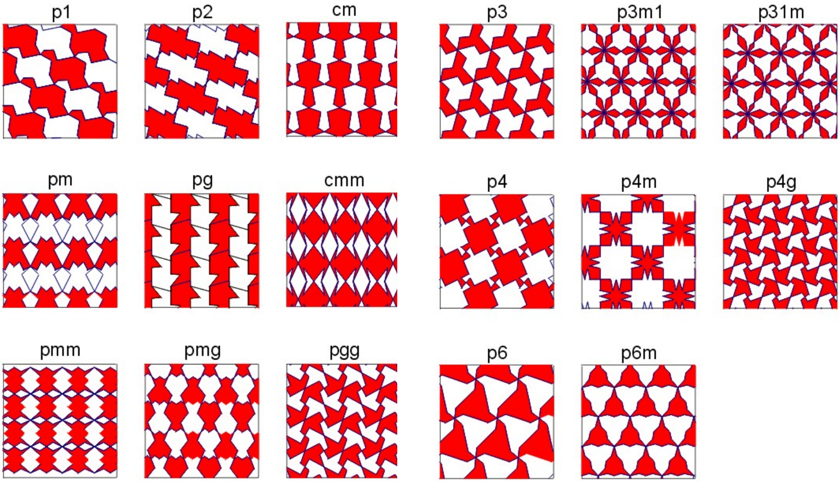
These were classified by Russian mathematician and crystallographer Evgraf Fedorov (1853–1919).

Theorem (1877)

There are 17 different wallpaper groups.

Mathematicians like to say "*there are only 17 different types of wallpapers*."

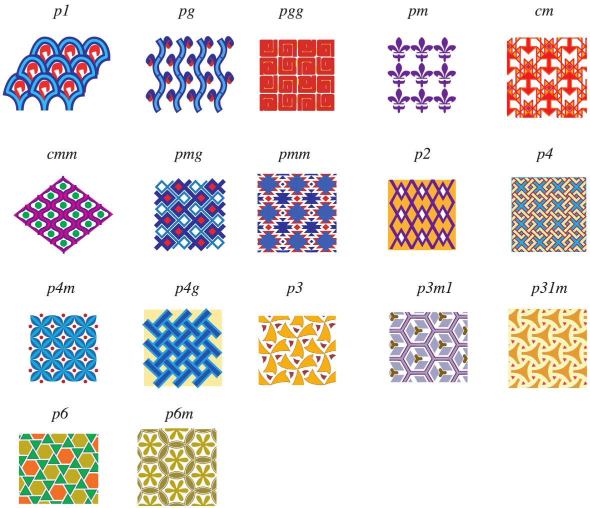
The 17 types of wallpaper patterns



Images by Patrick Morandi (New Mexico State University).

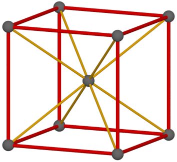
The 17 types of wallpaper patterns

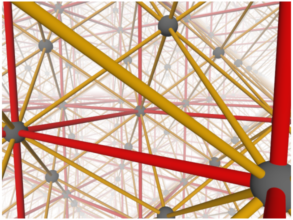
Here is another picture of all 17 wallpapers, with the official IUC notation for the symmetry group, adopted by the International Union of Crystallography in 1952.



Symmetry groups of crystals

Symmetry groups of crystals are called space, crystallographic, or Fedrov groups.





They were classified by Fedorov and Schöflies.

Theorem (1892)

There are 230 space groups.

In 1978, a group of mathematicians showed there were exactly 4895 four-dimensional symmetry groups.

In 2002, it was discovered that two were actually the same, so there's only 4894.