Visual Algebra

Lecture 2.8: Semidihedral and semiabelian groups

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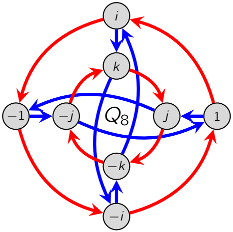
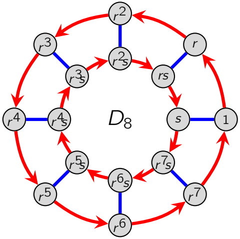
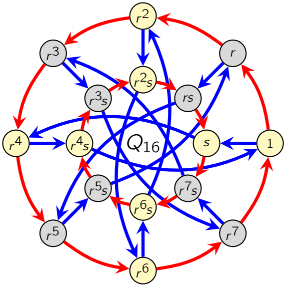
Generalizing the quaternion group

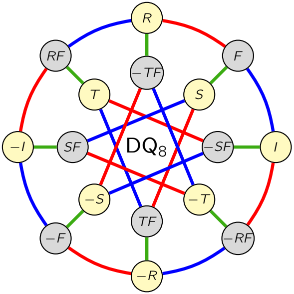
Last lecture, we started with the quaternion group, and using a dihedral group





we constructed the dicyclic and diquaternion groups:



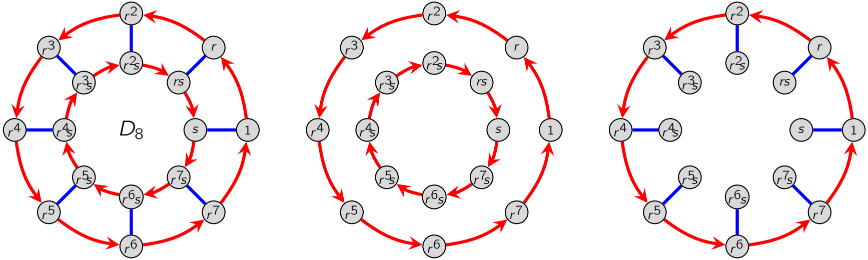


Generalizing the dihedral groups

We could have constructed the dicyclic groups by starting with a Cayley graph of $D_n = \langle r, f \rangle$.

Then, we could remove the blue arcs and investigate how they can be rewired.

But what if we kept those, but rewired the inner length-n red cycle?



In other words, we want to construct a group G that

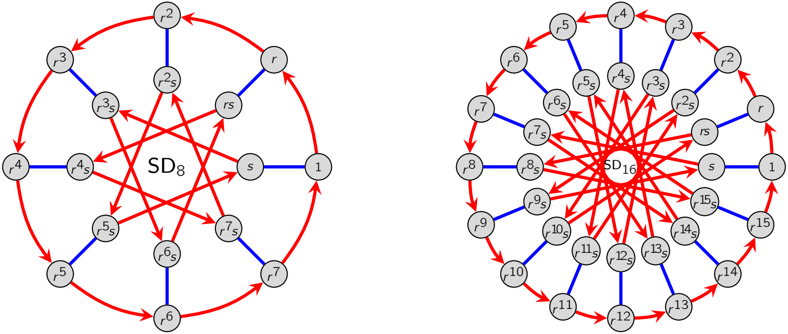
- has an element r of order n
- has an element $s \notin \langle r \rangle$ of order 2.

Equivalently, what can we replace the relation $srs = r^{n-1}$ with? That is,

$$G = \big\langle r, s \mid r^n = 1,\ s^2 = 1,\ ??? \big\rangle.$$

Semidihedral groups

If n is a power of 2, we can replace $srs = r^{n-1}$ with $srs = r^{n/2-1}$.



Definition

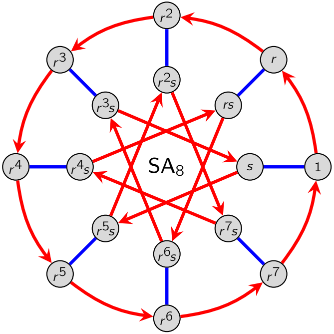
For each power of two, the semidihedral group of order 2^n is defined by

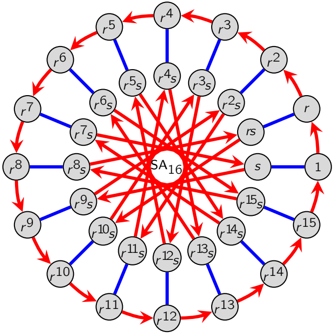
$$SD_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}-1} \rangle.$$

Do you see another way we can rewire these inner red arrows?

