Visual Algebra

Lecture 2.6: Symmetric and alternating groups

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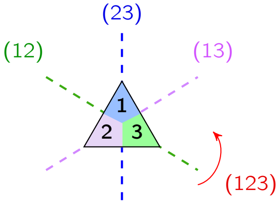
The symmetric group

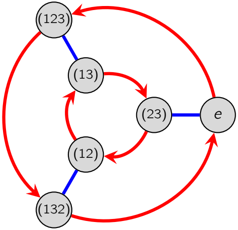
Recall that the symmetric group S_n is the group of all $n!$ permutations of $\{1, \ldots, n\}$.

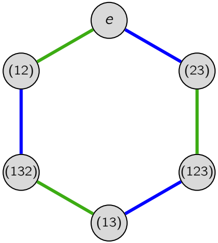
If we number the corners of an n-gon, every symmetry canonically defines a permutation.

However, not every permutation of the corners necessarily is a symmetry, unless $n = 3$.

Indeed, every permutation of $\{1, 2, 3\}$ can be realized as an element of D_3.







Remark

The groups D_n and S_n are isomorphic for $n = 3$, and non-isomorphic if $n > 3$.

The symmetric group

Instead of using configurations of the triangle, consider rearrangements of numbers:

$$\{\mathbf{123},\ \mathbf{132},\ \mathbf{213},\ \mathbf{231},\ \mathbf{312},\ \mathbf{321}\}.$$

Clearly, S_3 canonically rearranges these configurations.

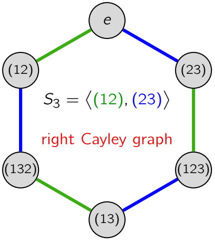
However, *there are two perfectly acceptable interpretations for "canonical."*

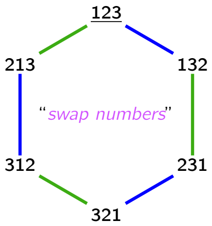
For example, (12) can be interpreted to mean

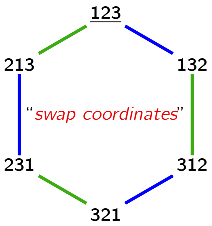
"*swap the numbers in the* 1^{st} *and* 2^{nd} *coordinates*."

Alternatively, (12) could mean

"*swap the numbers* 1 *and* 2*, regardless of where they are*."





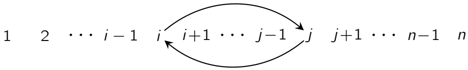


Later, we will understand this difference as a left group action vs. a right group action.

Transpositions

A transposition is a permutation that swaps two objects and fixes the rest, e.g.:

$\tau = (i\,j)$:



An adjacent transposition is one of the form $(i \;\; i{+}1)$.

The following result should be intuitive, if one thinks about rearranging n objects in a row.

Remark

There are three canonical types of generating sets for S_n:

- A transposition and an n-cycle, e.g.,:

$$S_n = \big\langle\, (1\,2),\;\; (1\,2 \cdots n{-}1 \;\; n) \big\rangle\,.$$

- Adjacent transpositions:

$$S_n = \big\langle\, (1\,2),\; (2\,3),\; \ldots,\; (n-1 \;\; n) \big\rangle\,.$$

- Overlapping transpositions:

$$S_n = \big\langle\, (1\,2),\; (1\,3),\; \ldots,\; (1 \;\; n) \big\rangle\,.$$

Polytopes and platonic solids

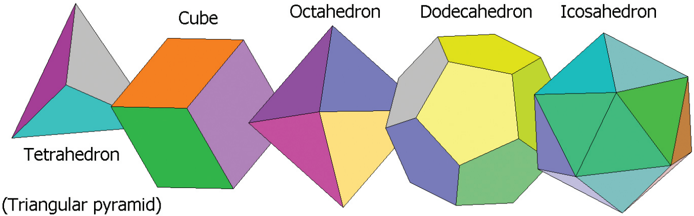
A polytope is a finite region of $\mathbb{R}^n$ enclosed by finitely many hyperplanes.

2D polytopes are *polygons*, and 3D polytopes are polyhedra.

The formal definition of a regular polytope involves a technical condition of its symmetry group.

Informally, it means all faces and all vertices are identical and indistinguishable – higher-dimensional analogues of regular polygons.

There are exactly five regular polyhedra, called Platonic solids.



Archimedean solids

More general than the Platonic solids are the Archimedean solids.

