Visual Algebra

Lecture 1.5: Cayley tables

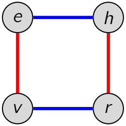
Dr. Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

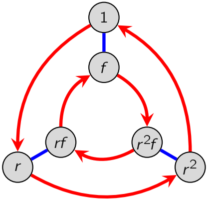
Cayley tables

In some sense, a Cayley graph is a type of "group calculator."

Another useful tool is something we all learned about in grade school.



	e	v	h	r
e	e	v	h	r
v	v	e	r	h
h	h	r	e	v
r	r	h	v	e



	1	r	r^2	f	rf	r^2f
1	1	r	r^2	f	rf	r^2f
r	r	r^2	1	rf	r^2f	f
r^2	r^2	1	r	r^2f	f	rf
f	f	r^2f	rf	1	r^2	r
rf	rf	f	r^2f	r	1	r^2
r^2f	r^2f	rf	f	r^2	r	1

We will call this a Cayley table.

Notational convention

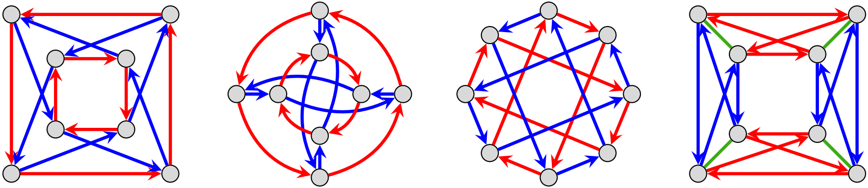
Since $ab \neq ba$ in general, we will say that the entry in row a and column b is ab.

Cayley tables can reveals patterns that are otherwise hidden.

Sometimes, these patterns only appear if we arrange elements in a certain order.

The quaterternion group

Here are four Cayley graphs of a new group called the quaternion group, Q_8.

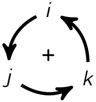


It's in no way clear that these even represent isomorphic groups.

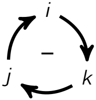
Notice how each one highlights different structural properties.

The first two Cayley graphs emphasize similarities and differences between Q_8 and **Sq**.

The group Q_8 is generated by "imaginary numbers" i, j, k, with $i^2 = j^2 = k^2 = -1$.



multiplying in this direction yields a positive result



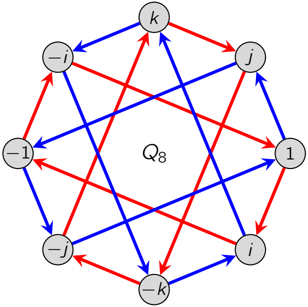
multiplying in this direction yields a negative result

The quaternion group

Two possible presentations for the quaternions are

$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle = \langle i, j \mid i^4 = j^4 = 1,\ iji = j \rangle .$$

This is one case where it's convenient to *not* use a minimal generating set.



	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

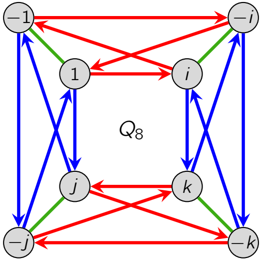
Remember how we said that some patterns in Cayley tables only appear if we arrange elements in a certain order?

The quaternion group

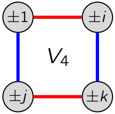
Rather than order elements as $1, i, j, k, -1, -i, -j, -k$ in

$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle = \langle i, j \mid i^4 = j^4 = 1,\ iji = j \rangle,$$

let's construct a Cayley table with them ordered $1, -1, i, -i, j, -j, k, -k$.



	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1



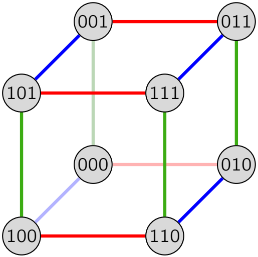
	± 1	$\pm i$	$\pm j$	$\pm k$
± 1	± 1	$\pm i$	$\pm j$	$\pm k$
$\pm i$	$\pm i$	± 1	$\pm k$	$\pm j$
$\pm j$	$\pm j$	$\pm k$	± 1	$\pm i$
$\pm k$	$\pm k$	$\pm j$	$\pm i$	± 1

Remark

"Collapsing" the group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ in this manner reveals the structure of V_4!

This is an example of taking a quotient of a group by a subgroup. We'll return to this!

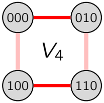
Another example of a quotient: **Light**$_3$



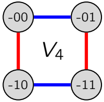
	000	100	010	110	001	101	011	111
000	000	100	010	110	001	101	011	111
100	100	000	110	010	101	001	111	011
010	010	110	000	100	011	111	001	101
110	110	010	100	000	111	011	101	001
001	001	101	011	111	000	100	010	110
101	101	001	111	011	100	000	110	010
011	011	111	001	101	010	110	000	100
111	111	011	101	001	110	010	100	000

“*subgroup of* **Light**$_3$

	000	100	010	110
000	000	100	010	110
100	100	000	110	010
010	010	110	000	100
110	111	010	100	000



“*quotients of* **Light**$_3$



	-00	-10	-01	-11
-00	-00	-10	-01	-11
-10	-10	-00	-11	-01
-01	-01	-11	-00	-10
-11	-11	-01	-10	-00

	--0	--1
--0	--0	--1
--1	--1	--0

Cayley tables

Proposition

An element cannot appear twice in the same row or column of a multiplication table.

Proof

Suppose that in row a, the element g appears in columns b and c. Algebraically, this means

$$ab = g = ac.$$

Multiplying everything on the left by a^{-1} yields

$$a^{-1}ab = a^{-1}g = a^{-1}ac \qquad \Longrightarrow \qquad b = c.$$

Thus, g (or any element) element cannot appear twice in the same row.

Verifying that g cannot appear twice in the same column is analogous. (Exercise) □

Question. *If we have a table where every element appears in every row and column once, is it a Cayley table for some group?*

Latin squares and forbidden Cayley tables

A table where every element appears in every row and column once is called a Latin square.

Here is an example of two Latin squares on a set of five elements, with identity element e.

	e	a	b	c	d
e	e	a	b	c	d
a	a	c	d	b	e
b	b	d	a	e	c
c	c	b	e	d	a
d	d	e	c	a	b

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c	b	d	e	a
d	d	c	a	b	e

Exploratory exercise

Can you construct a Cayley graph for either of these Latin squares? If not, what goes wrong?