Semiabelian groups

Still assuming n is a power of 2, let's replace $srs = r^{n/2-1}$ with $srs = r^{n/2+1}$.





Definition

For each power of two, the semiabelian group of order 2^n is defined by

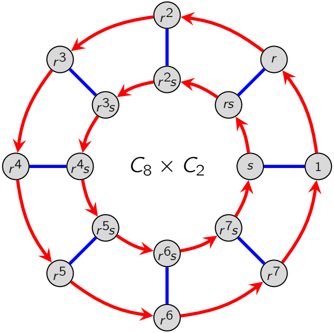
$$\mathrm{SA}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}+1} \rangle.$$

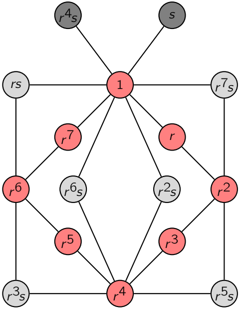
Do you see another way we can rewire these inner red arrows?

One more rewiring

Of course, there's one more way that we can rewire the dihedral group...

Here is its Cayley graph and cycle graph.





When this group has order 2^n, its presentation is

$$C_{2^n-1} \times C_2 = \langle r, s \mid r^{2^{n-1}} = s^2 = 1, \; srs = r \rangle.$$

Remarkably, this and the other three we've seen are the *only* possibilities:

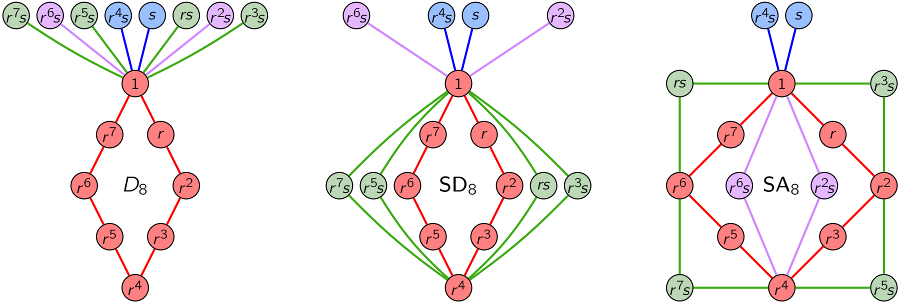
$srs = r^{-1}$ (dihedral), $\quad srs = r^{2^{n-2}-1}$ (semidihedral), $\quad srs = r^{2^{n-2}+1}$ (semiabelian).

Dihedral vs. semidihedral vs. semiabelian groups

In other words, there are exactly 4 groups of order 2^n with both:

- an element r of order 2^{n-1}
- an element $s \notin \langle r \rangle$ of order 2.

Let's compare the cycle graphs of the three non-abelian groups from this list:



Remark

The semiabelian group SA_n and the abelian group $C_n \times C_2$ have the same orbit structure!

This surprising fact has profound consequences that we'll see when we study subgroups.

Dihedral vs. semidihedral vs. semiabelian groups

Compare and contrast representations of the dihedral and semidihedral group:

$$D_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \mathrm{SD}_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & -\overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \zeta_n = e^{2\pi i/n}.$$

Now, compare and contrast those of the abelian and semiabelian group:

$$C_n \times C_2 \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \mathrm{SA}_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & -\zeta_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

Mnemonic: "semi-" = "*halfway around unit circle*" = $\zeta^{n/2} = -1$.

The groups SD_n and SA_n only exist when $n = 2^m$. In this case, we also have

$$Q_{2^{m+1}} = \mathrm{Dic}_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle,$$

called the generalized quaternion group.

Note that for *any* $n \in \mathbb{N}$, the matrices above generate *some* group.

Exploratory question

What groups do the above representations give if, e.g., n is odd, or not a power of 2?

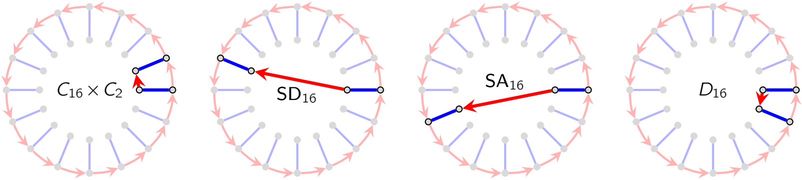
Non-abelian groups of order 2^n

We'll understand the following better when we study semi-direct products of groups.