These are non-regular convex uniform polyhedra built from regular polygons.

Though they can involve different polygons, all vertices are locally identical.

In the third century B.C.E., Archimedes classified all 13 such polyhedra.

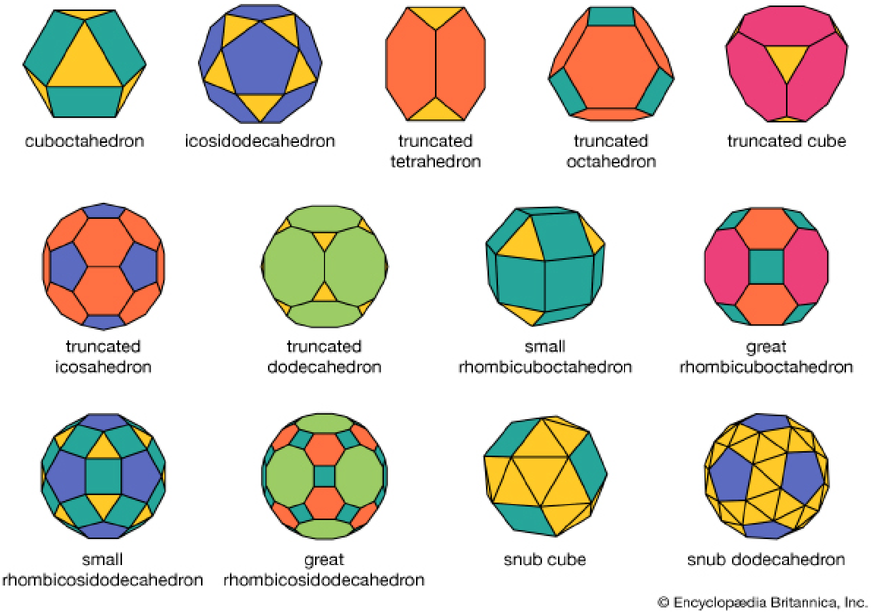
Five are "truncated versions" of the Platonic solids – formed by chopping off vertices.

The others consist of

- the chiral "snub cube" and "snub dodecahedron"
- "hybrids" such as the icosidodecahedron
- truncated versions of these hybrids.

The Cayley graph of S_4 can be arranged on the skeletons of several of these.

Archimedean solids



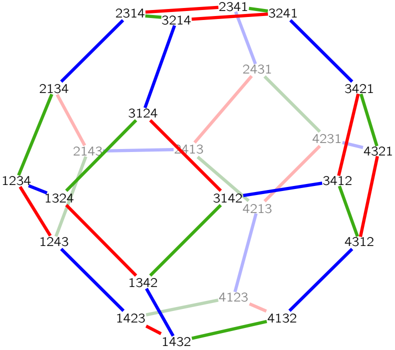
The left and right permutahedra

Definition

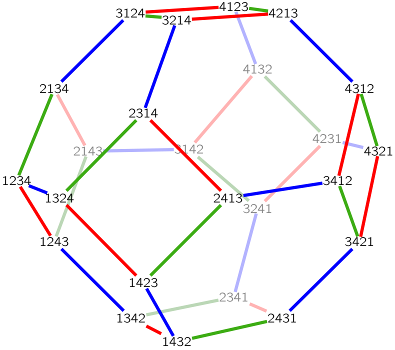
The (right) *n-permutahedron* is the convex hull of the $n!$ permutations of $(1, \ldots, n) \in \mathbb{R}^n$.

This is an $(n-1)$-dimensional polytope, as it lies on the hyperplane $x_1 + \cdots + x_n = \frac{(n-1)n}{2}$.
It is also the (right) Cayley graph of

$$S_4 = \langle (12), (23), (34) \rangle.$$



"*swap coordinates*"



"*swap numbers*"

Even and odd permutations

Remark

Even though every permutation in S_n can be written as a product of transpositions, there may be many ways to do this.

For example: $(132) = (12)(23) = (12)(23)(23)(23) = (12)(23)\,(12)\,(12)$.

Proposition

The parity of the number of transpositions of a fixed permutation is unique.

Definition

An even permutation in S_n can be written with an even number of transpositions. An odd permutation requires an odd number.

Remark

The product of:

- two even permutations is even
- two odd permutations is even
- an even an an odd permutation is odd.

