Visual Algebra

Lecture 2.7: Dicyclic and diquaternion groups

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Generalizing the quaternion group

The **quaternion group** Q_8 is generated by:

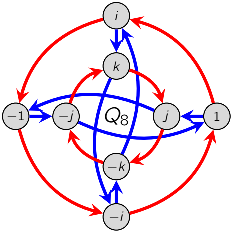
- a 4^{th} root of unity, $i = \zeta_4 = e^{2\pi i/4}$ ($2\pi/4$-rotation)
- the "imaginary number" j

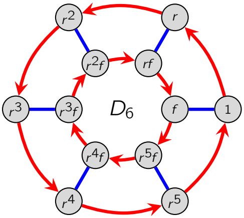
$$Q_8 = \langle i, j, k \rangle \cong \left\langle \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{R=R_4}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}}_{T=RS} \right\rangle.$$

The **dihedral group** is generated by

- an n^{th} root of unity, $r = \zeta_n = e^{2\pi i/n}$ ($2\pi/n$-rotation)
- a reflection f

$$D_n = \langle r, f \rangle \cong \left\langle \underbrace{\begin{bmatrix} \zeta_n & 0 \\ 0 & \bar{\zeta}_n \end{bmatrix}}_{R_n}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \right\rangle.$$



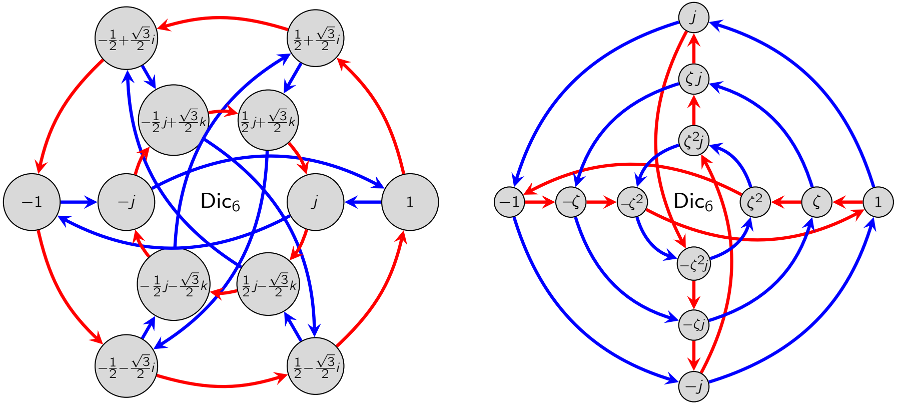


The dicyclic groups

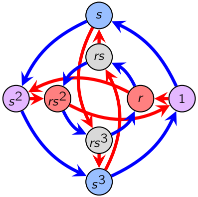
When n is even, we can replace ζ_4 with ζ_n in Q_8 to get the **dicyclic group**

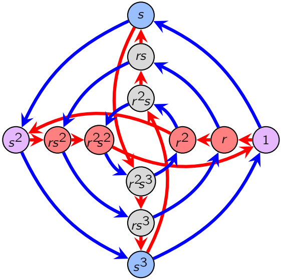
$$\mathrm{Dic}_n = \langle \zeta_n, j \rangle \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta_n} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong \langle r, s \mid r^n = s^4 = 1,\ r^{n/2} = s^2,\ rsr = s \rangle.$$

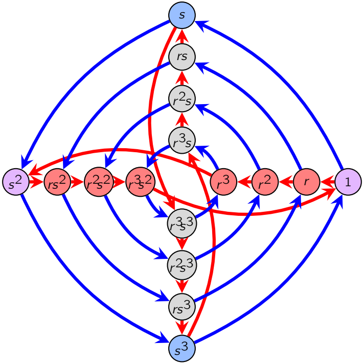
The multiplication rules $ij = k$ and $ji = -k$ remain unchanged.

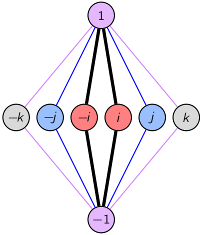


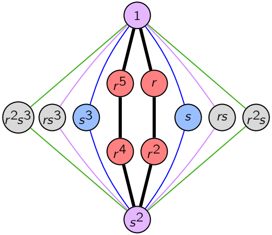
The dicyclic groups

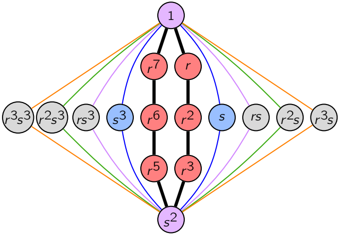










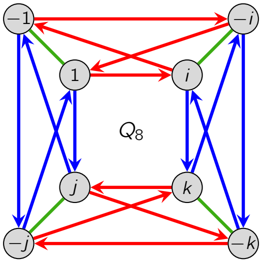


A quotient of the dicyclic group Dic_4

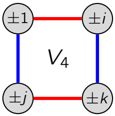
The quaternion group is $Q_8 = \langle \zeta_4, j \rangle = \{\pm 1, \pm i, \pm j, \pm k\} = \text{Dic}_4$.