Theorem

There are exactly four nonabelian groups of order 2^n that have an element r of order 2^{n-1}:

1. The **dihedral group** $D_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{-1} \rangle$.
2. The dicyclic group $\mathrm{Dic}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^4 = 1,\ r^{2^{n-2}} = s^2,\ rsr = s \rangle$.
3. The semidihedral group $\mathrm{SD}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}-1} \rangle$.
4. The semiabelian group $\mathrm{SA}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}+1} \rangle$.

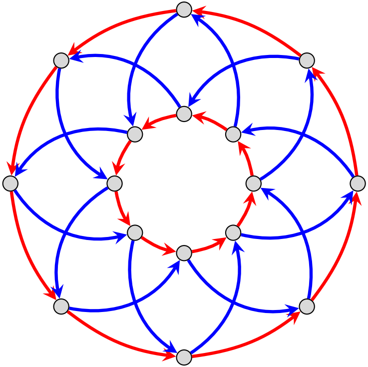


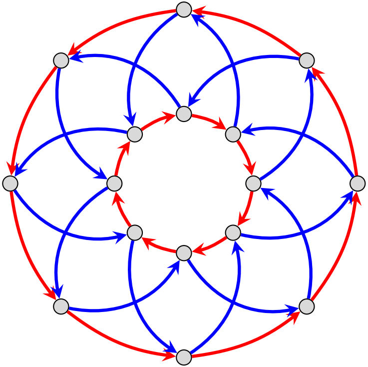
As we did before, we can ask:

what groups do these presentations describe when $2n$ is not a power of 2?

More fun group theory puzzles

Are these Cayley graphs of groups?





If so, what groups are they?

Since there is an element of order 8, there are only six possibilities:

$$C_{16}, \qquad C_8 \times C_2, \qquad D_8, \qquad \mathsf{SD}_8, \qquad \mathsf{SA}_8, \qquad Q_{16}.$$

More fun group theory puzzles

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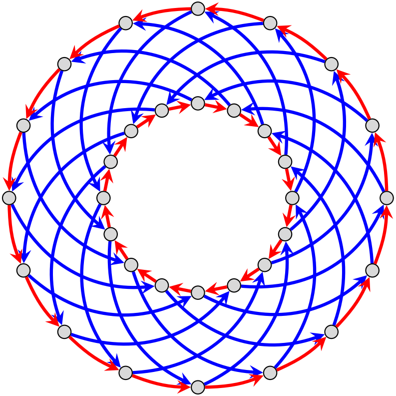
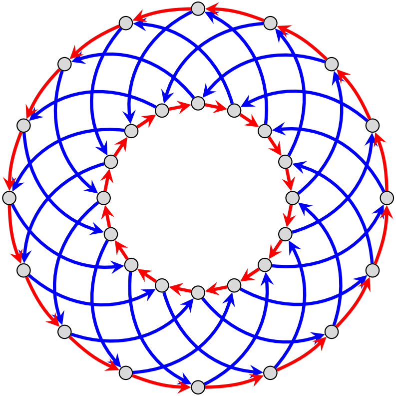




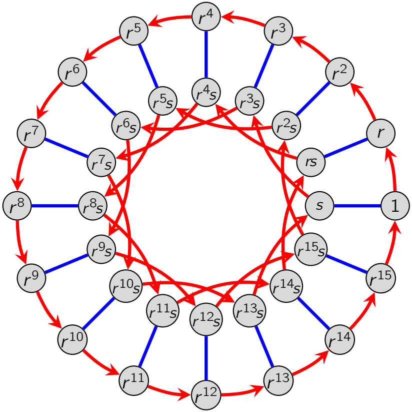
If so, what groups are they?

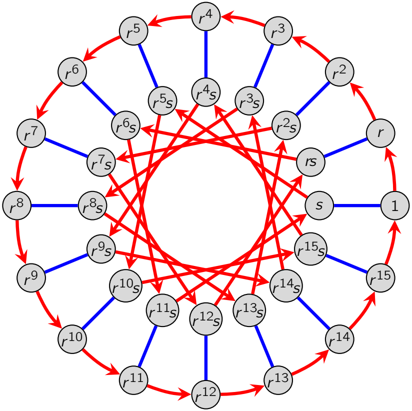
Since there is an element of order 16, there are only six possibilities:

$$C_{32}, \qquad C_{16} \times C_2, \qquad D_{16}, \qquad \mathsf{SD}_{16}, \qquad \mathsf{SA}_{16}, \qquad Q_{32}.$$

More fun group theory puzzles

Are these Cayley graphs of groups?





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