The alternating groups

Definition

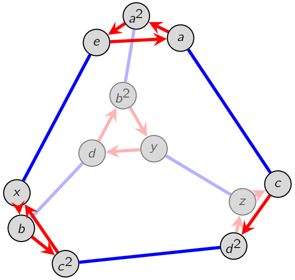
The set of even permutations in S_n is the alternating group, denoted A_n.

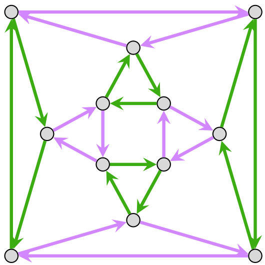
Proposition

Exactly half of the permutations in S_n are even, and so $|A_n| = \frac{n!}{2}$.

Rather than prove this using (messy) elementary methods now, we'll wait until we see the isomorphism theorems to get a 1-line proof.

Here are Cayley graphs for A_4 on a truncated tetrahedron and cuboctahedron.

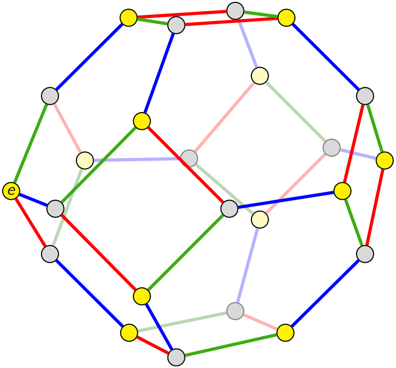




The appearance of A_4 in Cayley graphs for S_4

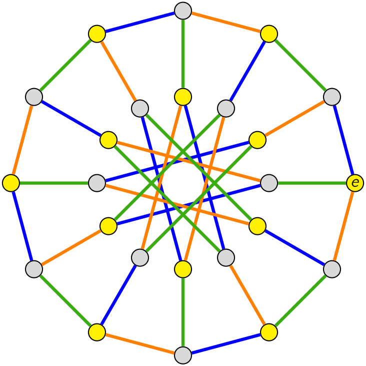
Let's highlight in yellow the even permutations in Cayley graphs for S_4.

$S_4 = \langle (12),\ (23),\ (34) \rangle$



truncated octahedron; "*permutahedron*"

$S_4 = \langle (12), (13), (14) \rangle$



"*Nauru graph*"

Notice that any two paths between yellow nodes has **even length**.

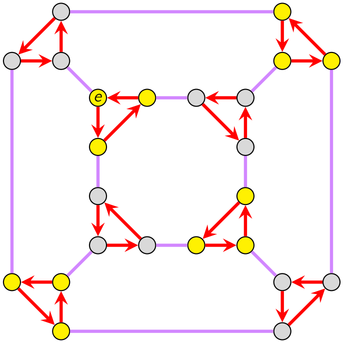
The appearance of A_4 in Cayley graphs for S_4

There are only five **cycle types** in S_4:

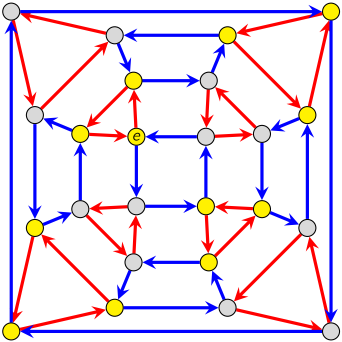
example element	e	(12)	(234)	(1234)	(12)(34)
parity	even	odd	even	odd	even
# elts	1	6	8	6	3

In both Cayley graphs, blue arrows flip the sign of the permutation; red arrows do not.

Once again, even permutations are highlighted in yellow.