Recall how we constructed a quotient of Q_8, which was

$$Q_8 / \langle -1 \rangle \cong V_4.$$



	1	−1	i	−i	j	−j	k	−k
1	1	−1	i	−i	j	−j	k	−k
−1	−1	1	−i	i	−j	j	−k	k
i	i	−i	−1	1	k	−k	−j	j
−i	−i	i	1	−1	−k	k	j	−j
j	j	−j	−k	k	−1	1	i	−i
−j	−j	j	k	−k	1	−1	−i	i
k	k	−k	j	−j	−i	i	−1	1
−k	−k	k	−j	j	i	−i	1	−1



	±1	±i	±j	±k
±1	±1	±i	±j	±k
±i	±i	±1	±k	±j
±j	±j	±k	±1	±i
±k	±k	±j	±i	±1

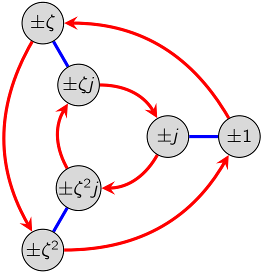
We can do a similar construction for dicyclic groups.

Note that $V_4 \cong D_2 = \langle r, f \mid r^2 = 1,\ f^2 = 1,\ rfr = f \rangle$.

A quotient of the dicyclic group D_n

The quotient of the dicyclic group Dic_6 by $\langle -1 \rangle = \{1, -1\}$ is

$$\text{Dic}_6 / \langle -1 \rangle \cong D_3.$$



	±1	±ζ	±ζ²	±j	±ζj	±ζ²j
±1	±1	±ζ	±ζ²	±j	±ζj	±ζ²j
±ζ	±ζ	±ζ²	±1	±ζj	±ζ²j	±j
±ζ²	±ζ²	±1	±ζ	±ζ²j	±j	±ζj
±j	±j	±ζ²j	±ζj	±1	±ζ²	±ζ
±ζj	±ζj	±j	±ζ²j	±ζ	±1	±ζ²
±ζ²j	±ζ²j	±ζj	±j	±ζ²	±ζ	±1

The product $(\pm\zeta j) \cdot (\pm\zeta^2 j) = \pm\zeta^2$ means:

"*the product of any element in* $\{\zeta j, -\zeta j\}$ *with any element in* $\{\zeta^2 j, -\zeta^2 j\}$ *is in* $\{\zeta^2, -\zeta^2\}$."

More generally, it will hold that $\text{Dic}_n / \langle -1 \rangle \cong D_{n/2}$.

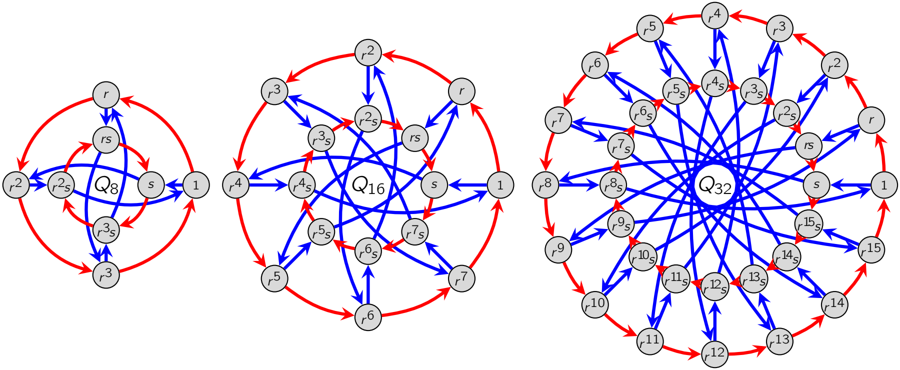
Generalized quaternion groups

When $n = 2^m$, the dicyclic group $\text{Dic}_{2^{m-1}}$ is called the generalized quaternion group, Q_{2^n}.

Remark

In a generalized quaternion group $\text{Dic}_n = Q_{2n}$, every nontrivial orbit $\langle g \rangle$ contains $r^{n/2} = -1$.

As we'll see, this gives Q_{2n} certain properties that general dicyclic groups lack.



