Visual Algebra

Lecture 2.3: Dihedral groups

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Dihedral groups

Definition

The dihedral group D_n is the group of symmetries of a regular n-gon. It has order $2n$.

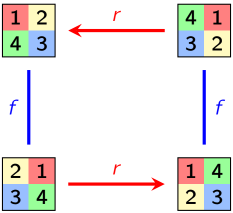
One possible choice of generators is

1. $r =$ counterclockwise rotation by $2\pi/n$ radians,
2. $f =$ flip across a fixed axis of symmetry.

Using these generators, one (of many) ways to write the elements of $D_n = \langle r, f \rangle$ is

$$D_n = \{ \underbrace{1, r, r^2, \dots, r^{n-1}}_{n \text{ rotations}}, \underbrace{f, rf, r^2 f, \dots, r^{n-1} f}_{n \text{ reflections}} \}.$$

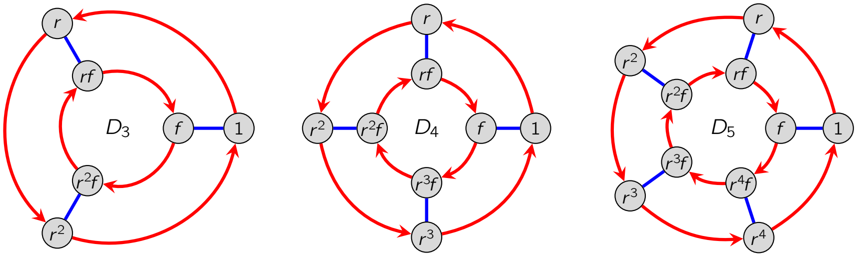
It is easy to check that $rf = fr^{-1}$:



Dihedral groups

Several different presentations for D_n are:

$$D_n = \langle r, f \mid r^n = 1,\ f^2 = 1,\ rfr = f \rangle = \langle r, f \mid r^n = 1,\ f^2 = 1,\ rf = fr^{n-1} \rangle.$$



Warning!

Many books denote the symmetries of the n-gon as D_{2n}.

A strong advantage to our convention is that we can write

$$C_n = \langle r \rangle = \{1, r, r^2, \dots, r^{n-1}\} \leq \langle r, f \rangle = D_n.$$

Dihedral groups

Another canonical way to generate D_n is with two reflections:

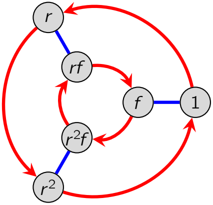
- $s := f$
- $t := fr = r^{n-1}f$

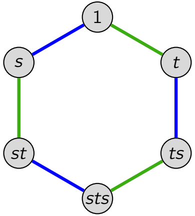
Composing these in either order is a rotation of $2\pi/n$ radians:

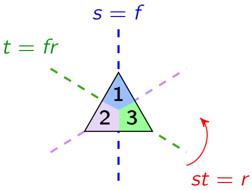
$$st = f(fr) = r, \qquad ts = (fr)f = (r^{n-1}f)f = r^{n-1}.$$

A group presentation with these generators is

$$D_n = \langle s, t \mid s^2 = 1,\ t^2 = 1,\ (st)^n = 1 \rangle = \{ \underbrace{1, st, ts, (st)^2, (ts)^2, \dots,}_{\text{rotations}} \underbrace{s, t, sts, tst, \dots}_{\text{reflections}} \}.$$





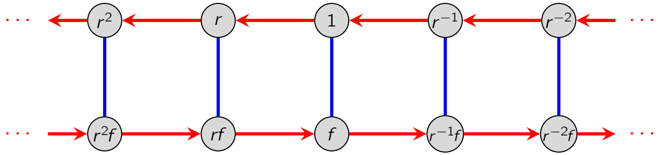


Dihedral groups

Definition

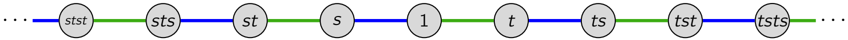
The infinite dihedral group, denoted D_∞, has presentation

$$D_\infty = \langle r, f \mid f^2 = 1,\ rfr = f \rangle.$$



We can also generate D_∞ with two reflections, $s := f$ and $t = fr$.

$$D_\infty = \langle s, t \mid s^2 = 1,\ t^2 = 1 \rangle = \{ \underbrace{1, st, ts, (st)^2, (ts)^2, \dots,}_{\text{"rotations"}} \underbrace{s, t, sts, tst, \dots}_{\text{"reflections"}} \}.$$

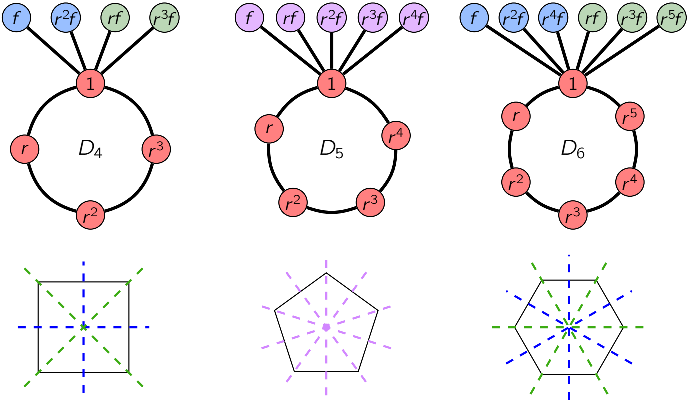


Cycle graphs of dihedral groups

The maximal orbits of D_n consist of

- 1 orbit of size n containing $\{1, r, \ldots, r^{n-1}\}$;
- n orbits of size 2 containing $\{1, r^k f\}$ for $k = 0, 1, \ldots, n-1$.

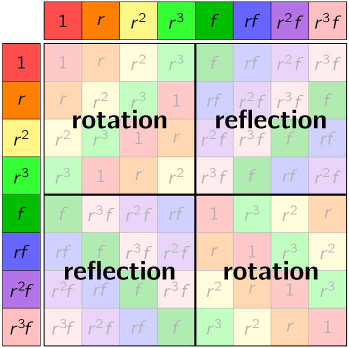
Unless n is prime, the size-n orbit will have smaller subsets that are orbits.



Cayley tables of dihedral groups

The separation of D_n into rotations and reflections is visible in its Cayley tables.

	1	r	r^2	r^3	f	rf	r^2f	r^3f
1	1	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	1	rf	r^2f	r^3f	f
r^2	r^2	r^3	1	r	r^2f	r^3f	f	rf
r^3	r^3	1	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	1	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	1	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	1	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	1



The partition of D_n as depicted above has the structure of group C_2.

"Shrinking" a group in this way is called a quotient.

It yields a group of order 2 with the following Cayley table:

	1	f
1	1	f
f	f	1

Representations of dihedral groups

Recall that the Klein 4-group can be represented by

$$V_4 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Moreover, a rotation of $2\pi/n$ radians can be

$$A_{2\pi/n} = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} \qquad \text{or} \qquad R_n := \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix} = \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}.$$

The canonical real representation of D_n with 2×2 matrices is

$$D_n \cong \left\langle \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

The canonical complex representations of D_n with 2×2 matrices is

$$D_n \cong \left\langle \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

Viewing the groups C_n and D_n as matrices makes our choice of calling the dihedral group D_n (rather than D_{2n}) much more natural!