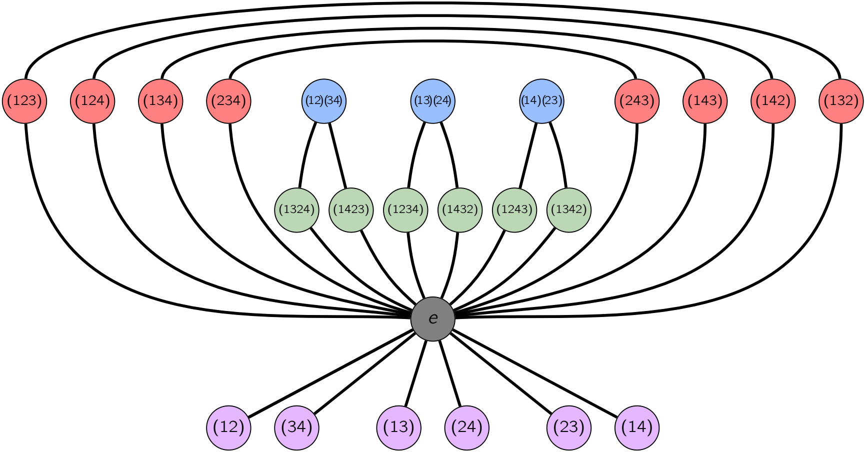


truncated cube



rhombicuboctahedron

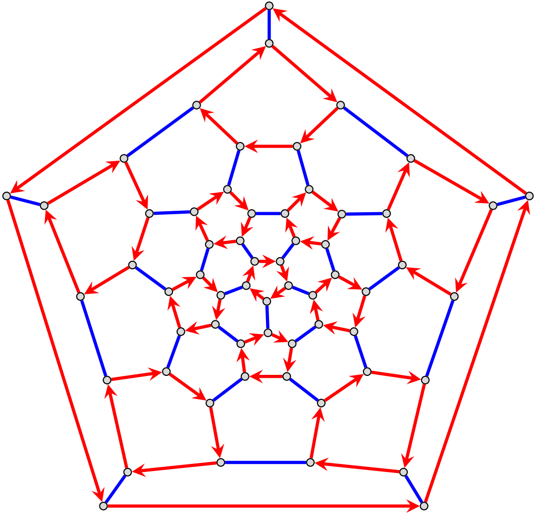
The cycle graph of S_4



A very important group

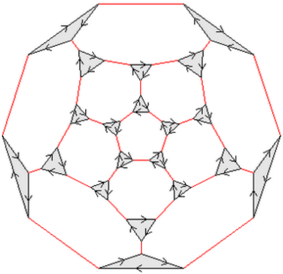
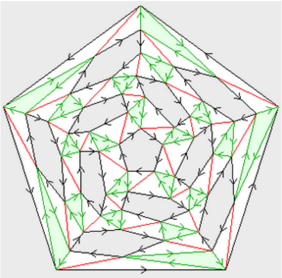
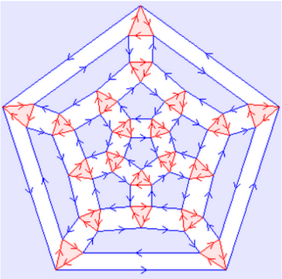
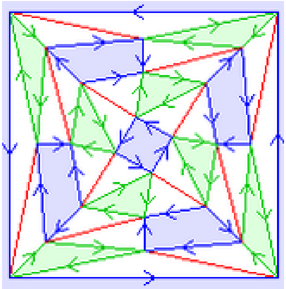
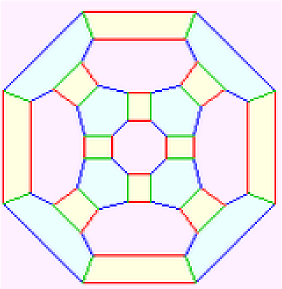
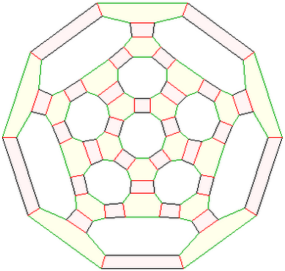
The group A_5 has special properties that we will learn about later.

Here is the Cayley graph of $A_5 = \langle (12345), (12)(34) \rangle$ on a truncated icosahedron.



More Cayley graphs on Platonic solids

Images from *Wedd's List*: `https://weddslist.com/groups/cayley-plat/`

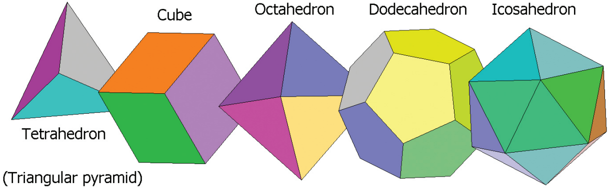







Symmetry groups of Platonic solids

Two-dimensional regular polytopes have rotation groups (C_n) and symmetry groups (D_n).

3D regular polytopes (Platonic solids) have these as well.

solid	**rotation group**	**symmetry group**
Tetrahedron	A_4	S_4
Cube	S_4	$S_4 \times C_2$
Octahedron	S_4	$S_4 \times C_2$
Icosahedron	A_5	$A_5 \times C_2$
Dodecahedron	A_5	$A_5 \times C_2$



There are higher-dimensional versions of the tetrahedron and cube, and their symmetry groups are S_n, and a group we haven't yet seen called $S_n \wr C_2$ (the "signed permutations").