The diquaternion group

Recall our standard representations of the quaternion and dihedral groups:

$$Q_8 = \langle i, j, k\rangle \cong \Bigg\langle \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{R=R_4}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}}_{T=RS} \Bigg\rangle, \qquad D_n = \langle r, f\rangle \cong \Bigg\langle \underbrace{\begin{bmatrix} \zeta_n & 0 \\ 0 & \bar{\zeta}_n \end{bmatrix}}_{R_n}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \Bigg\rangle.$$

Now, consider the group generated by adding the reflection matrix from D_n to Q_8.

This is the Pauli group on 1 qubit. We will call it the diquaternion group

$$\mathrm{DQ}_8 = \langle X, Y, Z\rangle = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\},$$

generated by the Pauli matrices from quantum mechanics and information theory:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to check that

$$XY = R \quad \text{"}i\text{"}, \qquad XZ = S \quad \text{"}j\text{"}, \qquad YZ = \overline{T} \quad \text{"}-k\text{"}.$$

This group can be constructed in other ways as well:

- as a semidirect product, $Q_8 \rtimes_2 C_2$, and $D_4 \rtimes_2 C_2$, and $(C_4 \times C_2) \rtimes_3 C_2$.
- as the "central product" $\mathrm{DQ}_8 = C_4 \circ D_4$.

The diquaternion group

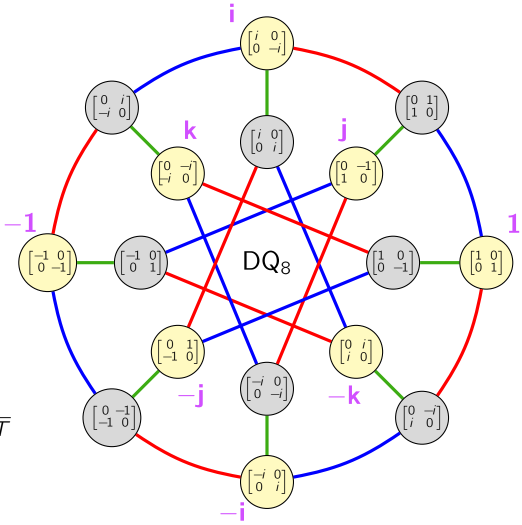
$$\mathrm{DQ}_8 = \langle X, Y, Z \mid X^2 = Y^2 = Z^2 = I,\ (XY)^4 = I,\ (XY)Z = Z(XY) \rangle$$

$$X = F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

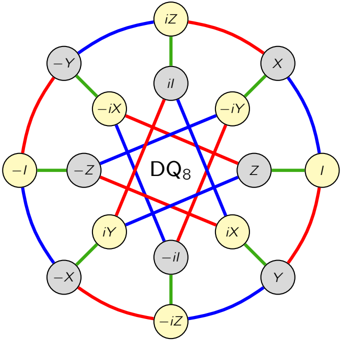
$XY = R, \quad XZ = S, \quad YZ = \overline{T}$

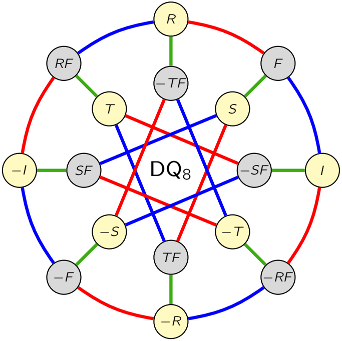


The diquaternion group

The diquaternion group is usually generated with Pauli matrices, $\mathrm{DQ}_8 = \langle X, Y, Z \rangle$.

We can also write it as $\mathrm{DQ}_8 = \langle R, S, T, F \rangle$ where $Q_8 = \langle R, S, T \rangle$ and $D_n = \langle R_n, F \rangle$.





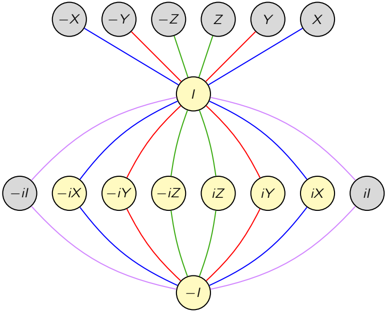
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad R = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

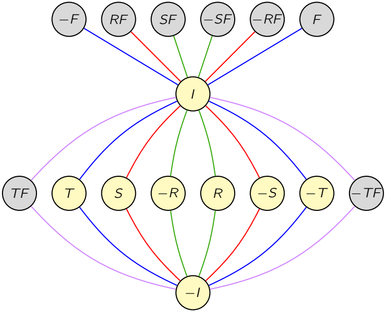
What group do you think the quotient $\mathrm{DQ}_8 / \langle -1 \rangle$ will be?

The diquaternion group

Here are two cycle graphs for

$$\mathrm{DQ}_8 = \langle X, Y, Z \rangle = \langle R, S, T, F \rangle.$$





$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad R = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Do you see a way to generalize this further? What if we use a different root of unity?

Generalized diquaternion groups

If $n=2^m$, replace $i=\zeta_4=e^{2\pi i/4}$ with $\zeta_n=e^{2\pi i/n}$ to get the generalized diquaternion group.

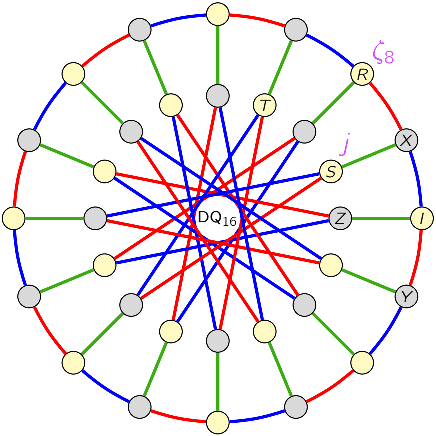
$$\mathrm{DQ}_n := \langle \zeta_n, j, \zeta_n j, f \rangle \cong \left\langle \underbrace{\begin{bmatrix} \zeta_n & 0 \\ 0 & \bar{\zeta}_n \end{bmatrix}}_{R=R_n}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -\zeta_n \\ \bar{\zeta}_n & 0 \end{bmatrix}}_{T=T_n}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \right\rangle \cong \mathrm{Dic}_n \rtimes_\theta C_2.$$

$$X = F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y := Y_8 = \begin{bmatrix} 0 & \bar{\zeta}_8 \\ \zeta_8 & 0 \end{bmatrix}$$

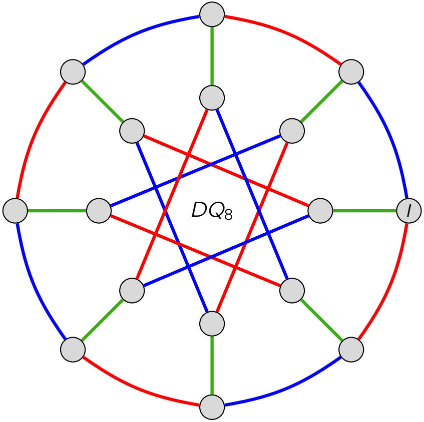
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

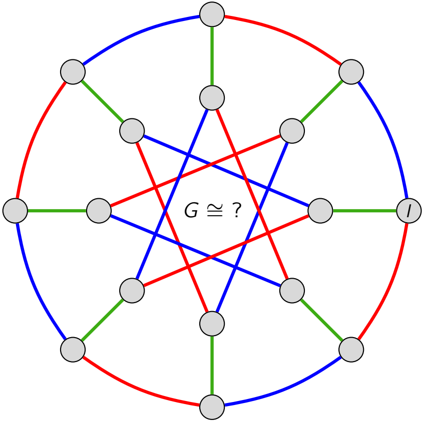
$$XY_8 = R_8, \qquad XZ = S, \qquad Y_8 Z = \overline{T}_8$$



A fun group theory puzzle

Do you see why these two groups cannot be isomorphic?





So, what group *is* the one on the right?

A fun group theory puzzle

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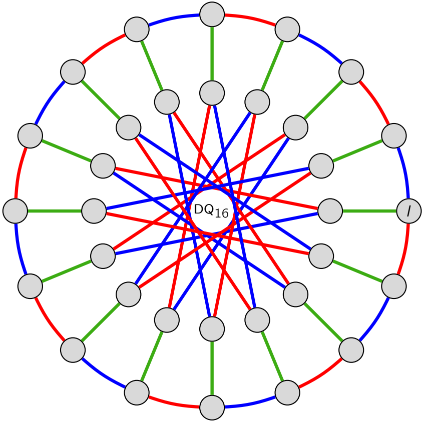

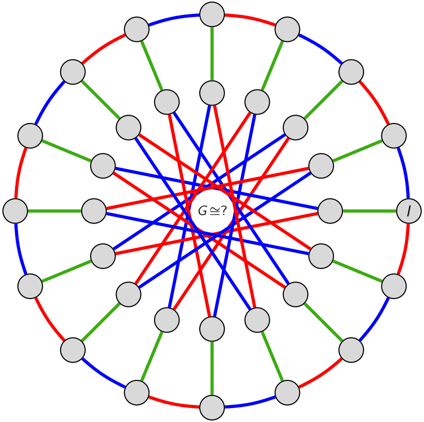

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